Physics 101 P

General Physics I

Problem Sessions - Week 8

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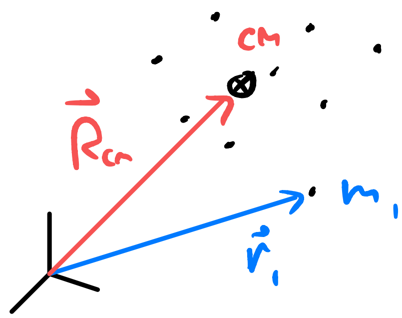
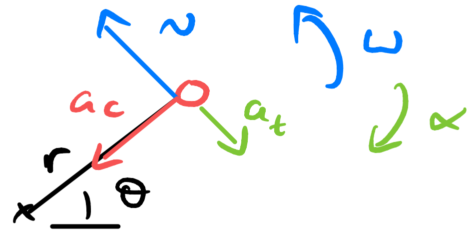
Systems of Particles — Rigid Bodies

Center of Mass – $\vec{R}_{cm} = \dfrac{\sum_{i=1}^{N} \vec{r}_i m_i}{\sum_{i=1}^{N} m_i}$



For a rigid body, $\vec{R}_{cm} = \dfrac{\int \vec{r}\, dm}{\int dm}$

Rotational Motion



Kinematics

Translation	Rotation	Relationship
x	θ	$s = r\theta$
v	ω	$v = r\omega$
a_t	α	$a_t = r\alpha$
a_c		$a_c = \frac{v^2}{r}$

$$x = x_0 + v_0 t + \frac{1}{2} a t^2 \qquad \theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$$

$$v^2 = v_0^2 + 2a\Delta x \qquad \omega^2 = \omega_0^2 + 2\alpha\Delta\theta$$

Note: θ must be in radians!

Dynamics

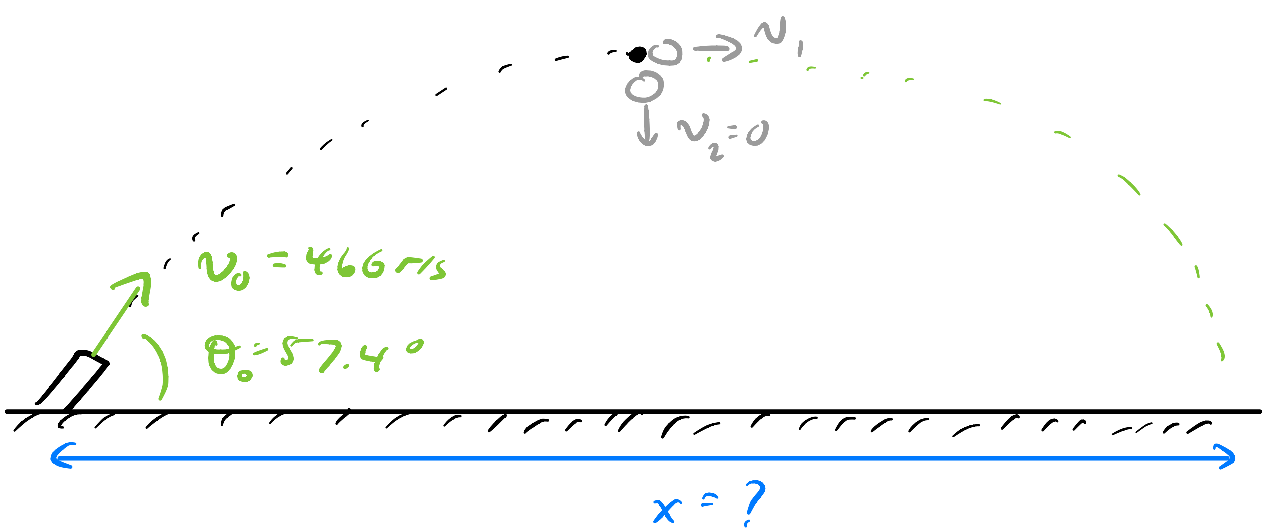
Translation	Rotation
m	$I = \sum_i m_i r_i^2$ $I = \int r^2 \, dm$
$\sum \vec{F} = m\vec{a}$	$\sum \vec{\tau} = I\alpha$
$W_{A\to B} = \int_A^B \vec{F}_{net} \cdot d\vec{r}$	$W_{A\to B} = \int_A^B \tau_{net} \, d\theta$
$K = \frac{1}{2} m v^2$	$K = \frac{1}{2} I \omega^2$
$P = \vec{F} \cdot \vec{v}$	$P = \tau \omega$

