Physics 303

Classical Mechanics II

Collision Theory

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Unbounded Orbits

So far in our discussion of Two-body systems, we have focused on bounded orbits. We will now examine unbounded orbits, which will lead us to the topic of scattering & collision theory

Recall the solution for a Kepler orbit w/ central force law $F(r) = -\gamma/r$,

$$r(\varphi) = \frac{c}{1+\epsilon\cos\varphi}$$

We saw for $\epsilon < 1 \Rightarrow E < 0$ and the orbit is bounded

Now we turn to $\epsilon \geq 1$, which from the analysis on the energy of the system,

$$E = \frac{\gamma^2\mu}{2\ell^2}(\epsilon^2 - 1) \geq 0$$

Notice that $\epsilon = 1$ is a transition, with $E = 0$.

For $\epsilon = 1$, $1 + \epsilon\cos\varphi = 0$ for $\varphi = \pm\pi$

so, $\varphi \to \pm\pi \Rightarrow r \to \infty$

So, the system is unbounded and approaches ∞.
Converting to Cartesian coordinates, we find

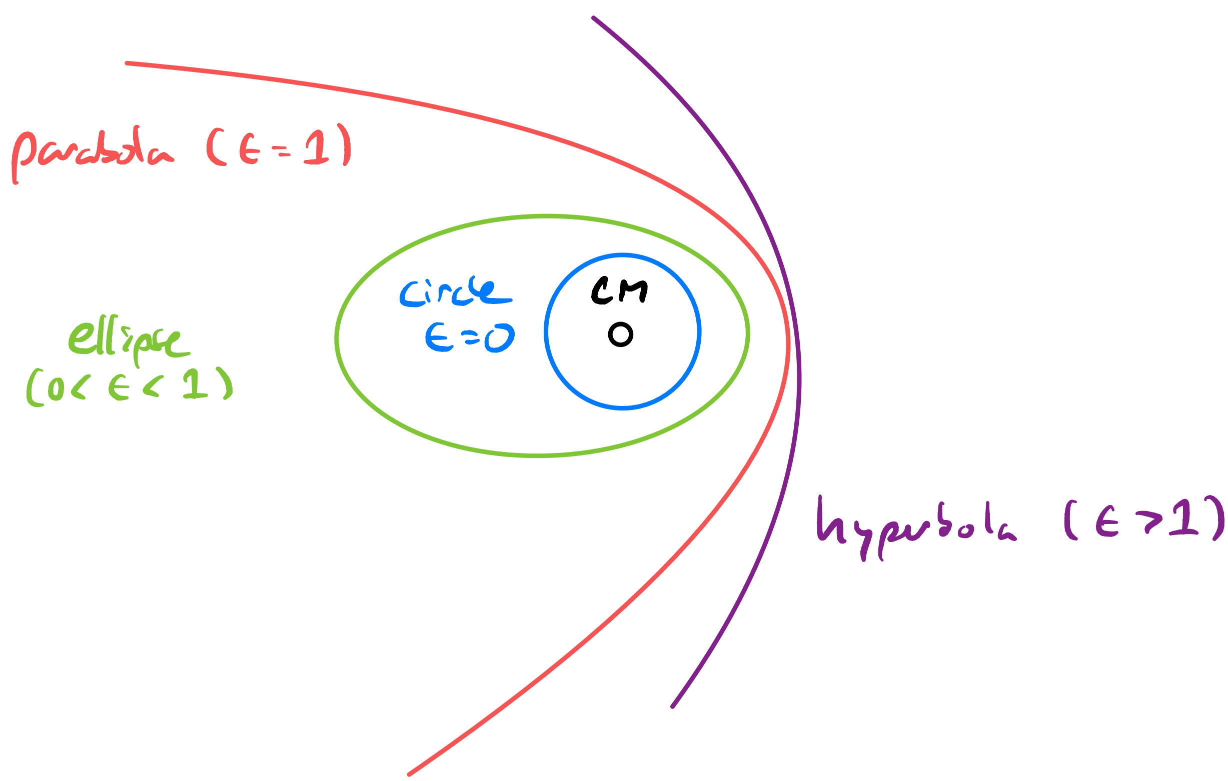
$$y^2 = c^2 - 2cx \quad \text{(parabola)}$$

If $\epsilon > 1$ ($E > 0$) then denominator vanishes when $1 + \epsilon \cos\varphi_{max} = 0$

So, as $\varphi \to \varphi_{max} \Rightarrow r \to \infty$, & the orbit is confined between angles $-\varphi_{max} < \varphi < \varphi_{max}$.
This is a hyperbolic geometry,

