Physics 101 P
General Physics I

Problem Sessions - Week 7

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Momentum

$$\vec{p} = m\vec{v}$$

If a system is composed of multiple bodies

$$\vec{P}_{tot} = \sum_{i=1}^{N} \vec{p}_i$$

$$= \sum_{i=1}^{N} m_i \vec{v}_i$$

Center of mass: $\vec{R}_{cm} = \dfrac{\sum_{i=1}^{N} m_i \vec{r}_i}{M}$ ~ $M = \sum_{i=1}^{N} m_i$

So,

$$\vec{P}_{tot} = \sum_{i=1}^{N} m_i \vec{v}_i$$

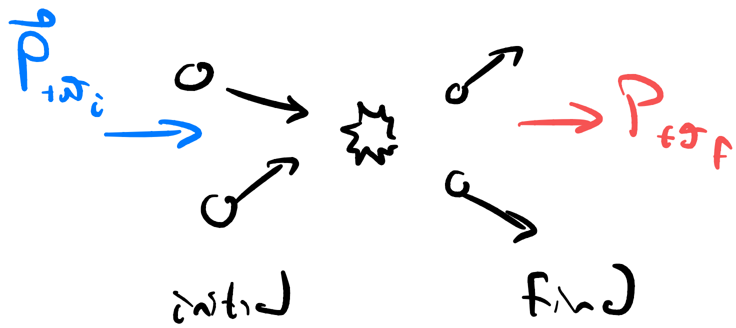
$$= \sum_{i=1}^{N} \frac{d}{dt}(m_i \vec{r}_i) = \frac{d}{dt}\left[\sum_{i=1}^{N} m_i \vec{r}_i\right]$$