$$\vec{\tau} = \vec{r} \times \vec{F}$$

Example

A shell is fired from a gun w/ muzzle velocity 466 m/s, at an angle of 57.4° w/ the horizontal. At the top of the trajectory, the shell explodes into two fragments of equal mass. One fragment, whose speed immediately after the explosion is zero, falls vertically. How far from the gun does the other fragment land, assuming level terrain?

Solution



Let's solve using center of mass.

the center of mass motion is

$$x = v_0 \cos\theta_0 t$$

$$y = v_0 \sin\theta_0 t - \frac{1}{2} g t^2$$

when the CM hits the ground,

$$0 = v_0 \sin\theta_0 T - \frac{1}{2} g T^2$$

$$\Rightarrow T \left(v_0 \sin\theta_0 - \frac{1}{2} g T \right) = 0$$

$$\Rightarrow T = \frac{2 v_0 \sin\theta_0}{g}$$

So,

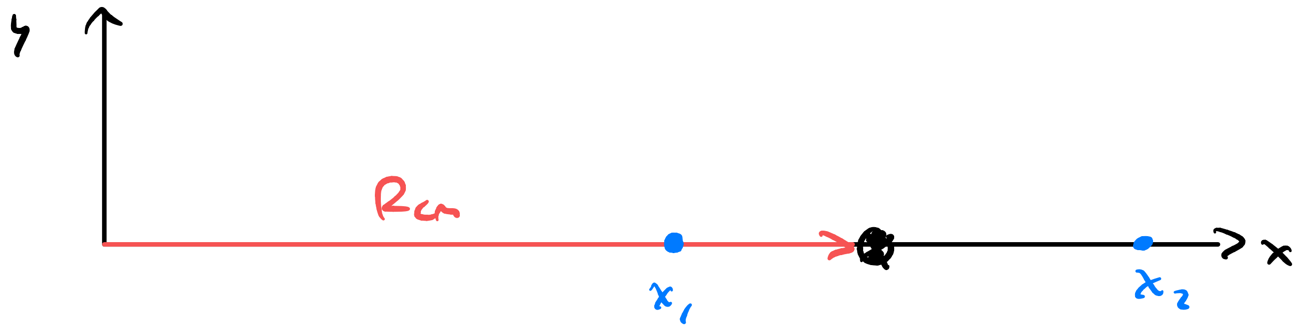
$$R_{cm} = x(T) = \frac{2 v_0^2 \sin\theta_0 \cos\theta_0}{g} = \frac{v_0^2 \sin 2\theta_0}{g}$$