$$\frac{(x-\delta)^2}{\alpha^2} - \frac{y^2}{\beta^2} = 1$$

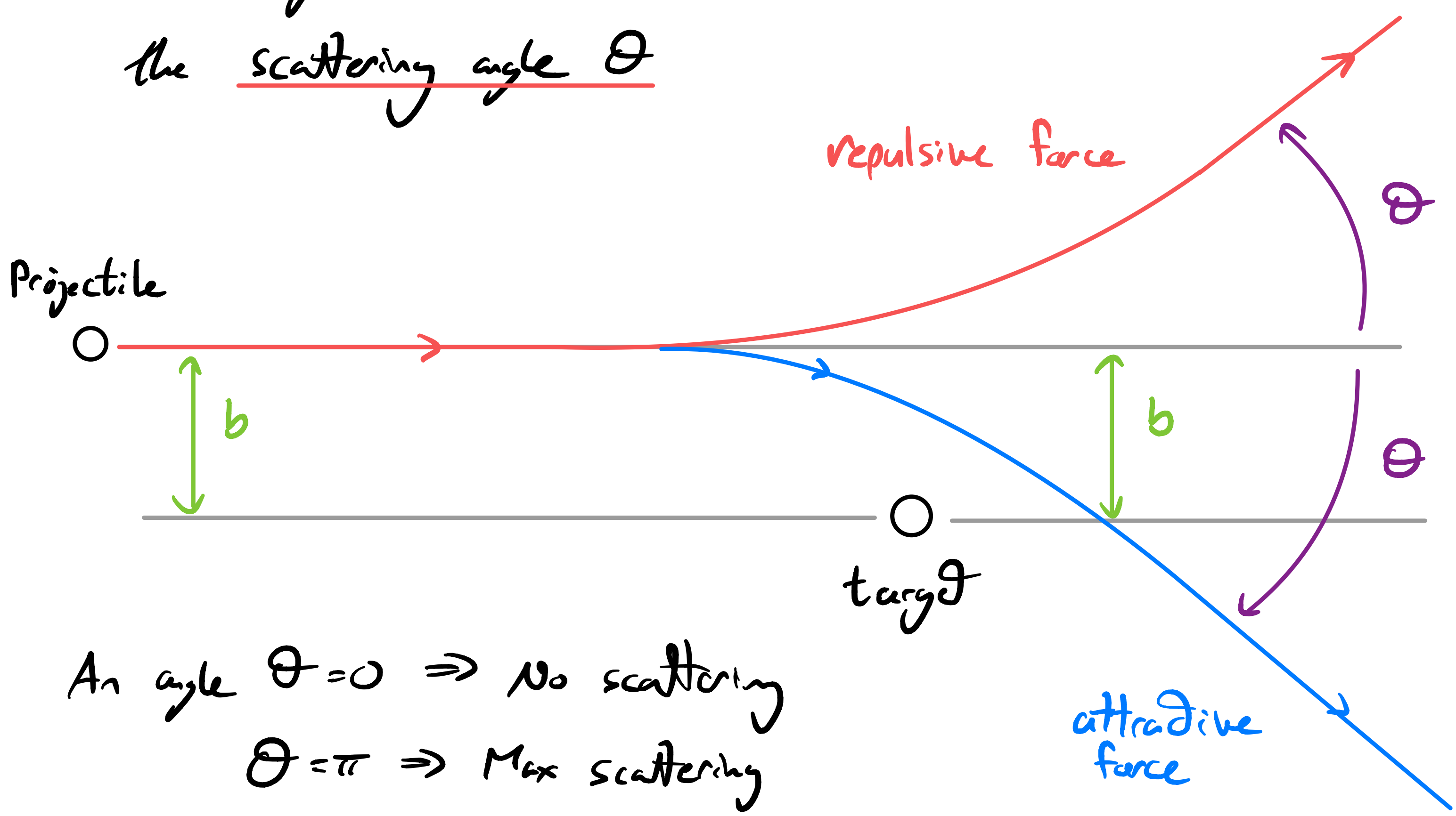
with α, β, δ in terms of ϵ & c (exercise)



Scattering

The previous example is part of a larger concept called scattering. In a scattering process, a particle (or projectile) approaches another particle (or target) from "infinitely" far away. If there is a potential energy $U(r)$ between the projectile & target (similar results hold for contact interactions), then the projectile is deflected.

The angle θ which it is deflected is called the scattering angle θ



An angle $\theta = 0 \Rightarrow$ No scattering

$\theta = \pi \Rightarrow$ Max scattering

(Head-on collision or back scattering)

Another useful quantity is the impact parameter b, which is the perpendicular distance from the projectiles incoming straight-line path to a parallel axis through the target's center.

$\Rightarrow$ b = distance of closest approach if there were no force.

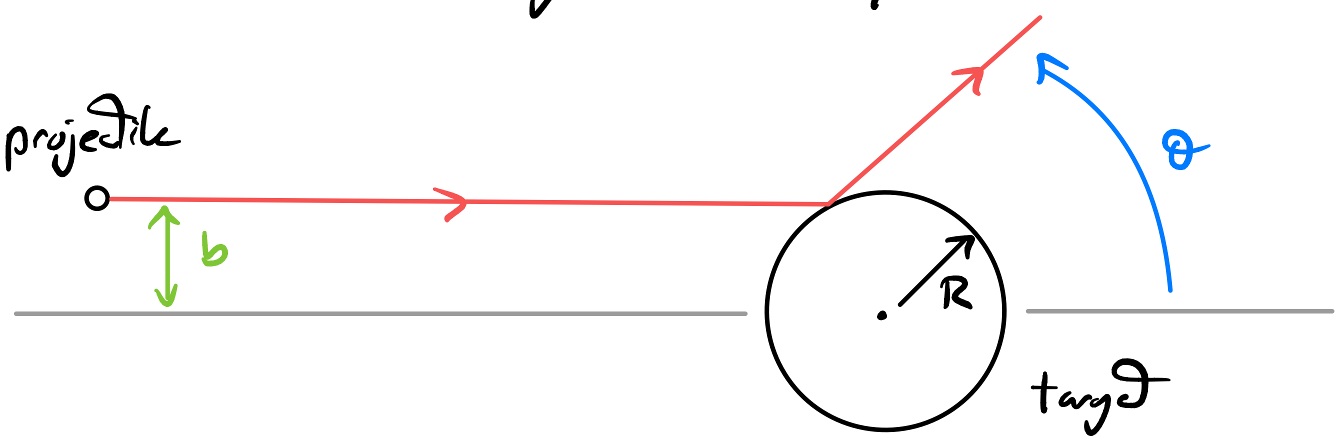
Scattering or Collision theory is especially important in particle & nuclear physics, where we learn about subatomic forces from collider experiments.

Generally, θ is easily measured, but b is not. Two perspectives:

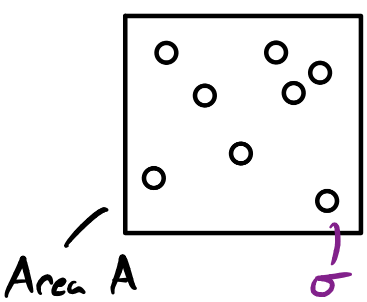
$$\boxed{\begin{array}{l}\text{know interaction} \Rightarrow \text{calculate } b \Rightarrow \text{predict } \theta \\ \text{Don't know interaction} \Rightarrow \text{measure } \theta \Rightarrow \text{determine } b\end{array}}$$

To effectively do this, we usually do not consider a single scattering experiment, but many projectiles on a bunch of targets $\Rightarrow$ Get Probabilistic determination.

Suppose we have an experiment where a projectile of negligible size interacts with a hard sphere with radius R through contact only.



We repeat this experiment many times on a target block



targets are randomly placed in the block of area A with number density n_{tar}.

$\Rightarrow$ # of targets $= n_A A$

targets / area

But, cross-sectional area, or cross-section σ, for a target is $\sigma = \pi R^2$

$\Rightarrow$ Total area of all targets is $n_{tar} A \sigma$

If we fire a beam of projectiles, probability of hit is

$$\text{Prob}_{\text{hit}} = \frac{\text{target Area}}{\text{total Area}} = \frac{n_{\text{tar}} A \sigma}{A} = n_{\text{tar}} \sigma$$

If beam contains N_{inc} incident particles, the number of scattered particles is

$$N_{sc} = \text{Prob}_{\text{hit}} N_{\text{inc}}$$

$$\Rightarrow \quad \boxed{N_{sc} = N_{\text{inc}} n_{\text{tar}} \sigma}$$

We generally know N_{inc} & n_{tar}, & can measure N_{sc}

$\Rightarrow$ get access to σ !

