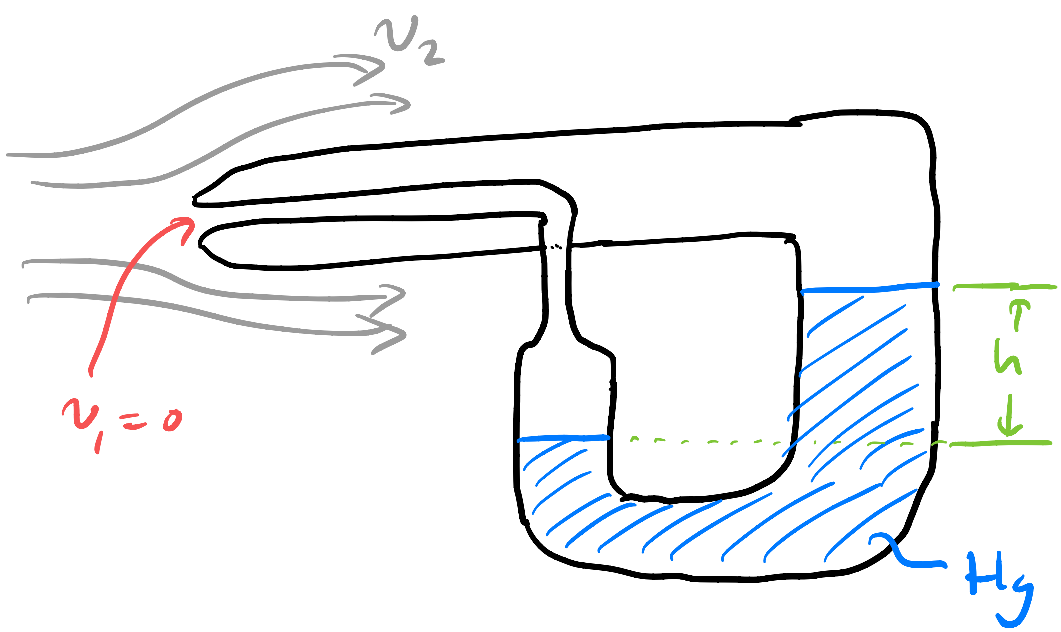
Physics 101 P
General Physics I

Problem Sessions - Week 12

A.W. Jackura —— William & Mary

Example

Suppose you have a pitot tube with a pressure reading at 11 mm·Hg at an air speed of 175 km/h. What will the pressure reading, in units of mm·Hg, when the wind speed is 305 km/h?



Solution

$$P = \rho g h = 11 \text{ mm Hg} \quad @ \quad v_2 = 175 \text{ km/h}.$$

if $v_2' = 305$ km/h, $P' = \rho g h' = ?$

Bernoulli's equation

$$\Delta P + \frac{1}{2}\rho \Delta v^2 + \rho g \Delta y = 0$$

Outside the tube, $\Delta y = 0$

& $\Delta P = P$ (gage pressure)

$\Delta v^2 = v_1^2 - v_2^2 = -v_2^2$ (since $v_1 = 0$)

$$\Rightarrow P = \frac{1}{2}\rho v_2^2$$

So, $P' = \frac{1}{2}\rho v_2'^2$

take ratio,

$$\frac{P'}{P} = \frac{v_2'^2}{v_2^2}$$

$$\Rightarrow P' = P\left(\frac{v_2'}{v_2}\right)^2$$

then,

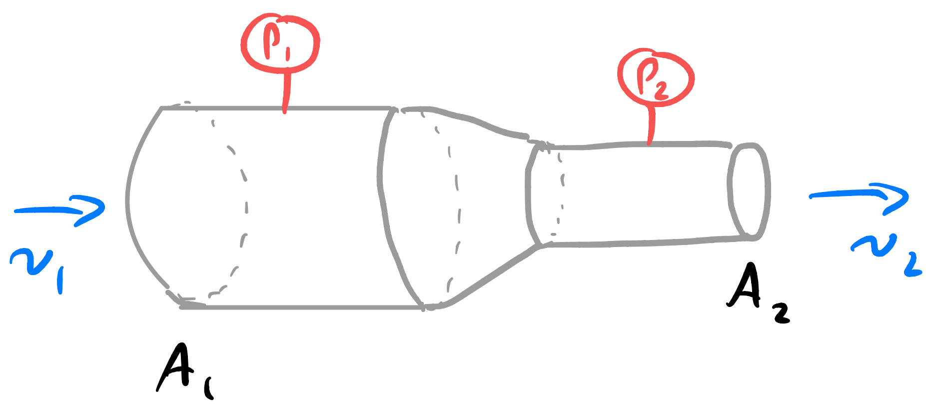
$$P' = P\left(\frac{v_2'}{v_2}\right)^2$$