Now, when the cm hits the ground, the relation to the fragments is

$$R_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

Fragments have equal mass $\Rightarrow$ $m_1 = m_2 = m$

$$\Rightarrow R_{cm} = \frac{1}{2}(x_1 + x_2)$$



Now, x_1 is located directly beneath of the explosion

$$\Rightarrow x_1 = \frac{R_{cm}}{2}$$

So, solve for x_2

$$\Rightarrow 2R_{cm} = \frac{R_{cm}}{2} + x_2$$

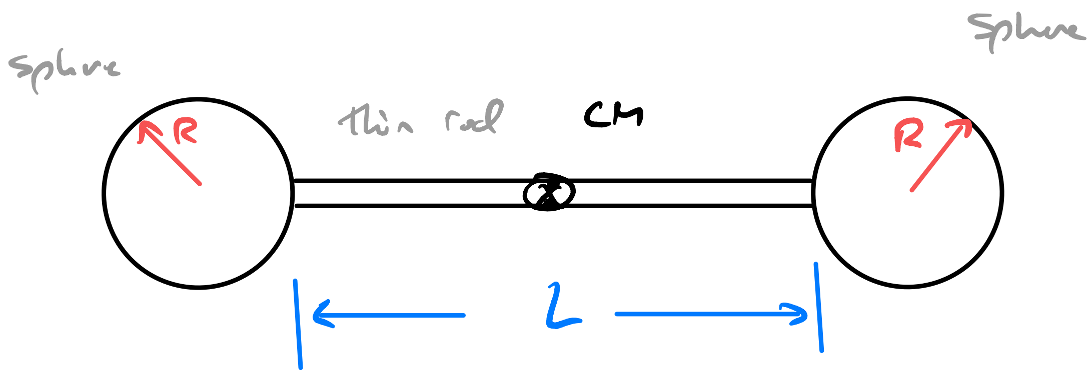
$$\Rightarrow x_2 = \frac{3}{2} R_{cm}$$

$$= \frac{3 v_0^2 \sin\theta_0 \cos\theta_0}{g}$$

$$\simeq 3.02 \times 10^4 \text{ m}$$

Example

Find the moment of inertia of the compound system shown, about the center of mass.



$R = 20$ cm

$L = 0.5$ m

$m_{sphere} = 2$ kg

$m_{rod} = 1$ kg

Solution

$$I = I_{sphere_1} + I_{sphere_2} + I_{rod}$$

All need to be about same axis of rotation

From table in book

$$I_{rod}\Big|_{cm} = \frac{1}{12} m_{rod} L^2$$

$$I_{sphere}\Big|_{cm} = \frac{2}{5} m_{sphere} R^2$$

Now, I_{rod} does not need to be altered,

but I_{sphere} we need to use parallel-axis theorem

$$I_{sphere} = I_{sphere}\Big|_{cm} + m_{sphere}\, d^2$$

where d is distance from cm to pivot point. here,

$$d = \left(R + \frac{L}{2}\right)$$

So,

$$I_{sphere} = \frac{2}{5} m_{sphere} R^2 + m_{sphere}\left(R + \frac{L}{2}\right)^2$$

$$= \frac{2}{5} m_{sphere} R^2 + m_{sphere} R^2$$

$$+ m_{sphere} RL + \frac{m_{sphere} L^2}{4}$$

$$= \frac{7}{5} m_{sphere} R^2 + m_{sphere} RL + \frac{m_{sphere} L^2}{4}$$

So,

$$I = 2I_{sphere} + I_{rod}$$

$$= \frac{14}{5} m_{sphere} R^2 + 2 m_{sphere} RL + \frac{1}{2} m_{sphere} L^2$$

$$+ \frac{1}{12} m_{rod} L^2$$

$$I = \frac{14}{5} m_{sphere} R^2 + 2 m_{sphere} R L + \frac{1}{2}\left(m_{sphere} + \frac{1}{6} m_{rod}\right) L^2$$