The cross-section σ is the effective area of the target to interact with the projectile.

Often, we measure hits in some time interval Δt

$\Rightarrow$ R_{inc} = rate of incident particles $= \dfrac{N_{\text{inc}}}{\Delta t}$

R_{sc} = rate of scattered particles $= \dfrac{N_{sc}}{\Delta t}$

$$\Rightarrow \quad \boxed{R_{sc} = R_{\text{inc}} n_{\text{tar}} \sigma}$$

Since $\sigma = \pi R^2$, $[\sigma] = L^2$ = Area.

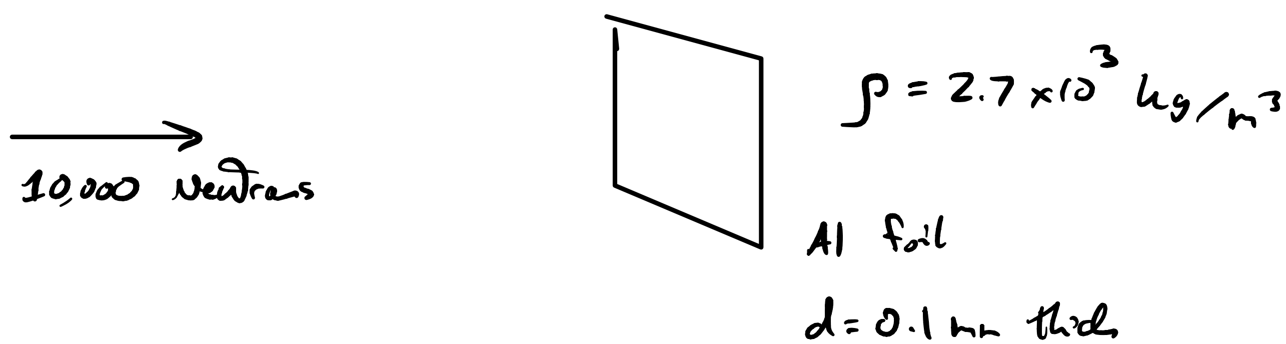
For particle & nuclear physics, $R_{nucleus} \sim 10^{-14}$ m

$$\Rightarrow \sigma \sim 10^{-28} m$$

Define a barn a $1 \text{ barn} = 10^{-28} m^2$

Example

10,000 neutrons are fired on an Al foil target of 0.1 mm thick. The cross-section is $\sigma_{n-Al} = 1.5$ barns. How many neutrons are scattered?



Solution

$$N_{sc} = N_{inc} \cdot n_{target} \cdot \sigma$$

Now, $N_{inc} = 10^4$, $\sigma = 1.5$ barns $= 1.5 \times 10^{-28} m^2$

Number density is $n_{tar} = \dfrac{\# \text{ Al atoms}}{\text{area}}$

But, area mass density $= \rho \cdot d$

The Atomic mass of Al is $27 \times 1.66 \times 10^{-27}$ kg

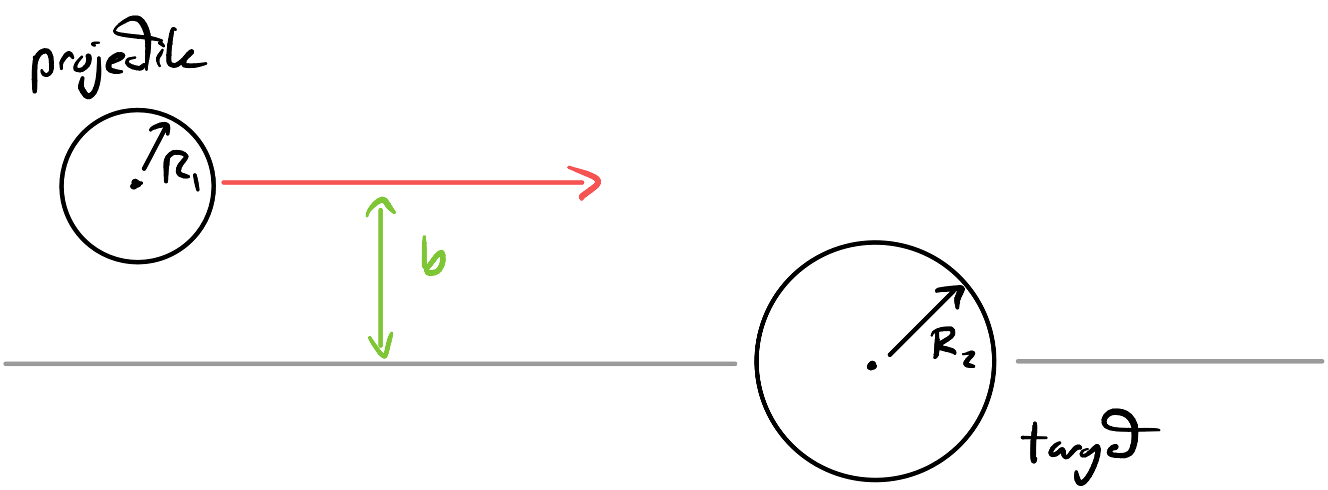
$$\Rightarrow \quad n_{tar} = \frac{\rho d}{m} = \frac{(2.7\times10^{3}\,kg/m^{3})(10^{-4}\,m)}{27\times 1.66\times10^{-27}\,kg} \approx 6.0\times10^{24}\,m^{-2}$$

So,

$$N_{sc} = N_{inc}\, n_{tar}\, \sigma$$
$$= 10^{4} \cdot 6.0\times10^{24}\,m^{-2} \cdot 1.5\times10^{28}\,m^{2} = 9$$

9 scattered events for every 10,000 ! ∎

Consider now the scattering of two hard spheres.
projectile has radius R_1, target R_2