$$= (11 \text{ mm Hg})\left(\frac{305}{175}\right)^2$$

$$\simeq 33.4 \text{ mm}\cdot\text{Hg}. \qquad \blacksquare$$

Example

A fluid of constant density flows through a reduction in a pipe. Find an equation for the change in pressure, in terms of v_1, A_1, A_2, & the density. Find v_2 in terms of ΔP, A_1, A_2, & the density.



Solution

Bernoulli's eqn.

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho g y_1$$

$$= P_2 + \frac{1}{2}\rho v_2^2 + \rho g y_2$$

Now, $y_1 = y_2$

$$P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2$$

$$\Rightarrow \quad \Delta P = P_2 - P_1$$

$$= \frac{1}{2}\rho\left(v_1^2 - v_2^2\right)$$

Now, mass flow rate is conserved

$$Q = v \cdot A = \text{constant}$$

So, $v_1 A_1 = v_2 A_2 \Rightarrow v_2 = v_1 \dfrac{A_1}{A_2}$

$$\Rightarrow \quad \Delta P = \frac{1}{2}\rho\left(v_1^2 - v_2^2\right)$$

$$= \frac{1}{2}\rho\left(v_1^2 - v_1^2 \frac{A_1^2}{A_2^2}\right)$$

$$= \frac{1}{2}\rho v_1^2\left(1 - \left(\frac{A_1}{A_2}\right)^2\right) \qquad \blacksquare$$

$$\Rightarrow \quad \boxed{\Delta P = \frac{1}{2}\rho v_1^2\left(1 - \left(\frac{A_1}{A_2}\right)^2\right)}$$

Now, $v_2 = v_1 \dfrac{A_1}{A_2}$

From Bernoulli, $\Delta P = \frac{1}{2} \rho v_1^2 \left(1 - \dfrac{A_1^2}{A_2^2}\right)$

$$\Rightarrow v_1^2 = \frac{2\Delta P}{\rho} \frac{1}{1-\left(\frac{A_1}{A_2}\right)^2}$$

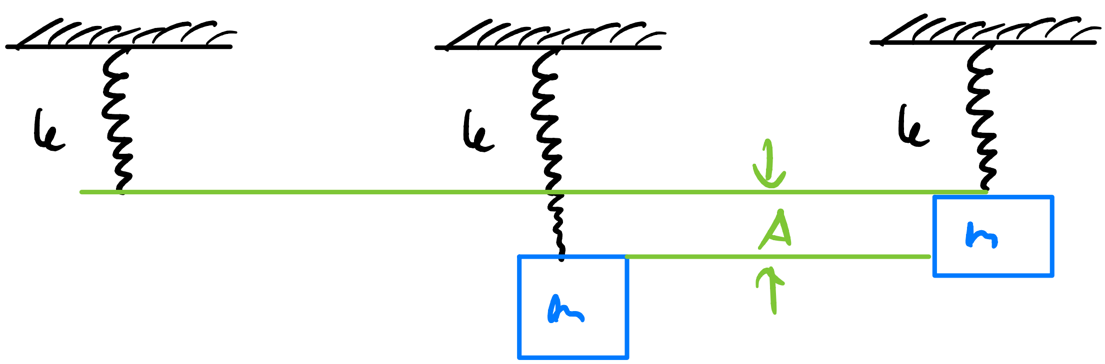
So,

$$v_2 = \sqrt{\frac{2\Delta P}{\rho}} \sqrt{\frac{\left(\frac{A_1}{A_2}\right)^2}{1-\left(\frac{A_1}{A_2}\right)^2}}$$

$$\Rightarrow \boxed{v_2 = \sqrt{\frac{2\Delta P}{\rho} \cdot \frac{A_1^2}{A_2^2 - A_1^2}}} \quad \blacksquare$$

Example

A spring, with spring constant $k = 7.5$ N/m, hangs vertically from a bracket at its unweighted equilibrium weight. An object with mass $m = 0.15$ kg is attached to the lower end of the spring and is gently lowered until the spring reaches a new equilibrium length. The mass is raised until the spring returns to the original length, and is released from rest resulting in vertical oscillations. What is the oscillation amplitude in meters? What is the maximum velocity of the mass?