$$= M \frac{d\vec{R}_{cm}}{dt} = M \vec{V}_{cm}$$

Collisions

$$\vec{P}_{tot_i} = \vec{P}_{tot_f}$$

Momentum conservation



- If a collision conserves energy

  ⇒ Elastic collision

- If energy not conserved

  ⇒ Inelastic collision

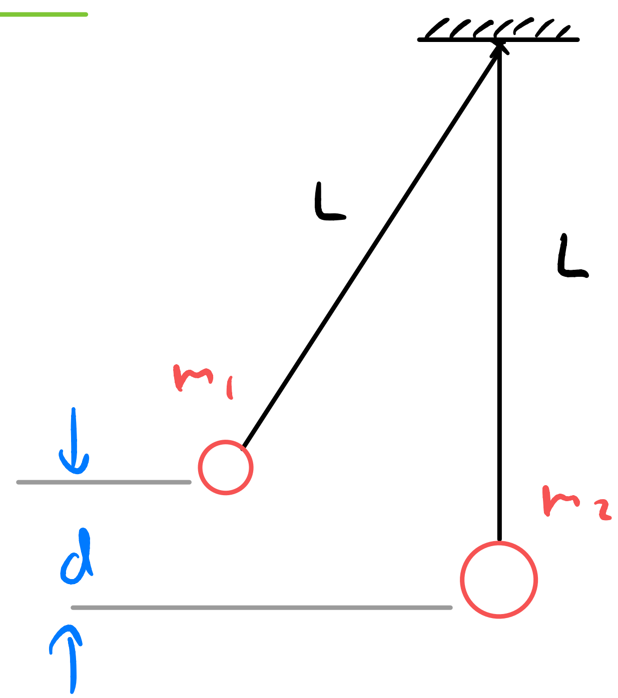
  ↳ If max energy lost

  ⇒ Total Inelastic collision

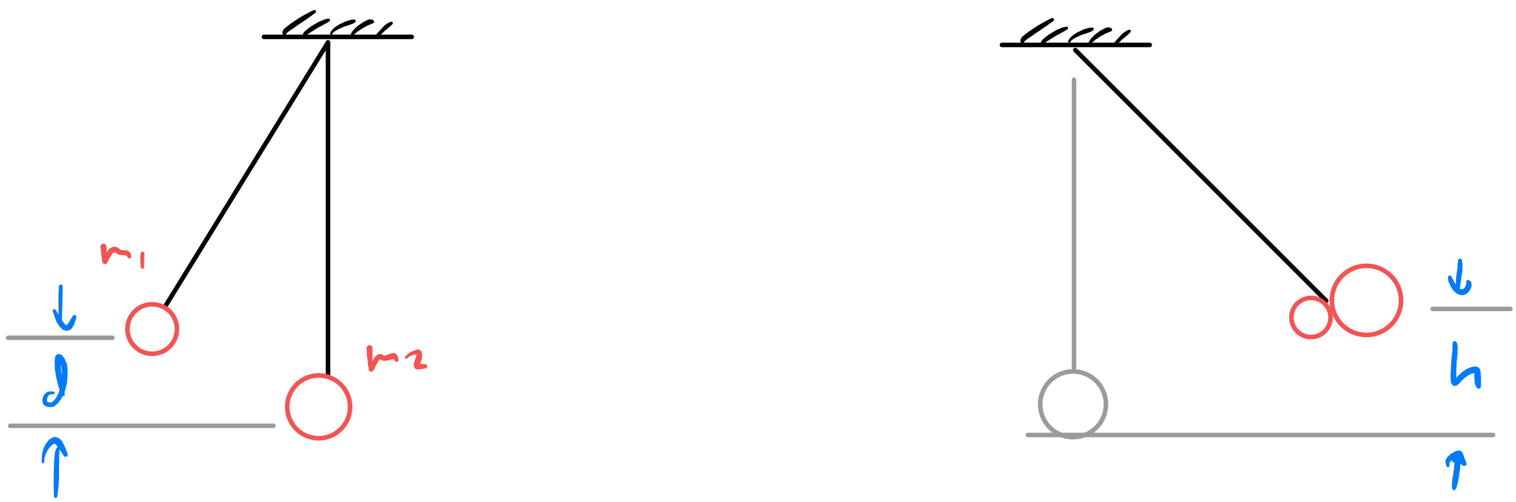
Example

Two pendulums each of length L are initially situated as shown. The first pendulum is released from a height d & strikes the second. Assume the collision is completely inelastic and neglect the mass of the strings & any friction effects. How high does the center of mass rise after the collision?

Solution



The motion can be divided into two stages:



Before

Set $y=0$ at m_2 position

use conservation of energy for m_1

$$K_i + U_i = K_f + U_f$$

Initial

$K_i = 0$

$U_i = mgd$

Final

$K_f = \frac{1}{2} m_1 v^2$

$U_f = 0$

$$\Rightarrow \quad mgd = \frac{1}{2} m_1 v^2 \quad \Rightarrow \quad v = \sqrt{2gd}$$

Now, total momentum is conserved during collision

$$\vec{P}_{before} = \vec{P}_{after}$$

$$\vec{P}_{before} = m_1 v \, \hat{i}$$

$$\vec{P}_{after} = (m_1 + m_2) v' \, \hat{i}$$

inelastic collision!

So, $\quad m_1 v = (m_1 + m_2) v'$

$$\Rightarrow \quad v' = \frac{m_1}{m_1 + m_2} v$$

Also, since $v = \sqrt{2gd}$

$$\Rightarrow \quad v' = \frac{m_1}{m_1 + m_2} \sqrt{2gd}$$

Now, relate collision kinematics to motion after collision

<u>After</u>

total energy again conserved for (m_1+m_2) massive object

$$K_i' + U_i' = K_f' + U_f'$$

<u>Initial</u>

$K_i' = \frac{1}{2}(m_1+m_2)v'^2$

$U_i' = 0$

<u>final</u>

$K_f' = 0$

$U_f' = (m_1+m_2)gh$

$$\Rightarrow \quad \frac{1}{2}(m_1+m_2)v'^2 = (m_1+m_2)gh$$

$$\Rightarrow \quad h = \frac{v'^2}{2g}$$

Now use

$$v' = \frac{m_1}{m_1+m_2}\sqrt{2gd}$$

to find

$$h = \frac{v'^2}{2g} = \left(\frac{m_1}{m_1+m_2}\right)^2 d$$

$$\boxed{h = \left(\frac{m_1}{m_1+m_2}\right)^2 d}$$

Some extremes:

if $m_1 = m_2 \Rightarrow h = \left(\frac{1}{2}\right)^2 d = \frac{d}{4}$

if $m_1 \ll m_2 \Rightarrow \frac{m_1}{m_2} \ll 1$

$$\Rightarrow \frac{m_1}{m_1+m_2} = \frac{m_1}{m_2}\left(\frac{1}{1+\frac{m_1}{m_2}}\right)$$

$$\simeq \frac{m_1}{m_2}\left(1 - \frac{m_1}{m_2}\right) + \cdots$$

$\Rightarrow h \simeq \left(\frac{m_1}{m_2}\right)^2 d$, since $\frac{m_1}{m_2} \ll 1$, $\frac{h}{d} \ll 1$

if $m_1 \gg m_2 \Rightarrow \frac{m_2}{m_1} \ll 1$

$$\Rightarrow \frac{m_1}{m_1+m_2} = \frac{1}{1+\frac{m_2}{m_1}} \simeq 1 - \frac{m_2}{m_1}$$