Two two spheres make contact iff $b \le R_1 + R_2$

The cross-section is

$$\sigma = \pi (R_1 + R_2)^2$$

Following the same arguments as before, find

$$N_{sc} = N_{inc.} \, n_{tar} \cdot \sigma$$

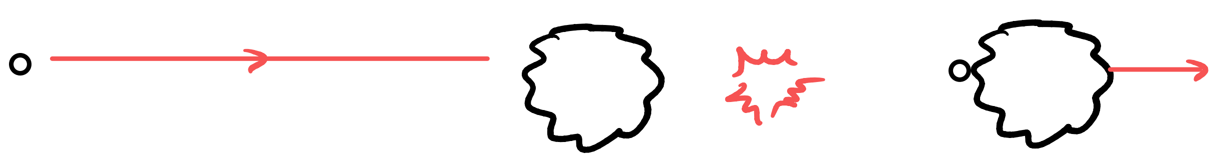
$$\hookrightarrow = \pi (R_1 + R_2)^2$$

$\Rightarrow$ $\boxed{\text{The cross-section is the effective area of the target AND the projectile!}}$

Other cross-sections exists

Capture

Imagine a target of putty



This is called capture or absorption cross-section

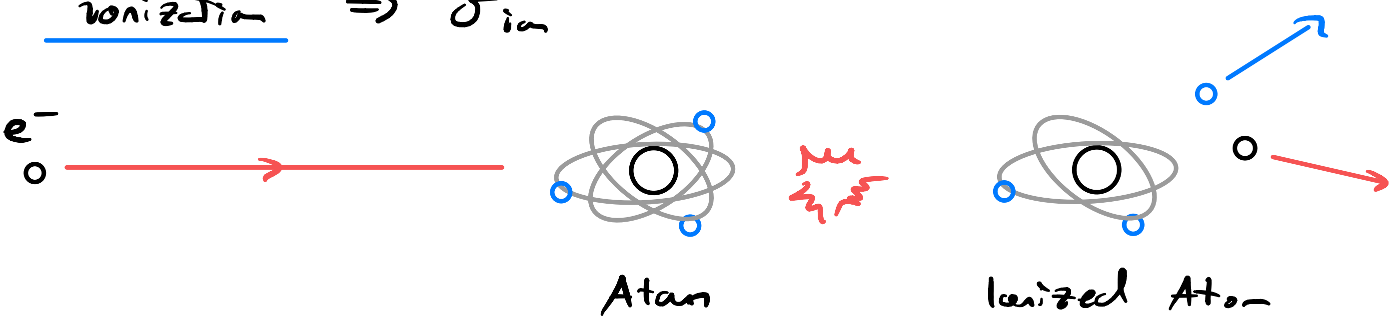
$$N_{cap} = N_{inc} \cdot n_{tar} \cdot \sigma_{cap}$$

A real collision would have both scattering & absorption

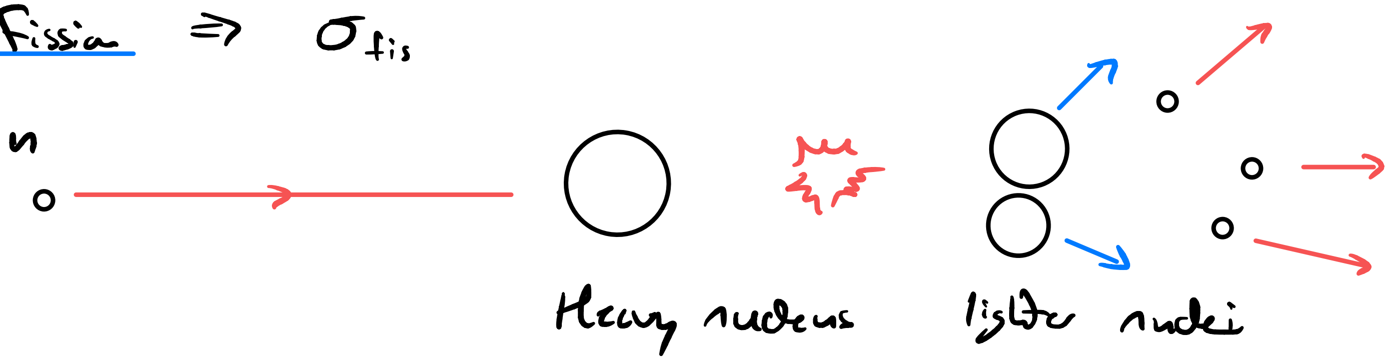
$$\sigma_{tot} = \sigma_{sc} + \sigma_{cap}$$

In atomic or nuclear systems, have many others

ionization $\Rightarrow$ σ_{ion}



Fission $\Rightarrow$ σ_{fis}



etc.

Differential Cross-Sections

The cross-section is a measure of the number of events of a scattering process. What if we also measured direction of the scattered particle?

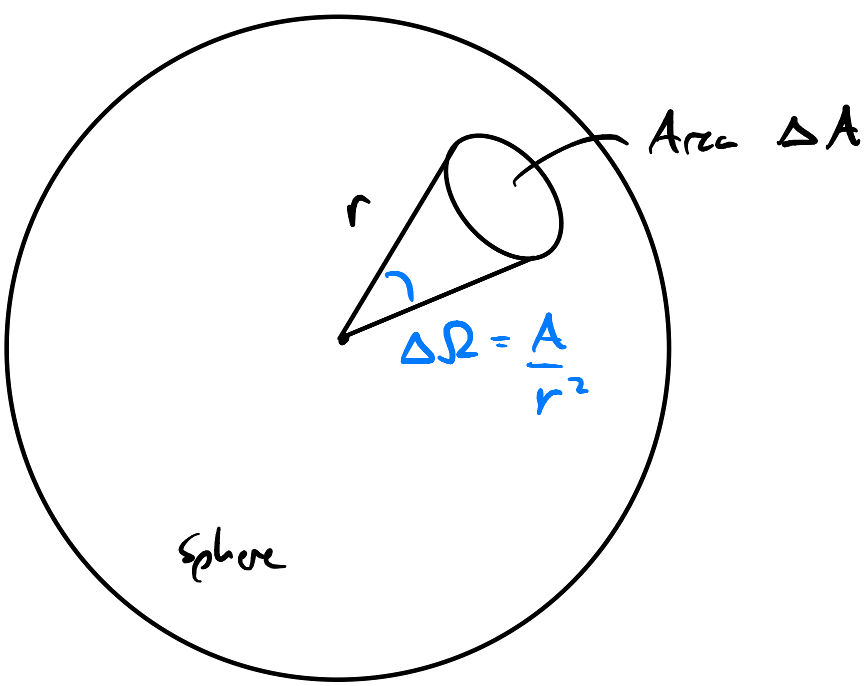
$\Rightarrow$ Differential Cross-section