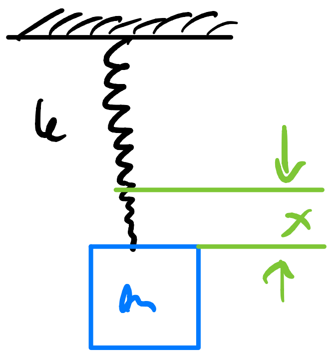


Solution

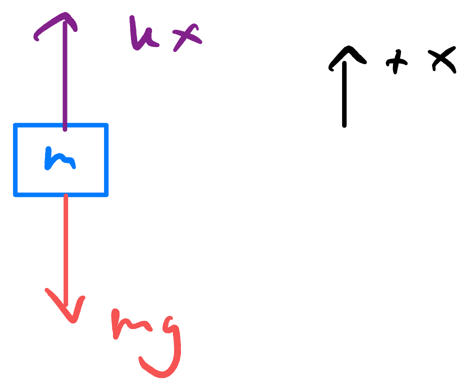
Force from spring. $\vec{F} = -k(\vec{x} - \vec{x}_0)$

equilibrium point

Focus on point where mass hangs, extending the spring



FBD



$$\sum \vec{F} = \vec{0}$$

$$kx - mg = 0 \Rightarrow x = \frac{mg}{k}$$

So, amplitude is this stretched length

$$A = x = \frac{mg}{k} \simeq 0.196 \text{ m} \quad \blacksquare$$

Now, max velocity happens when mass reaches equilibrium position.

From conservation of energy ($E_i = E_f$)

$$E_i = U_i + K_i$$

$$= \frac{1}{2} k A^2 + 0$$

$$E_f = U_f + K_f$$

$$= 0 + \frac{1}{2} m v_{max}^2$$

$$\Rightarrow \quad k A^2 = m v_{max}^2$$

$$\Rightarrow \quad v_{max} = A \sqrt{\frac{k}{m}}$$

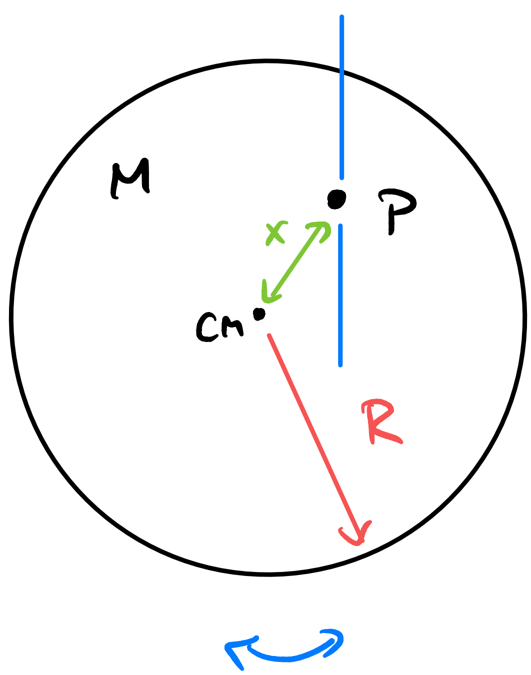
$$\simeq 1.4 \text{ m/s} \quad \blacksquare$$

Example

A uniform disk of radius $R = 1\,m$ and mass $M = 2\,kg$ is free to swing by a pivoting distance $x = 0.2\,m$ from the center. It undergoes harmonic motion under the influence of gravity. What is the disk's period T in seconds for small oscillations about the pivot point? In terms of R, what is the expression for the distance x_m for which the period is a minimum.

