Physics 101 P
General Physics I

Problem Sessions - Week 2

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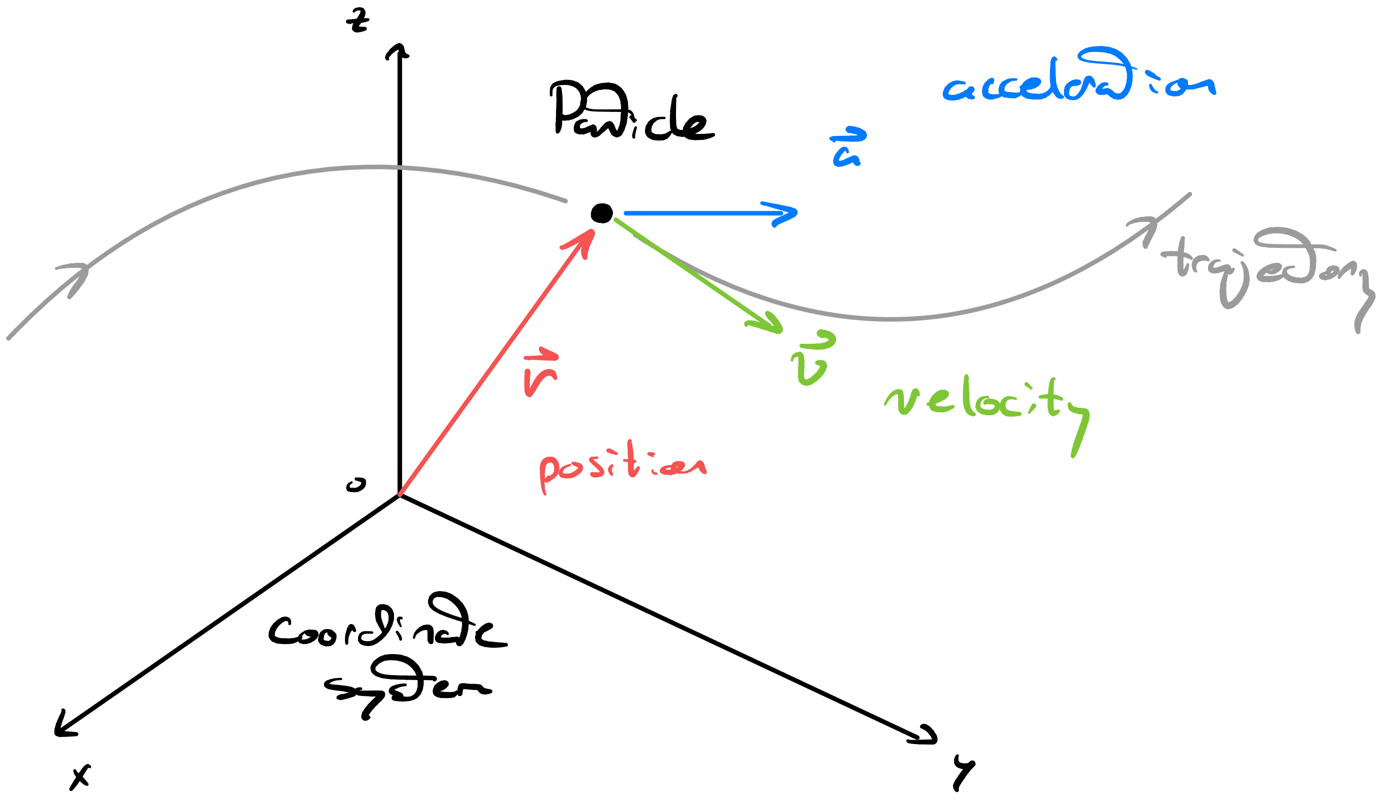
Motion

Kinematics is the study of motion (without regards for its cause)



$$\vec{v} = \frac{d\vec{r}}{dt} \quad \text{velocity}$$

$$\vec{a} = \frac{d\vec{v}}{dt} \quad \text{acceleration}$$

Example

The position of a particle is given by

$$x(t) = A e^{Bt} + Ct$$

where A, B, C are known constants.