so, $h \simeq \left(1 - \frac{m_2}{m_1}\right)^2 d \simeq d$

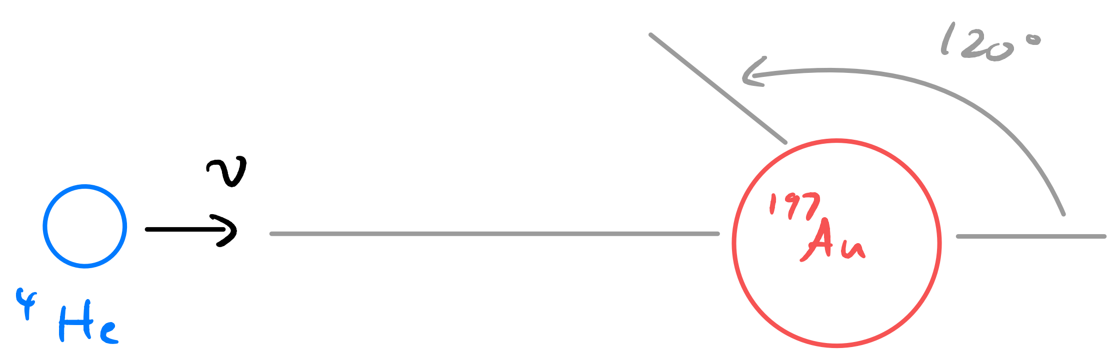
Example

Scattering of helium-4 nuclei from gold-197 nuclei.

The energy of an incoming helium-4 nucleus is 8×10^{-13} J, & the masses of the helium & gold nuclei are 6.68×10^{-27} kg & 3.29×10^{-25} kg, respectively. (N.B. mass ratio 4 to 197)

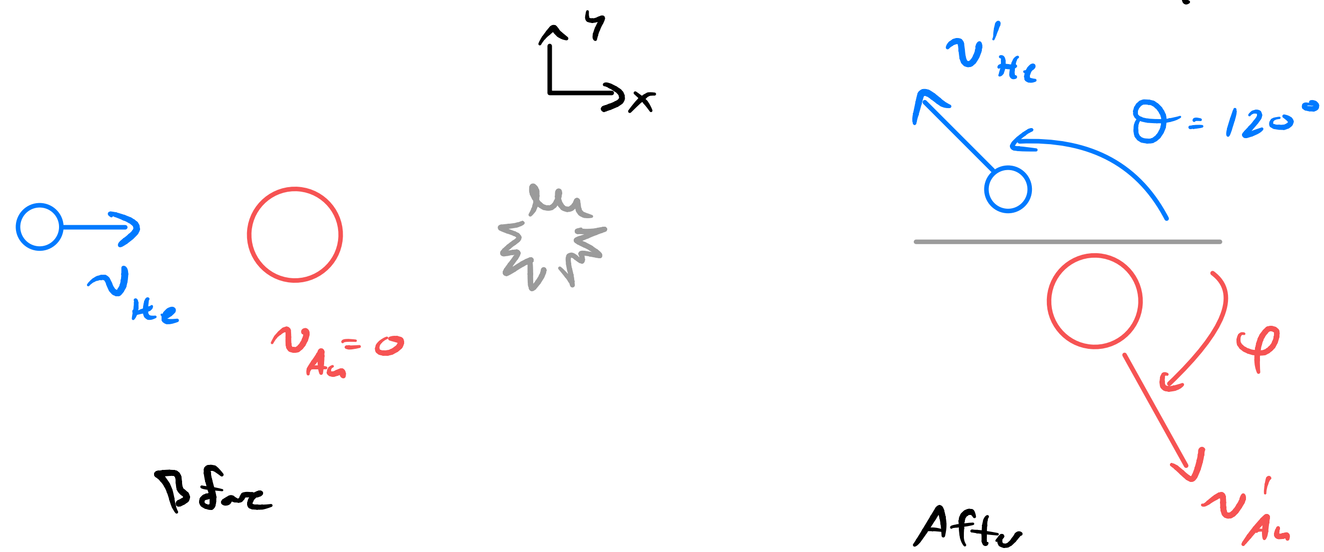
(a) If the helium nucleus scatters to an angle of $120°$ during an elastic collision with a gold nucleus, calculate the helium nucleus's final speed & the final velocity of the gold nucleus.

(b) What is the final kinetic energy of the helium nucleus?



Solution

Elastic collisions conserve momentum & energy



Momentum

$$x: \quad m_{He} v_{He} = m_{He} v'_{He} \cos\theta + m_{Au} v'_{Au} \cos\varphi$$

$$y: \quad 0 = m_{He} v'_{He} \sin\theta - m_{Au} v'_{Au} \sin\varphi$$

Energy

$$\frac{1}{2} m_{He} v_{He}^2 = \frac{1}{2} m_{He} v'^2_{He} + \frac{1}{2} m_{Au} v'^2_{Au}$$

Define mass ratio $\frac{1}{\alpha} = \frac{m_{He}}{m_{Au}} = \frac{4}{197}$

$$\Rightarrow \alpha = \frac{m_{Au}}{m_{He}}$$

So,

$$v_{He} = v_{He}' \cos\theta + \alpha v_{Au}' \cos\varphi \qquad (1)$$

$$0 = v_{He}' \sin\theta - \alpha v_{Au}' \sin\varphi \qquad (2)$$

$$v_{He}^2 = v_{He}'^2 + \alpha v_{Au}'^2 \qquad (3)$$

Goal: Need v_{He}', v_{Au}', φ.

3 eqns., 3 unknowns, solve!