Solution

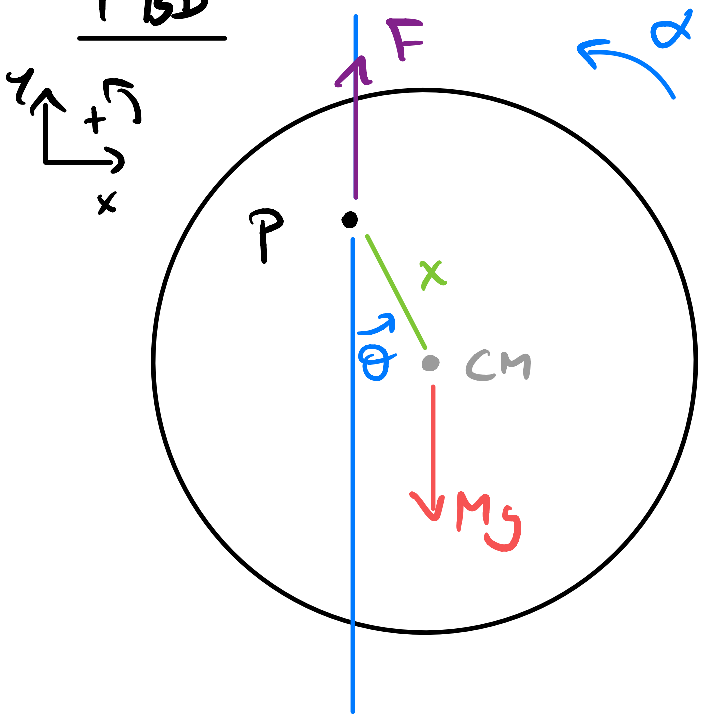
We want oscillations about point P.



$$T = \frac{2\pi}{\omega}$$

Let's solve using Newton's laws

FBD



$$\sum \tau_P = I_P \alpha$$

$$-Mgx\sin\theta = I_P \alpha$$

So,
$$\alpha = -\frac{Mgx}{I_P}\sin\theta$$

For small angles,
$$\sin\theta \simeq \theta$$

So,
$$\alpha = -\frac{Mgx}{I_P}\theta$$

For simple harmonic motion, $\alpha = \frac{d^2\theta}{dt^2} = -\omega^2\theta$

So,
$$\omega^2 = \frac{Mgx}{I_P}$$

Now, $I_p = I_{cm} + M x^2$

where $I_{cm} = \frac{1}{2} M R^2$

So,

$$\omega^2 = \frac{M g x}{\frac{1}{2} M R^2 + M x^2}$$

$$= \frac{g x}{x^2 + \frac{1}{2} R^2}$$

Now, $T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{x^2 + \frac{1}{2} R^2}{g x}}$

$$= 2\pi \sqrt{\frac{x}{g}\left(1 + \frac{1}{2}\frac{R^2}{x^2}\right)}$$

Note, $T = 2\pi \sqrt{\frac{x}{g}}$ is period of ideal pendulum of length x.

So,

$$T = 2\pi\sqrt{\frac{x}{g}\left(1 + \frac{1}{2}\left(\frac{R}{x}\right)^2\right)}$$

$$= 2\pi\sqrt{\frac{0.2\,m}{9.8\,m/s}\left(1 + \frac{1}{2}\left(\frac{1}{0.2}\right)^2\right)}$$

$$\simeq 4.58\ s \blacksquare$$

the minimum period is when

$$\left.\frac{dT}{dx}\right|_{x=x_m} = 0$$

Now, $T^2 = 4\pi^2\left[\frac{x}{g}\left(1 + \frac{1}{2}\left(\frac{R}{x}\right)^2\right)\right]$

and $\frac{dT^2}{dx} = 2T\frac{dT}{dx} = 0$

$\hookrightarrow T \neq 0$

So,

$$\left.\frac{dT^2}{dx}\right|_{x=x_m} = \frac{d}{dx}\left\{4\pi^2\left[\frac{x}{g}\left(1+\frac{1}{2}\left(\frac{R}{x}\right)^2\right)\right]\right\}_{x=x_m}$$

$$= 4\pi^2\left[\frac{1}{g} + \frac{1}{2}\frac{R^2}{g}\frac{d}{dx}\left(\frac{1}{x}\right)\right]_{x=x_m}$$

$$= 4\pi^2\left[\frac{1}{g} - \frac{1}{2}\frac{R^2}{g}\frac{1}{x_m^2}\right]$$

$$= \frac{4\pi^2}{g}\left[1 - \frac{1}{2}\left(\frac{R}{x_m}\right)^2\right] = 0$$

$$\Rightarrow \frac{1}{2}\left(\frac{R}{x_m}\right)^2 = 1$$

or, $R = \sqrt{2}\, x_m \Rightarrow x_m = \frac{1}{\sqrt{2}} R$

$$\simeq 0.707 \text{ m} \blacksquare$$