(a) What are the dimensions of A, B, & C?

(b) What is the particles velocity?

(c) What is the particles acceleration?

Solution

(a) Know, $[x] = L$, $[t] = T$

Also, $[e^{Bt}] = 1 \Rightarrow [B][t] = 1$

$$\Rightarrow [B] = \frac{1}{T} \blacksquare$$

then,

$$[x] = [A]\cdot 1 + [C]\cdot T$$

$$\Rightarrow [A] = L \blacksquare$$

$$[C] = \frac{L}{T} \blacksquare$$

(b) Recall $v(t) = \dfrac{dx}{dt}$

$$= \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t}$$

Useful derivatives to know:

if a = constant

- $\dfrac{da}{dx} = 0$
- $\dfrac{d}{dx}(a x^n) = a n x^{n-1}$
- $\dfrac{d}{dx}(e^{ax}) = a e^{ax}$
- $\dfrac{d}{dx}(\sin(ax)) = a \cos(ax)$
- $\dfrac{d}{dx}(\cos(ax)) = -a \sin(ax)$
- $\dfrac{d}{dx}(\ln(ax)) = \dfrac{1}{x}$

So,

$$v = \frac{dx}{dt}$$

$$= \frac{d}{dt}\left(A e^{Bt} + Ct\right)$$

$$= \frac{d}{dt}\left(A e^{Bt}\right) + \frac{d}{dt}(Ct)$$

$$= AB e^{Bt} + C$$

$$\Rightarrow \boxed{v(t) = AB e^{Bt} + C}$$

(c)

$$a = \frac{dv}{dt}$$

$$= \frac{d}{dt}\left(AB e^{Bt} + C\right)$$

$$= \frac{d}{dt}\left(AB e^{Bt}\right) + \frac{d}{dt}C$$

$$= AB^2 e^{Bt}$$

$$\Rightarrow \boxed{a(t) = AB^2 e^{Bt}}$$

Example

The acceleration of a particle is constant,

$$a = A .$$

If the particle starts from the origin at rest, what is its velocity & position as a function of time?

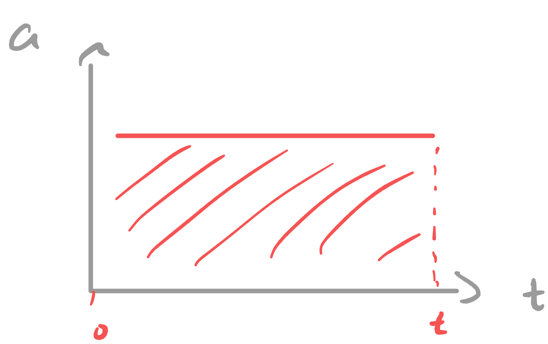
Solution

if $a = A = \text{constant}$, we want to invert $a = \frac{dv}{dt}$ & $v = \frac{dx}{dt}$ for v & x

Subject to the constraint $x(t=0) = x_0 = 0$
$v(t=0) = v_0 = 0$

$$a = \frac{dv}{dt} \Rightarrow dv = a\,dt$$

Integrate! $\displaystyle\int_{v_0}^{v} dv = \int_0^t a\,dt$



Useful integrals:

if $a = \text{constant}$,

- $\displaystyle\int_{x_1}^{x_2} a\,dx = ax\Big|_{x_1}^{x_2} = a(x_2 - x_1)$

- $\displaystyle\int_{x_1}^{x_2} a x^n\,dx = \frac{a x^{n+1}}{n+1}\Big|_{x_1}^{x_2} = \frac{a}{n+1}\left(x_2^{n+1} - x_1^{n+1}\right)$

- $\displaystyle\int_{x_1}^{x_2} \frac{1}{x}\,dx = \ln(x)\Big|_{x_1}^{x_2} = \ln\left(\frac{x_2}{x_1}\right)$