Square (1) & (2)

$$(v_{He} - v_{He}' \cos\theta)^2 = \alpha^2 v_{Au}'^2 \cos^2\varphi$$

$$\hookrightarrow v_{He}^2 + v_{He}'^2 \cos^2\theta - 2 v_{He} v_{He}' \cos\theta = \alpha^2 v_{Au}'^2 \cos^2\varphi \qquad (1')$$

$$v_{He}'^2 \sin^2\theta = \alpha^2 v_{Au}'^2 \sin^2\varphi \qquad (2')$$

Add (1') & (2'), use $\cos^2\theta + \sin^2\theta = 1$

$$\Rightarrow v_{He}^2 + v_{He}'^2 - 2 v_{He} v_{He}' \cos\theta = \alpha^2 v_{Au}'^2$$

? ? ?

Now, use (3) to eliminate $v_{Au}'^2$

$$\alpha v_{Au}'^2 = v_{He}^2 - v_{He}'^2$$

$$v_{He}^2 + v_{He}'^2 - 2 v_{He} v_{He}' \cos\theta = \alpha (v_{He}^2 - v_{He}'^2)$$

$$\Rightarrow \quad (1+\alpha) v_{He}'^2 - 2 v_{He} \cos\theta \, v_{He}' + (1-\alpha) v_{He}^2 = 0$$

or

$$v_{He}'^2 - \frac{2 v_{He} \cos\theta}{1+\alpha} v_{He}' + \left(\frac{1-\alpha}{1+\alpha}\right) v_{He}^2 = 0$$

Solve quadratic for v_{He}'

$$v_{He}' = \frac{v_{He}\cos\theta}{1+\alpha} \pm \frac{1}{2}\sqrt{\frac{4 v_{He}^2 \cos^2\theta}{(1+\alpha)^2} - \frac{4(1-\alpha)}{1+\alpha} v_{He}^2}$$

$$= \frac{v_{He}\cos\theta}{1+\alpha} \pm \frac{v_{He}\cos\theta}{1+\alpha}\sqrt{1 - \frac{(1-\alpha^2)}{\cos^2\theta}}$$

$$= \frac{v_{He}\cos\theta}{1+\alpha}\left[1 \pm \sqrt{1 + \frac{\alpha^2 - 1}{\cos^2\theta}}\right]$$

Substitute numbers, what is v_{He}?

$$K_{He} = \frac{1}{2} m_{He} v_{He}^2 \Rightarrow v_{He} = \sqrt{\frac{2 K_{He}}{m_{He}}}$$

$$\simeq 1.548 \times 10^7 \text{ m/s}$$

So,

$$v_{He}' \simeq 1.5 \times 10^7 \text{ m/s}$$

other solution is negative ...

Now, we know $v'_{He} = 1.5 \times 10^7$ m/s

So, from (3),

$$v'_{Au} = \sqrt{\frac{v_{He}^2 - v_{He}'^2}{\alpha}}$$

$$\simeq 5.36 \times 10^5 \text{ m/s}$$

Finally, from (2)

$$\sin\varphi = \frac{v'_{He}}{\alpha v'_{Au}} \sin\theta$$

$$\simeq 0.492$$

$$\Rightarrow \quad \varphi = \sin^{-1}(0.492)$$

$$\simeq 29.5^\circ$$

So,

$$\boxed{\begin{aligned} v'_{He} &\simeq 1.5 \times 10^7 \text{ m/s} \\ v'_{Au} &\simeq 5.4 \times 10^5 \text{ m/s} \\ \varphi &\simeq 29.5^\circ \end{aligned}}$$

(b) $K_{He_f} = \frac{1}{2} m_{He} v_{He}'^2$

$\simeq 7.53 \times 10^{-13}$ J

So, $\dfrac{K_{He_f}}{K_{He_i}} = \dfrac{7.53 \times 10^{-13} \text{ J}}{8 \times 10^{-13} \text{ J}} \simeq 0.94$

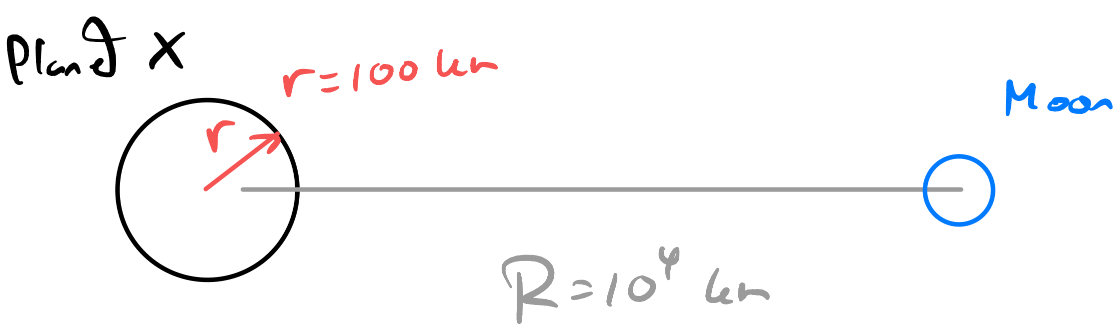
So, 6% of energy transferred to gold nucleus.