Consider elastic scattering, e.g., projectile off a hard sphere target or Coulomb repulsion (Rutherford scattering)

Define z-axis along beam direction, so (θ, φ) are polar angles of emerged particle. But, we only measure a small cone of emerged angles

$$[\theta, \theta + d\theta] \ \& \ [\varphi, \varphi + d\varphi]$$

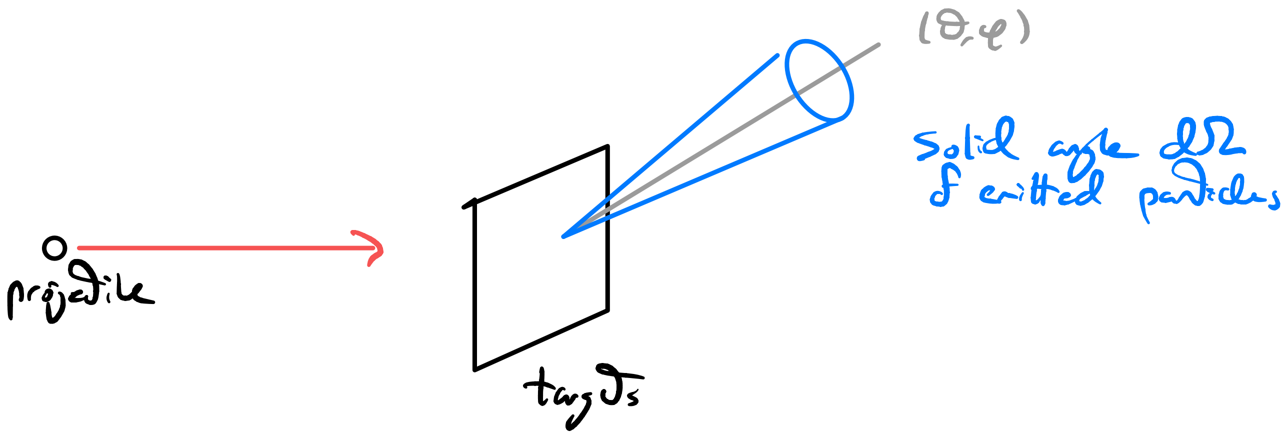
Introduce solid angle $\quad \Delta\Omega = \dfrac{\Delta A}{r^2}$ (steradians)



The solid angle of whole sphere is 4π sr

Take infinitesimal cone $\Rightarrow$ $dA = r^2 \sin\theta \, d\theta \, d\varphi$

$$\Rightarrow \boxed{d\Omega = \sin\theta \, d\theta \, d\varphi}$$



$$\Rightarrow \quad N_{sc}(\text{into } d\Omega) = N_{inc} \cdot n_{tar} \cdot d\sigma(\text{into } d\Omega)$$

$$= \#\text{ of particles emitted in cone } d\Omega$$

effective cross-section is then

$$d\sigma(\text{into } d\Omega) = \boxed{\frac{d\sigma}{d\Omega}} \, d\Omega$$

Differential cross-section

$$\Rightarrow \boxed{N_{sc}(\text{into } d\Omega) = N_{inc} \cdot n_{tar} \cdot \frac{d\sigma}{d\Omega}(\theta, \varphi) \cdot d\Omega}$$

If we add up all N_{sc} (into $d\Omega$) over all $d\Omega$, we must get

$$N_{sc} = N_{inc} \cdot n_{tar} \cdot \sigma$$

$$\Rightarrow \quad \sigma = \int \frac{d\sigma}{d\Omega}(\theta,\varphi)\, d\Omega$$

$$= \int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\varphi \;\frac{d\sigma}{d\Omega}(\theta,\varphi)$$

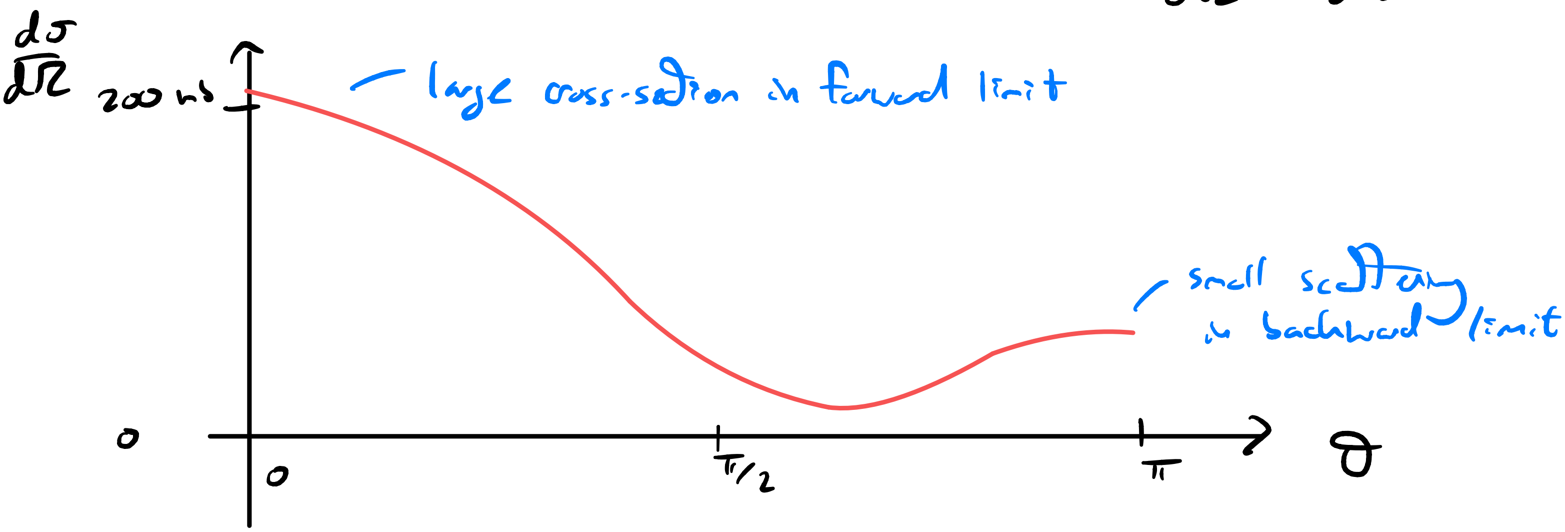
Example

Neutrons scatter off a target at several MeV.
The differential cross-section is measured to be

$$\frac{d\sigma}{d\Omega}(\theta,\varphi) = \sigma_0 \left(1 + 3\cos\theta + 3\cos^2\theta \right)$$

where $\sigma_0 \simeq 30$ mb/sr

The angular distribution is axially symmetric, $\frac{d\sigma}{d\Omega} \neq \frac{d\sigma}{d\Omega}(\varphi)$