So, since $a = \text{constant}$,

$$\int_{v_0}^{v} dv = At \Rightarrow v(t) = At$$

Similarly, $\dfrac{dx}{dt} = v \Rightarrow dx = v\,dt$

$$\Rightarrow \int_{x_0}^{x} dx = \int_0^t v(t)\,dt$$

$$= \int_0^t At\,dt = \frac{At^2}{2}$$

So, $x = x_0 + \frac{1}{2}At^2$

so, $x_0 = 0$

$$\Rightarrow \quad x(t) = \frac{1}{2}At^2$$

So, for $a = A = \text{const}$

& $x = 0$, $v = 0$ @ $t = 0$,

$$\boxed{\begin{aligned} v(t) &= At \\ x(t) &= \frac{1}{2}At^2 \end{aligned}}$$

Recall in general for constant acceleration motion,

$$a = \text{constant}$$

$$v = v_0 + at$$

$$x = x_0 + v_0 t + \frac{1}{2}at^2$$

Example

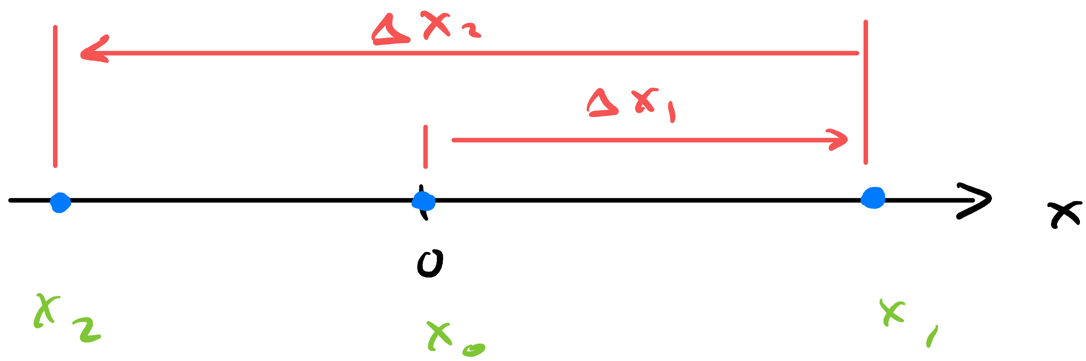
A car moves 72 km to the right, & then reverses direction & travels 120 km to the left.

(a) What is the displacement of the car?

(b) What is the distance traveled by the car?

Solution

(a) Set origin at starting point of the car.



$$\Delta x_1 = x_1 - x_0 = 72 \text{ km}$$

$$\Delta x_2 = x_2 - x_1 = -120 \text{ km}$$

Total displacement

$$\Delta x = x_2 - x_0$$

$$= x_2 - x_1 + x_1 - x_0$$

$$= \Delta x_2 + \Delta x_1$$

$$= -120 \text{ km} + 72 \text{ km}$$

$$= -48 \text{ km}$$

$$\boxed{\Delta x = -48 \text{ km}}$$

(b) Total distance $= d$

$$d = |\Delta x_1| + |\Delta x_2|$$

$$= 72 \text{ km} + 120 \text{ km}$$

$$= 192 \text{ km}$$

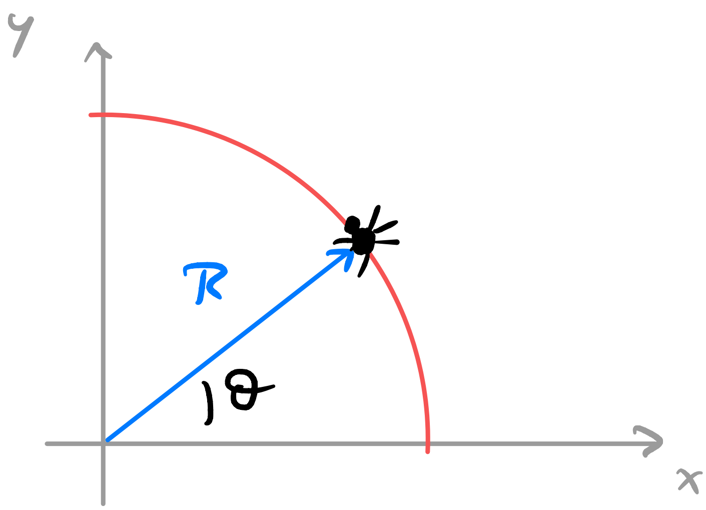
$$\boxed{d = 192 \text{ km}}$$

N.B. $d \neq |\Delta x|$

Example

A bug travels on a quarter circle of radius R. Find the displacement & the distance traveled by the bug.