Example

A small moon is orbiting a planet with a period of 10 days, and the distance between the moon & planet is 10^4 km. Deduce the mass of the planet.

The planet is further observed to have a radius of 100 km. What is the acceleration due to gravity at the surface of the planet?

Solution

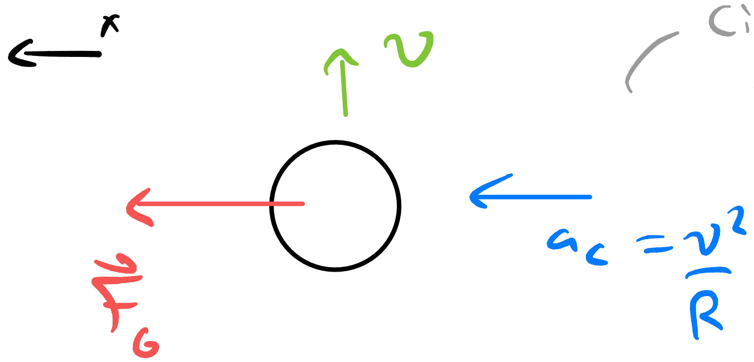


$T_{moon} = 10$ days.

Let M_X = mass of planet X

M_M = mass of moon

FBD of moon



Circular orbit
⇒ centripetal motion

$$\sum \vec{F} = m\vec{a}$$

$$F_G = M_m a_c$$

& Newton's Law of Gravity

$$F_G = \frac{G M_x M_m}{R^2}$$

$$\Rightarrow \quad \frac{G M_x \cancel{M_m}}{R^2} = \cancel{M_m} a_c$$

Moon mass cancels

Solve for M_x

$$M_x = \frac{R^2 a_c}{G} = \frac{R^2 v^2}{G R} = \frac{R v^2}{G}$$

So, if we know the speed of the moon, we have the mass of the planet X.

$$M_X = \frac{R v^2}{G}$$

Know period $T = 10$ days

$$T = 10 \text{ days} \cdot \left(\frac{24 \text{ h}}{1 \text{ day}}\right) \cdot \left(\frac{3600 \text{ s}}{1 \text{ h}}\right)$$

$$\simeq 8.64 \times 10^5 \text{ s}$$

Circular motion

$$v = \frac{2\pi R}{T} = \frac{2\pi \cdot 10^4 \text{ km}}{8.64 \times 10^5 \text{ s}}$$

$$\simeq 0.0727 \text{ km/s}$$

$$= 72.7 \text{ m/s}$$

So,

$$M_x = \frac{R v^2}{G}$$

$$= \frac{10^7 \text{ m} \cdot (72.7 \text{ m/s})^2}{\left(6.67 \times 10^{-11} \ \frac{\text{N m}^2}{\text{kg}^2}\right)}$$