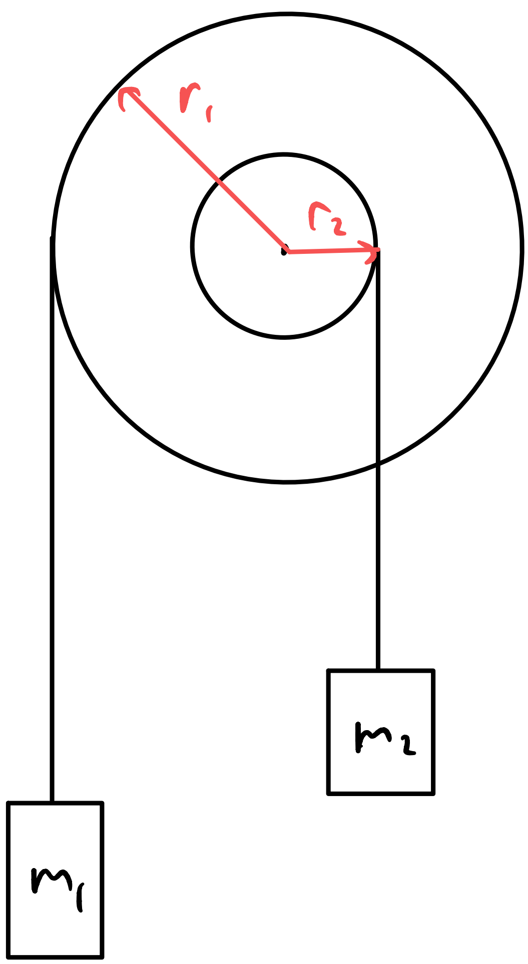
So, find

$$I \simeq 0.895 \ kg \cdot m^2$$

Example

A pulley of moment of inertia 2.0 $kg \cdot m^2$ is mounted on a wall as shown. Light strings are wrapped around two circumferences of the pulley and weights are attached.

What are the (a) angular acceleration of the pulley, & (b) the linear acceleration of the weights?



$r_1 = 50$ cm

$r_2 = 20$ cm

$m_1 = 1$ kg

$m_2 = 2$ kg

Solution

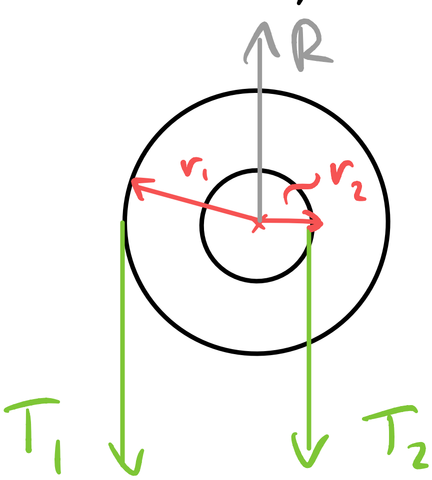
Let $I = 2\ \mathrm{kg \cdot m^2}$ be the moment of inertia of the pulley about the CM.

Now, use Newton's Laws

$$\sum \vec{F} = m\vec{a}$$

$$\sum \tau = I\alpha$$

$\underline{\text{FBD}_{\text{pulley}}}$

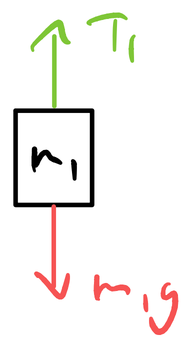


assume $\circlearrowleft \alpha$

$+\circlearrowleft$ $\underline{\sum \tau = I\alpha}$ about CM

$$T_1 r_1 - T_2 r_2 = I\alpha$$

$\underline{\text{FBD}_{m_1}}$

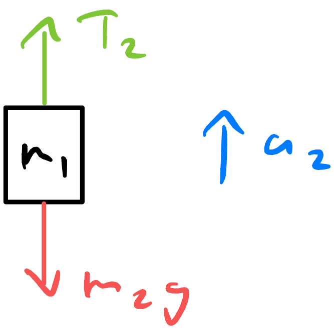


$\downarrow a_1$

$\uparrow +$ $\underline{\sum \vec{F} = m\vec{a}}$

$$T_1 - m_1 g = -m_1 a_1$$

$\underline{FBD_{m_2}}$ $\quad \uparrow^{+}$ $\quad \underline{\sum \vec{F} = m\vec{a}}$



$$T_2 - m_2 g = m_2 a_2$$

So, 3 - equations

$$T_1 r_1 - T_2 r_2 = I\alpha \qquad (1)$$

$$T_1 - m_1 g = -m_1 a_1 \qquad (2)$$

$$T_2 - m_2 g = m_2 a_2 \qquad (3)$$

Need $T_1, T_2, \alpha, a_1, a_2$

Note: $a_1 \neq a_2$ since not connected by same string!