Solution



Displacement

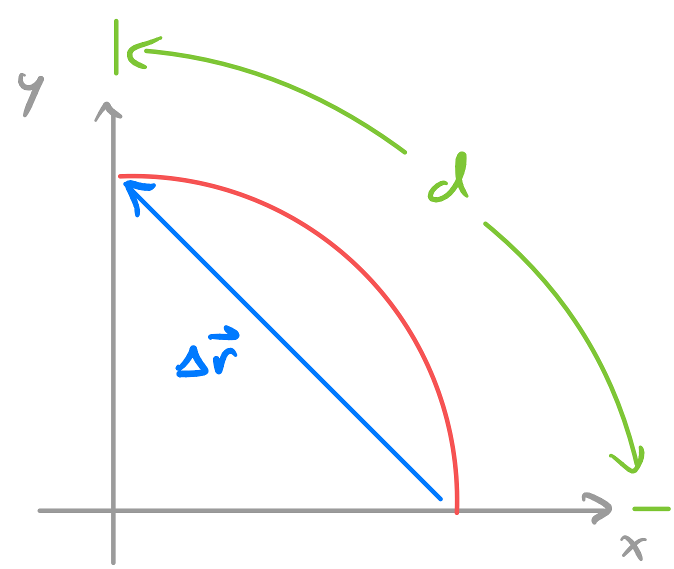
$$\Delta\vec{r} = \vec{r}_f - \vec{r}_i$$