$$\simeq 7.92 \times 10^{20} \text{ kg} \quad \blacksquare$$

On the surface of planet X

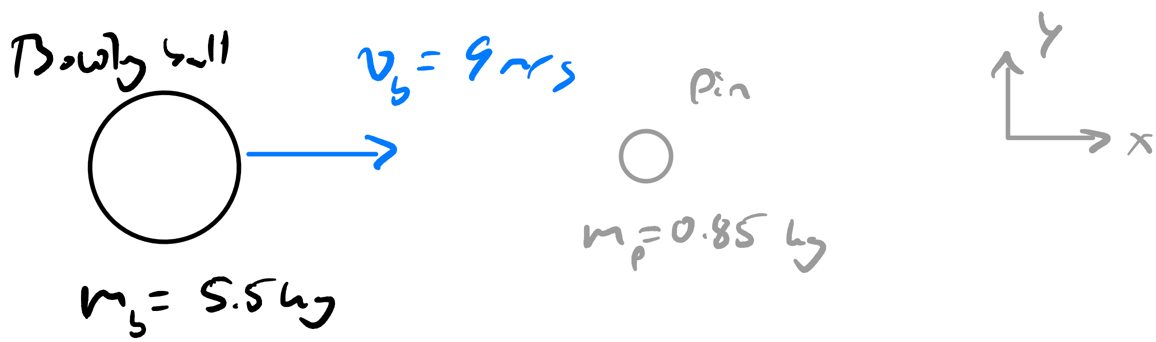
$$g_x = \frac{G M_x}{r^2}$$

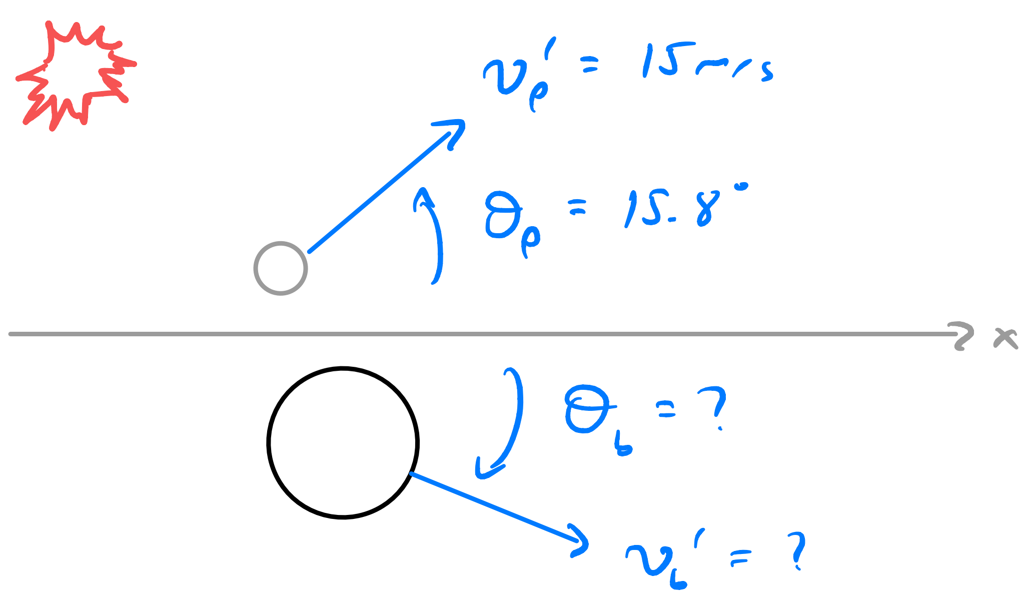
$$\simeq 5.28 \text{ m/s}^2 \quad \blacksquare$$

Example

A 5.5 kg bowling ball moving at 9 m/s collides with a 0.85 kg bowling pin, which is scattered at an angle of 15.8° to the initial direction of the bowling ball and with a speed of 15 m/s. Calculate the final velocity of the bowling ball. Is the collision elastic?

Solution





Momentum is conserved

$$\vec{P}_{tot} = \vec{P}_{tot}'$$

$$x: \quad m_b v_b = m_p v_p' \cos\theta_p + m_b v_b' \cos\theta_b \qquad (1)$$

$$y: \quad 0 = m_p v_p' \sin\theta_p - m_b v_b' \sin\theta_b \qquad (2)$$

Solve (2) for v_b'

$$v_b' = v_p' \frac{m_p}{m_b} \frac{\sin\theta_p}{\sin\theta_b} \qquad (2')$$

Substitute (2') into (1)

$$m_b v_b = m_p v_p' \cos\theta_p + \cancel{m_b}\left(v_p' \frac{m_p}{\cancel{m_b}} \frac{\sin\theta_p}{\sin\theta_b}\right)\cos\theta_b$$

$$m_b v_b = m_p v_p' \cos\theta_p + m_p v_p' \sin\theta_p \frac{1}{\tan\theta_b}$$

Solve for $\tan\theta_b$

$$\Rightarrow \tan\theta_b = \frac{m_p v_p' \sin\theta_p}{m_b v_b - m_p v_p' \cos\theta_p}$$

$$\tan\theta_b = \frac{m_p v_p' \sin\theta_p}{m_b v_b - m_p v_p' \cos\theta_p}$$

$$\simeq 0.093$$

$$\Rightarrow \quad \theta_b \simeq 5.33^\circ \quad \blacksquare$$

From (2')

$$v_b' = v_p' \frac{m_p}{m_b} \frac{\sin\theta_p}{\sin\theta_b}$$

$$\simeq 6.79 \text{ m/s} \quad \blacksquare$$

Is the collision elastic?