The total cross-section is

$$\sigma = \int \frac{d\sigma}{d\Omega}\, d\Omega = \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta \, \frac{d\sigma}{d\Omega}$$

$$= 2\pi\sigma_0 \int_0^{\pi} d\theta \sin\theta \left[1 + 3\cos\theta + 3\cos^2\theta\right]$$

$$= 2\pi\sigma_0 \int_{-1}^{1} d\cos\theta \left[1 + 3\cos\theta + 3\cos^2\theta\right]$$

$$= 2\pi\sigma_0 \left[\cos\theta + \frac{3}{2}\cos^2\theta + \cos^3\theta\right]_{-1}^{1}$$

$$= 2\pi\sigma_0 \left[2 + 0 + 2\right]$$

$$= 8\pi\sigma_0$$

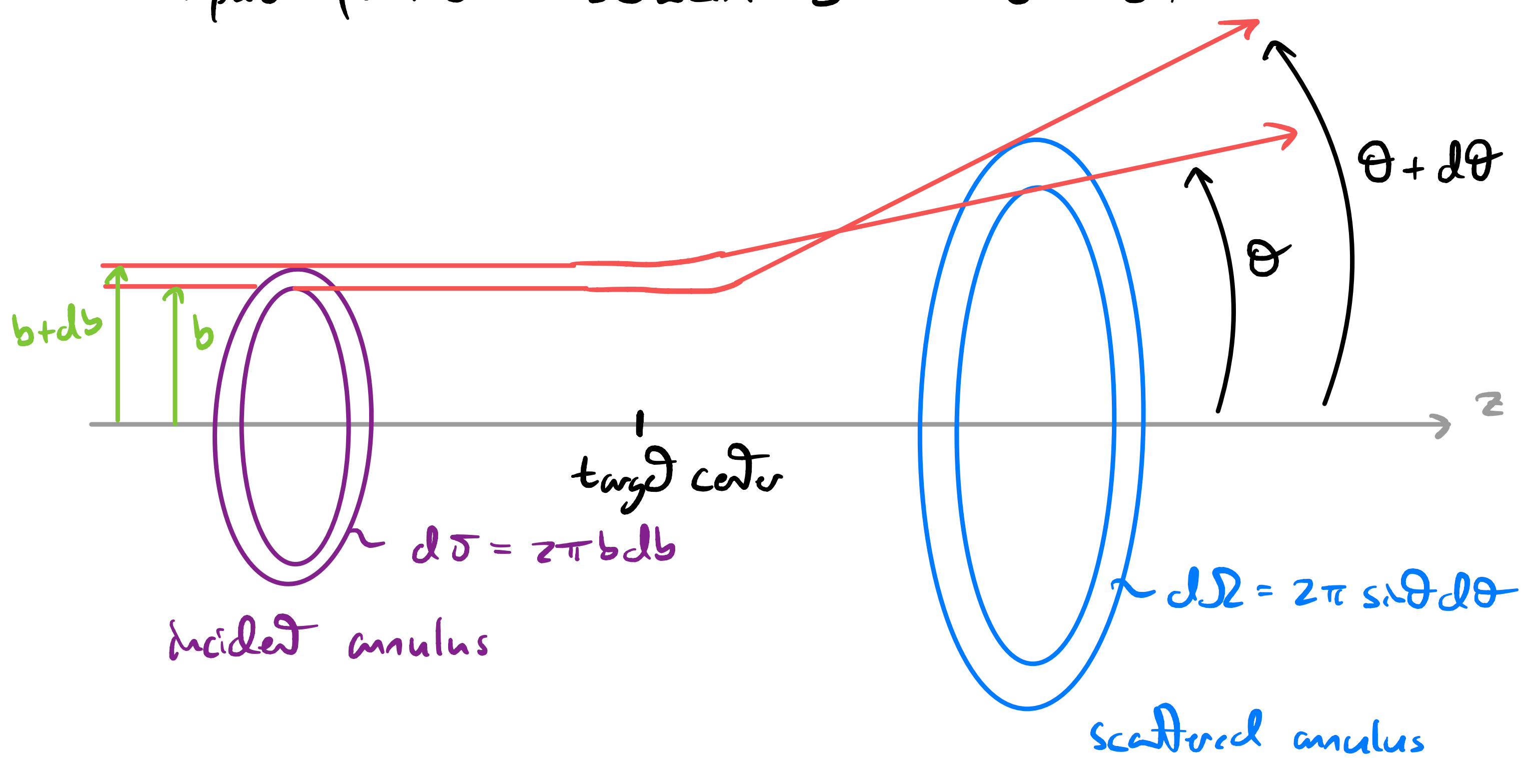
So,

$$\sigma = 8\pi\sigma_0 \simeq 754 \text{ mb.} \qquad \blacksquare$$

We now want to connect the differential cross-section to the impact parameter. For simplicity, let us assume that the scattering is axially symmetric, that is it is φ-independent.

Consider a projectile incident on a target with impact parameter b. By calculating the particles trajectory, we can (in principle) calculate the scattering angle $\theta = \theta(b)$.
Alternatively, by solving for b, $b = b(\theta)$.

Consider now projectiles on a target with impact parameters between b and $b + db$.