$$\vec{r}_i = R\,\hat{\imath} \qquad \text{(initial)}$$

$$\vec{r}_f = R\,\hat{\jmath} \qquad \text{(final)}$$

$$\Rightarrow \boxed{\Delta\vec{r} = -R(\hat{\imath} - \hat{\jmath})}$$

$$\text{Distance} = d = \frac{C}{4}$$ — circumference of circle



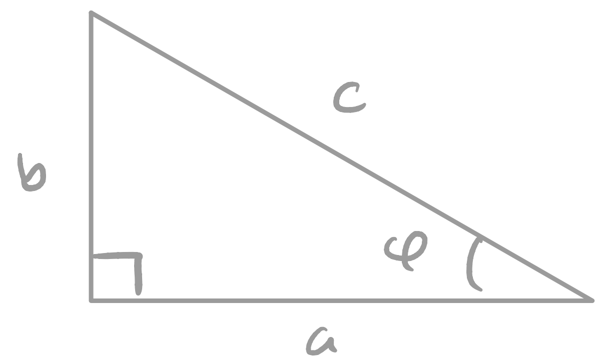
$$C = 2\pi R$$

$$\Rightarrow \boxed{d = \frac{\pi R}{4}}$$

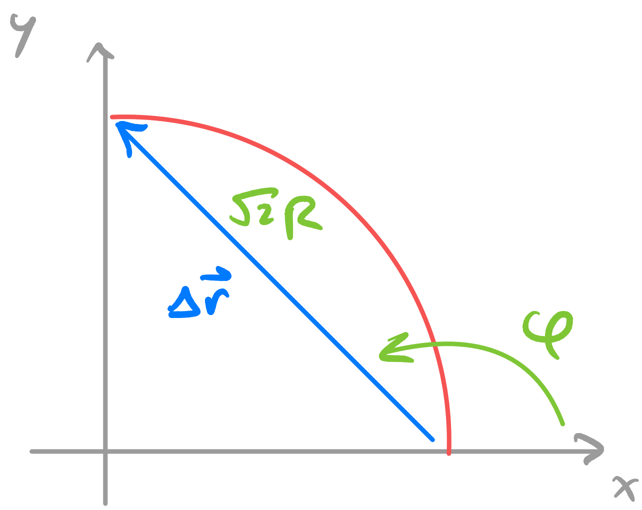
Can also express displacement as

$$|\Delta\vec{r}| = \sqrt{R^2 + R^2}$$
$$= \sqrt{2}\,R$$

$$\varphi = \tan^{-1}\left(\frac{R}{-R}\right) = 135^\circ$$



$$\tan\varphi = \frac{b}{a}$$



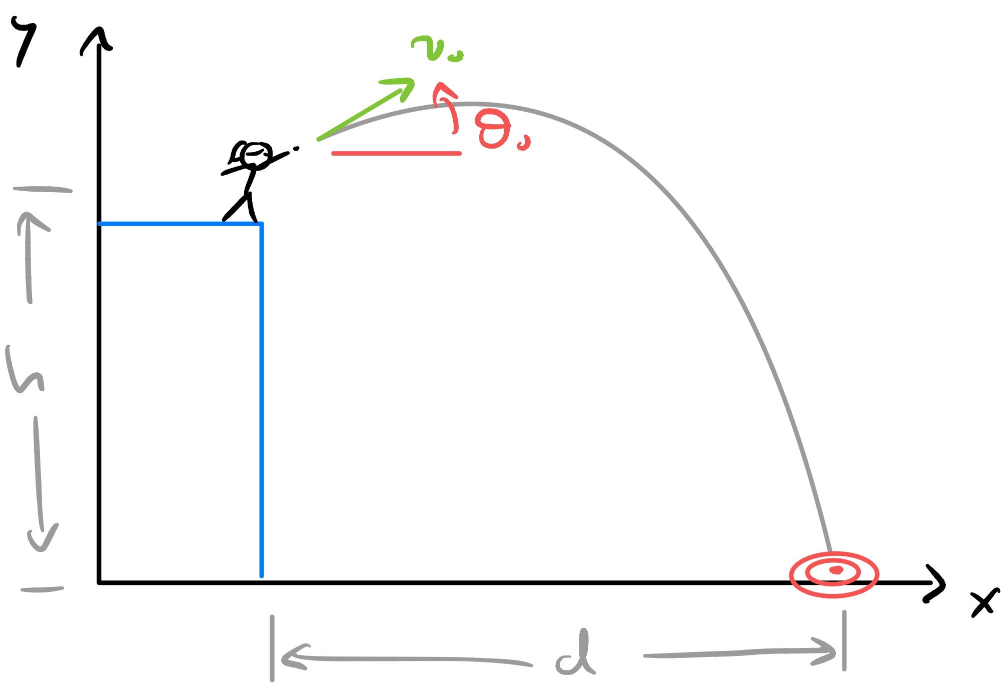
N.B. $|\Delta\vec{r}| \neq d$!

$$|\Delta\vec{r}| = \sqrt{2}\,R \quad , \quad d = \frac{\pi R}{2}$$

Example

Alice throws a ball off a building a height h. She throws at an angle θ_0, and hits a target which is a distance d from the base of the building.

At what speed v_0 must Alice throw the ball to hit the target?



Solution

Break motion into x & y directions

$$x = x_0 + v_{0x} t + \frac{1}{2} a_x t^2$$

$$y = y_0 + v_{0y} t + \frac{1}{2} a_y t^2$$

in x: $\quad x_0 = 0$

$$v_{0x} = v_0 \cos\theta_0$$

$$a_x = 0$$

in y: $\quad y_0 = h$

$$v_{0y} = v_0 \sin\theta_0$$

$$a_y = -g$$

$$\Rightarrow \begin{cases} x = v_0 \cos\theta_0 \, t \\ y = h + v_0 \sin\theta_0 \, t - \frac{1}{2} g t^2 \end{cases}$$

Now, want v_0 such that $y=0$, $x=d$

(Alice hits the target).