BG,

$$a_1 = r_1 \alpha \qquad (4)$$

$$a_2 = r_2 \alpha \qquad (5)$$

So,

$$T_1 r_1 - T_2 r_2 = I\alpha \qquad (1)$$

$$T_1 - m_1 g = -m_1 r_1 \alpha \qquad (2^*)$$

$$T_2 - m_2 g = m_2 r_2 \alpha \qquad (3^*)$$

Now, (2^*) & (3^*) into (1)

$$\Rightarrow \quad m_1 (g - r_1 \alpha) r_1 - m_2 (g + r_2 \alpha) r_2 = I \alpha$$

$$r_1 m_1 g - r_2 m_2 g - m_1 r_1^2 \alpha - m_2 r_2^2 \alpha = I \alpha$$

$$(r_1 m_1 - r_2 m_2) g = (I + m_1 r_1^2 + m_2 r_2^2) \alpha$$

Solve for α

$$\alpha = \left(\frac{r_1 m_1 - r_2 m_2}{I + m_1 r_1^2 + m_2 r_2^2} \right) g$$

$$\simeq 0.42 \ \frac{\text{rad}}{s^2} \qquad \blacksquare$$

$\uparrow$

$+$ sign $\Rightarrow$ rotates counter clockwise

Now,

$$a_1 = r_1 \alpha$$

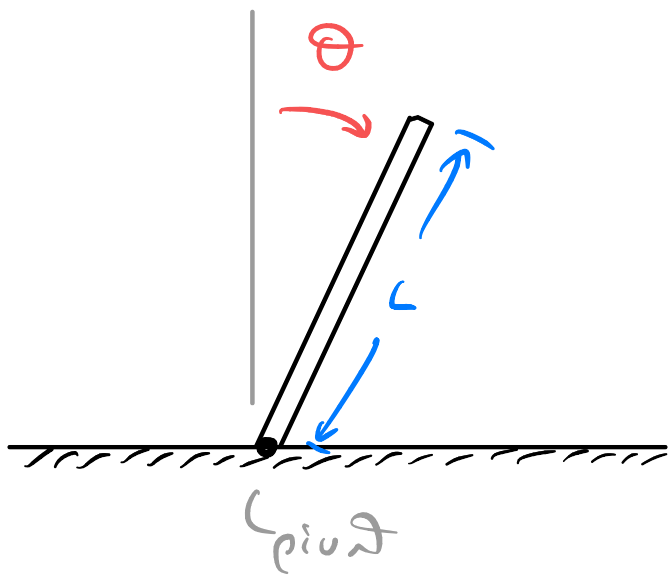
$$\simeq 0.21 \ m/s^2 \quad \blacksquare$$

$$a_2 = r_2 \alpha$$

$$\simeq 0.084 \ m/s^2 \quad \blacksquare$$

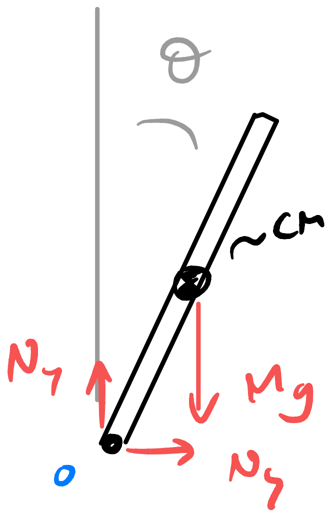
Example

A uniform thin rod of mass M and length L is falling as shown. What is its angular acceleration as a function of L, g, & M?



Solution

Let us apply $\sum \vec{F} = m\vec{a}$ and $\sum \tau = I\alpha$ about the pivot point



$$+\circlearrowright \sum \tau \Big|_O = I_O \alpha$$

$$Mg \frac{L}{2} \sin\theta = I_O \alpha$$

$$\Rightarrow \quad \alpha = \frac{MgL}{2I_O} \sin\theta$$

What about I_O?

$$I_O = I_{cm} + M\left(\frac{L}{2}\right)^2$$ parallel axis theorem

& from table, $I_{cm} = \frac{1}{12} M L^2$

$$\Rightarrow \quad I_O = \frac{1}{12} M L^2 + \frac{1}{4} M L^2$$

$$= \frac{4}{12} M L^2 = \frac{1}{3} M L^2$$

So,

$$\alpha = \frac{3g}{2L} \sin\theta \quad \blacksquare$$