The incident annulus has cross-section

$$d\sigma = 2\pi b \, db$$

The particles are scattered at a solid angle

$$d\Omega = 2\pi \sin\theta \, d\theta$$

↳ axial symmetry

Therefore, the differential cross-section is

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|}$$

→ ensures probability definition

So, path is : find trajectory $\theta = \theta(b)$

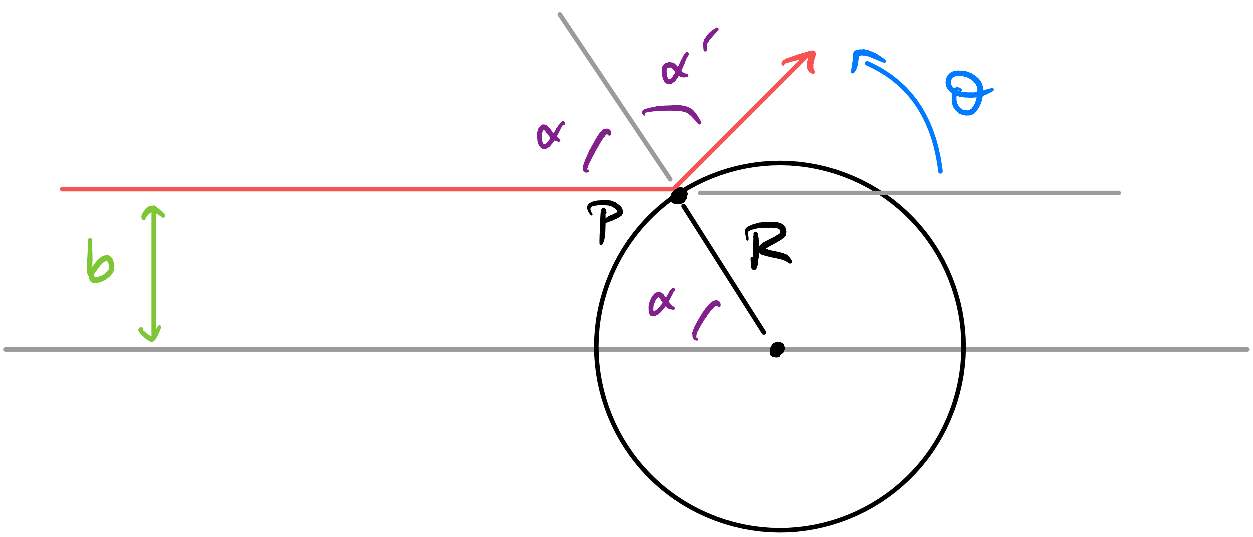
$\Rightarrow$ invert to find $b = b(\theta)$

$\Rightarrow$ Differentiate $db/d\theta$

Example

Find the differential cross-section for the scattering of a point particle off a fixed rigid sphere of radius R.

Solution: We must first find the trajectory $\theta = \theta(b)$.



Let α be the angle of the line connecting the target center to the strike point P.
From this line, α' the angle toward the direction after collision.

$$\therefore \quad \theta = \pi - \alpha - \alpha'$$

We now need α & α'.

First, note the law of reflection: $\alpha = \alpha'$.

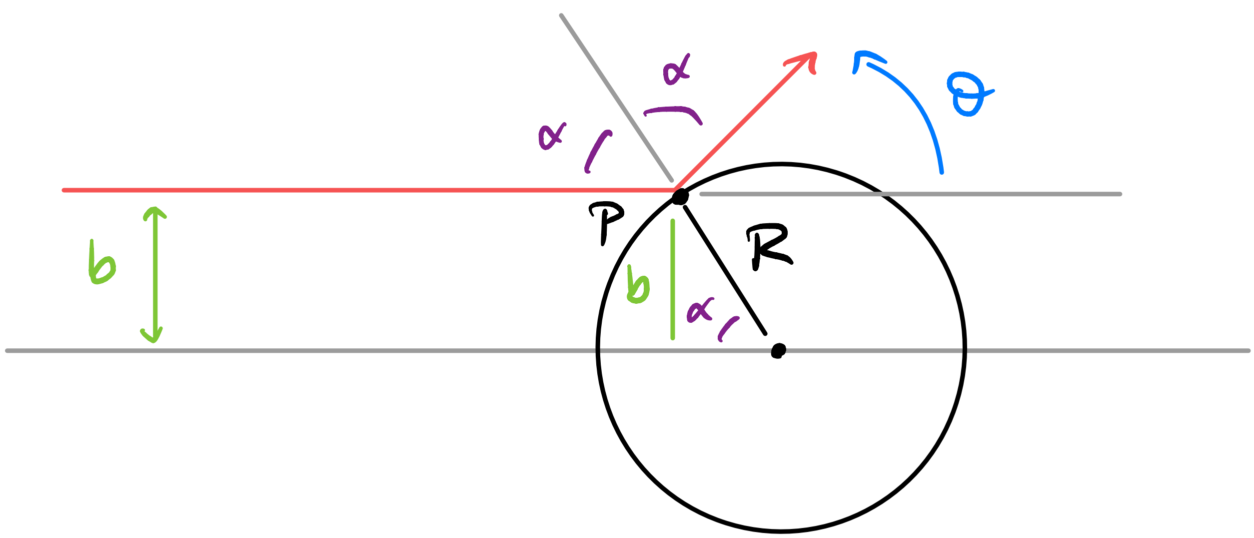
To see this, let v & v' be the incoming and outgoing speeds. The collision is elastic

$$\Rightarrow \quad v = v'.$$

Conservation of angular momentum $\Rightarrow$ $mvR\sin\alpha = mv'R\sin\alpha'$

$$\Rightarrow \quad \sin\alpha = \sin\alpha' \quad \Rightarrow \quad \alpha = \alpha'$$

Thus, $\theta = \pi - 2\alpha$



From the triangle, $\sin\alpha = \dfrac{b}{R} \quad \Rightarrow \quad b = R\sin\alpha$

So,
$$b = R\sin\alpha = R\sin\left(\frac{\pi - \theta}{2}\right) = R\cos\frac{\theta}{2}$$

$$\Rightarrow \quad \frac{db}{d\theta} = -\frac{R}{2}\sin\frac{\theta}{2}$$

So, we have

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

$$= \frac{R\cos\theta/2}{\sin\theta}\left|R\,\frac{\sin\theta/2}{2}\right|$$

$$= \frac{R^2}{2}\,\frac{\cos\theta/2\,\sin\theta/2}{\sin\theta} \quad \leftarrow \sin\theta = 2\sin\theta/2\cos\theta/2$$

$$= \frac{R^2}{4}$$

Notice that $\frac{d\sigma}{d\Omega} = \frac{R^2}{4}$ is independent of θ!

So, we find

$$\sigma = \int d\Omega\,\frac{d\sigma}{d\Omega} = \pi R^2 \qquad \blacksquare$$

Rutherford scattering

We will now consider the scattering experiment that led to the discovery of the atomic nucleus.