Let $t = T$ be the time when ball hits.

$$\begin{cases} d = v_0 \cos\theta_0 T & (1) \\ 0 = h + v_0 \sin\theta_0 T - \frac{1}{2} g T^2 & (2) \end{cases}$$

two equations, two unknowns (v_0 & T)

solve for v_0!

from (1), $T = \dfrac{d}{v_0 \cos\theta_0}$ (1^*)

Sub (1^*) into (2)

$$\Rightarrow 0 = h + v_0 \sin\theta_0 \left(\frac{d}{v_0 \cos\theta_0} \right) - \frac{1}{2} g \frac{d^2}{v_0^2 \cos^2\theta_0}$$

$$= h + d \tan\theta_0 - \frac{g d^2}{2 v_0^2 \cos^2\theta_0}$$

solve for v_0

$$\Rightarrow \frac{g d^2}{2 v_0^2 \cos^2\theta_0} = h + d \tan\theta_0$$

$$\frac{2v_0^2\cos^2\theta_0}{gd^2} = \frac{1}{h+d\tan\theta_0}$$

$$\Rightarrow \quad v_0^2 = \frac{d^2}{\cos^2\theta_0}\left(\frac{g}{2(h+d\tan\theta_0)}\right)$$

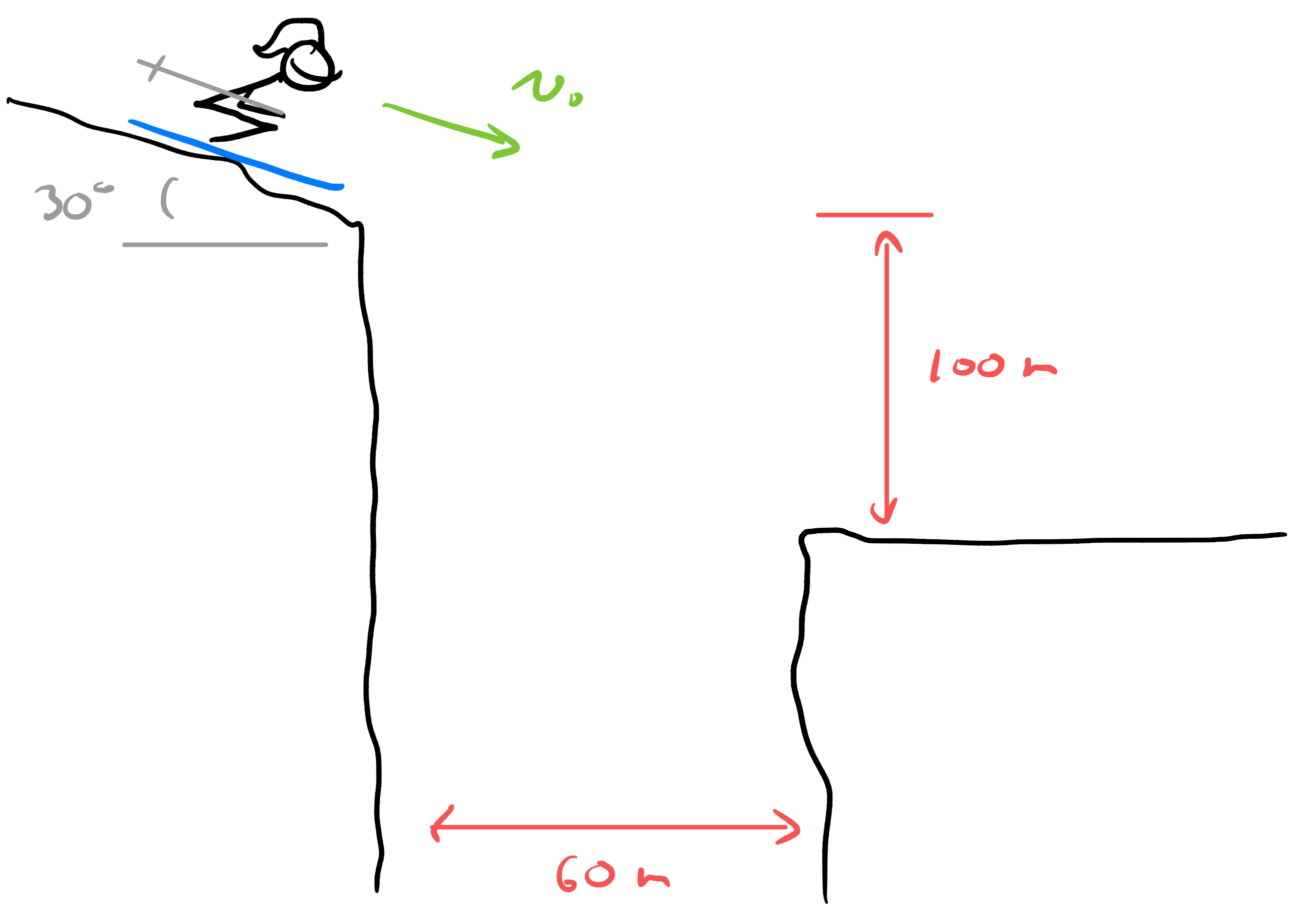
So,

$$\boxed{v_0 = \frac{d}{\cos^2\theta_0}\sqrt{\frac{g}{2(h+d\tan\theta_0)}}}$$

Why not negative sign?

Example

Trying to escape his pursuers, secret agent Alice skis off a slope inclined at $30°$ below the horizontal at 60 km/h. To survive and land on the snow 100 m below, she must clear a gorge 60 m wide. Does she make it?