$$K_i = \frac{1}{2} m_b v_b^2$$

$$\simeq 222.75 \text{ J}$$

$$K_f = \frac{1}{2} m_p v_p'^2 + \frac{1}{2} m_b v_b'^2$$

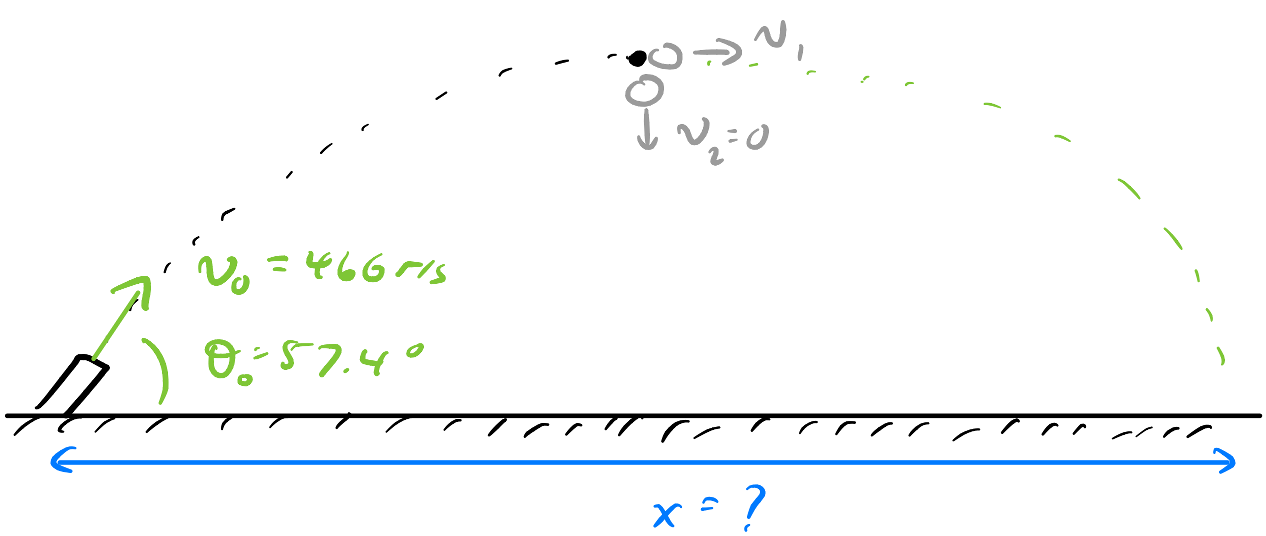
$$\simeq 222.41 \text{ J}$$

$$\Rightarrow \quad K_f < K_i \quad \Rightarrow \quad \text{Inelastic!} \quad \blacksquare$$

Example

A shell is fired from a gun w/ muzzle velocity 466 m/s, at an angle of 57.4° w/ the horizontal. At the top of the trajectory, the shell explodes into two fragments of equal mass. One fragment, whose speed immediately after the explosion is zero, falls vertically. How far from the gun does the other fragment land, assuming level terrain?

Solution



At top of trajectory, $v_y = 0$

$$v_y = v_{0y} - gt$$

$$\Rightarrow \quad 0 = v_0 \sin\theta_0 - g t_h$$ ↖ t @ top of trajectory

$$t_h = \frac{v_0 \sin\theta_0}{g}$$

$$v_x = v_{0x}$$
$$= v_0 \cos\theta_0$$

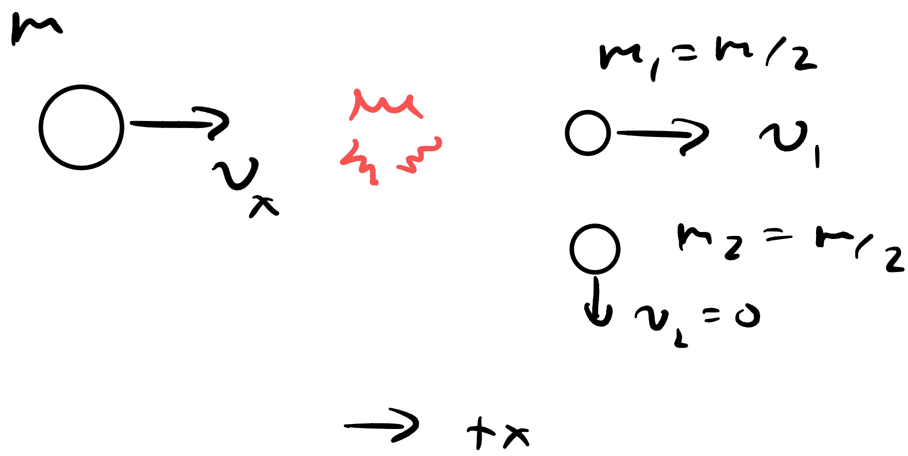
range at top of trajectory

$$x_h = v_0 \cos\theta_0 \, t_h$$
$$= \frac{v_0^2 \sin\theta_0 \cos\theta_0}{g}$$

height at top of trajectory

$$y_h = h = v_0 \sin\theta_0 \, t_h - \frac{1}{2} g t_h^2$$
$$= v_0 \sin\theta_0 \left(\frac{v_0 \sin\theta_0}{g} \right) - \frac{1}{2} g \left(\frac{v_0^2 \sin^2\theta_0}{g^2} \right)$$
$$= \frac{v_0^2 \sin^2\theta_0}{2g}$$

Now, explosion!