Solution

Let $v_0 = 60$ km/h

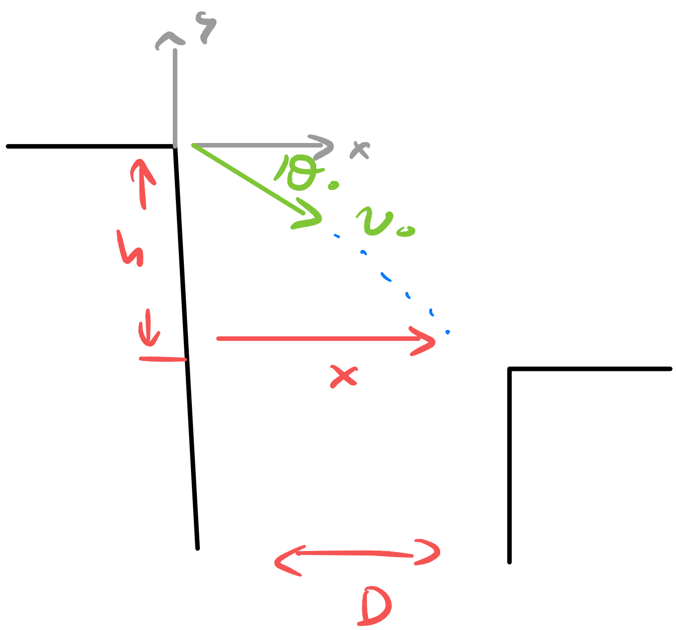
$\theta_0 = 30^\circ$

$h = 100$ m

$D = 60$ m

Want to know if trajectory is such that $x > D$.

Set coordinate system at point before "take off"



So, $v_{0x} = v_0 \cos\theta_0$

$v_{0y} = -v_0 \sin\theta_0$

Equations of motion

$$x = v_{0x} t$$

$$y = v_{0y} t - \frac{1}{2} g t^2$$

So, solving for t when $y = -h$

$$-h = v_{0y} t - \frac{1}{2} g t^2$$

$$\Rightarrow \frac{1}{2} g t^2 - v_{0y} t - h = 0$$

$$t = \frac{+v_{0y} \pm \sqrt{v_{0y}^2 - 4\left(\frac{1}{2} g\right)(-h)}}{2\left(\frac{1}{2} g\right)}$$

$$= \frac{v_{0y} \pm \sqrt{v_{0y}^2 + 2gh}}{g}$$

Now, $v_{0y} = -v_0 \sin\theta_0$

$$v_0 = 60 \frac{km}{h} = 60 \frac{\times 10^3 m}{h} \cdot \left(\frac{1 h}{3600 s}\right)$$

$$= 16.7 \text{ m/s}$$

$$\Rightarrow v_{0y} = -8.35 \text{ m/s}$$

Therefore, $t = \underline{\underline{3.75 \text{ s}}}$ or -5.45 s

So,

$$x\Big|_{t=t_{hit}} = v_{0x}\cos\theta\, t_{hit}$$

$$\simeq 54.2 \text{ m}$$

$\Rightarrow$ Alice does not make it $\ddot{\frown}$