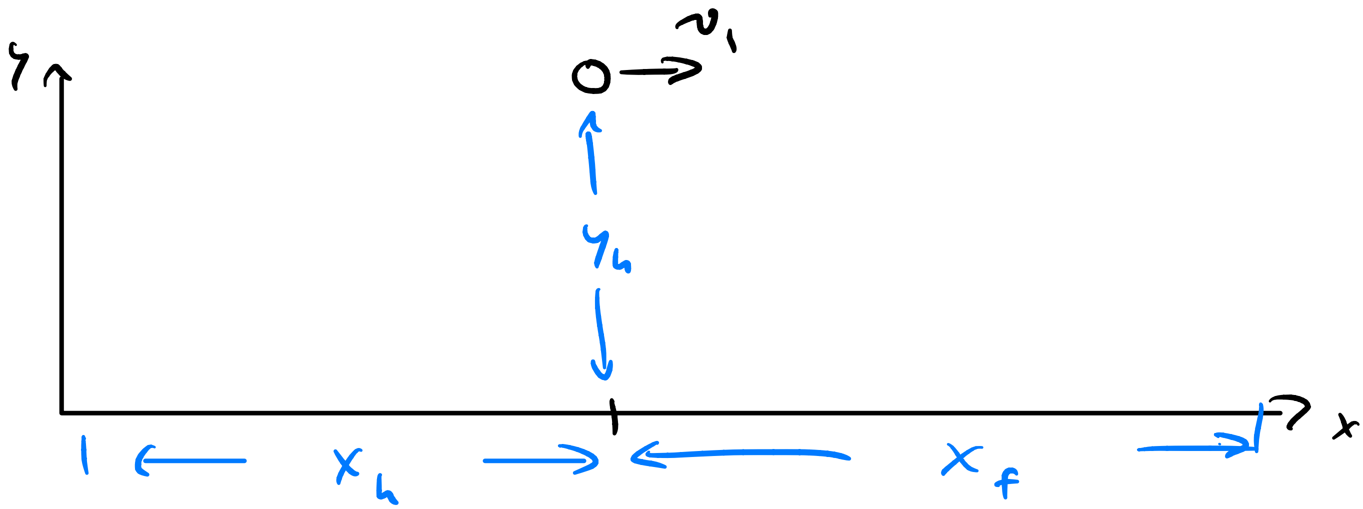
Momentum is conserved during explosion

$$m v_x = m_1 v_1$$

$$= \frac{m}{2} v_1$$

$$\Rightarrow \quad v_1 = 2 v_x$$

Now, we can map the trajectory of this fragment



$$x_f = v_1 t_{hit}$$

$$y_f = 0 = y_0 - \frac{1}{2} g t_{hit}^2$$

Now, $y_0 = y_h \Rightarrow t_{hit} = \sqrt{\frac{2y_h}{g}}$

but, $y_h = \frac{v_0^2 \sin^2\theta_0}{2g}$

$$\Rightarrow t_{hit} = \sqrt{\frac{2y_h}{g}}$$

$$= \sqrt{\frac{2}{g} \frac{v_0^2 \sin^2\theta_0}{2g}}$$

$$= \frac{v_0 \sin\theta_0}{g}$$

$v_1 = 2v_x$ $\quad$ $v_x = v_0 \cos\theta_0$

So, $x_f = \frac{v_1 v_0 \sin\theta_0}{g} = \frac{2 v_x v_0 \sin\theta_0}{g} = \frac{2 v_0^2 \sin\theta_0 \cos\theta_0}{g}$

Total distance : $x = x_f + x_h$

$$x_h = \frac{v_0^2 \sin\theta_0 \cos\theta_0}{g}$$

Sol,

$$x = \frac{3 v_0^2 \sin\theta_0 \cos\theta_0}{g}$$

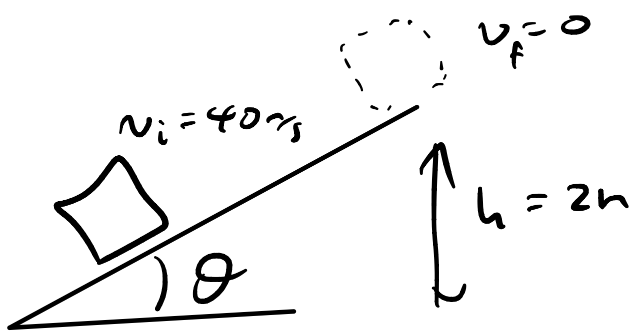
$$\simeq 3.02 \times 10^4 \text{ m} \quad \blacksquare$$

Example

A box with mass $m = 10$ kg was kicked up a ramp with an incline of $\theta = 30°$ & speed $v = 40$ m/s. It reached a height $h = 2$m before it stops.

How much work is done by friction?

Solution



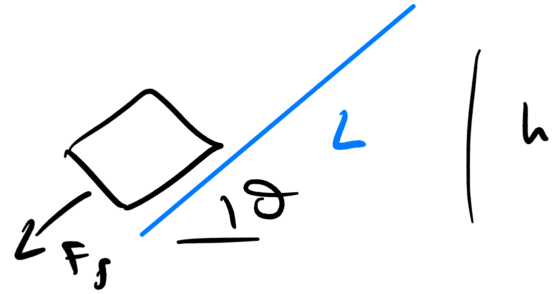
$$W_{NC} = \Delta E$$

$$= (K_f + U_f) - (K_i + U_i)$$

$$= mgh - \frac{1}{2}mv^2$$

$$= -7800 \text{ J} \quad \blacksquare$$

What is the force of friction?



$$W_{NC} = -F_f \, L$$

So, $L = \dfrac{h}{\sin\theta}$

$$\Rightarrow \quad F_f = -\frac{W_{NC}}{L} = -\frac{W_{NC}\sin\theta}{h}$$

$$\simeq 1950 \text{ N}$$