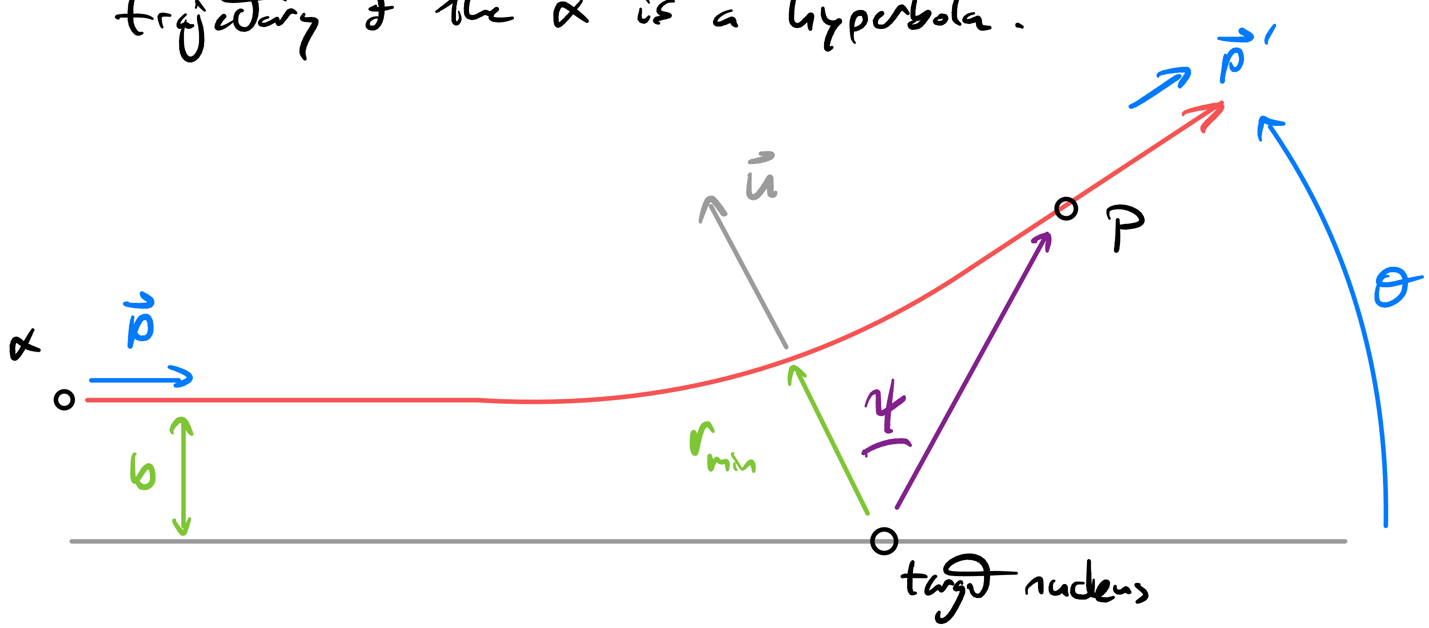
Rutherford scattering consists of alpha particles (He^{++}) off of gold nuclei in a thin gold foil.

The force is that of Coulomb's law

$$F = \frac{k \overset{\alpha\text{-charge}}{q}\; \overset{\text{gold nucleus charge}}{Q}}{r^2} \equiv \frac{\gamma}{r^2}$$

if the speeds of the α's are large enough to penetrate the gold's electron cloud

Since this force is the same as discussed in Kepler orbits, we know immediately that the trajectory of the α is a hyperbola.



Let $\vec{u}$ be unit vector in direction from target's center to the point of closest approach, r_{min}

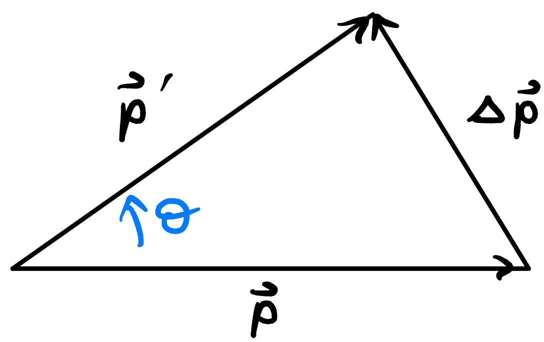
$\Rightarrow$ orbit is symmetric about this point

If ψ is angle of projectile wrt center as measured from $\vec{u}$, it is bounded by $\psi \in [-\psi_0, \psi_0]$ where $\lim_{\psi \to \psi_0} r \to \infty$

So, scattering angle is $\theta = \pi - 2\psi_0$

We now relate θ to b.

The change in momentum is



$$\Delta\vec{p} = \vec{p}\,' - \vec{p}$$

Where $\vec{p}$ & $\vec{p}\,'$ are momenta long before & after collision.

Collision is elastic $\Rightarrow$ $T = T'$ $\Rightarrow$ $|\vec{p}| = |\vec{p}\,'|$

So,
$$(\Delta\vec{p})^2 = (\vec{p}\,' - \vec{p}) \cdot (\vec{p}\,' - \vec{p})$$
$$= |\vec{p}\,'|^2 + |\vec{p}|^2 - 2\vec{p} \cdot \vec{p}\,'$$

But, $\vec{p}\cdot\vec{p}\,' = |\vec{p}||\vec{p}\,'|\cos\theta$

So,
$$(\Delta\vec{p})^2 = 2|\vec{p}|^2 - 2|\vec{p}|^2\cos\theta \qquad (|\vec{p}| = |\vec{p}\,'|)$$
$$= 2|\vec{p}|^2(1-\cos\theta)$$
$$= 4|\vec{p}|^2\sin^2\theta/2$$

$$\Rightarrow \quad |\Delta\vec{p}| = 2|\vec{p}|\sin\theta/2 \qquad (1)$$

However, from Impulse-Momentum theorem,

$$\Delta\vec{p} = \int_{-\infty}^{\infty} dt\, \vec{F}$$

Since $\Delta\vec{p} = |\Delta\vec{p}|\vec{u}$

$$\Rightarrow \quad |\Delta\vec{p}| = \int_{-\infty}^{\infty} dt\, F_u \quad , \; F_u = \vec{F}\cdot\vec{u}$$

At point P, $\vec{F} = \frac{\gamma}{r^2}\hat{r}$, with $\hat{r}\cdot\vec{u} = \cos\psi$

$$\Rightarrow \quad F_u = \frac{\gamma}{r^2}\cos\psi$$

using $dt = \frac{dt}{d\psi}d\psi = \frac{d\psi}{\dot{\psi}}$

$$\Rightarrow \quad |\Delta\vec{p}| = \int_{-\psi_0}^{\psi_0} d\psi\, F_u/\dot{\psi}$$

Finally, angular momentum conservation

$$m r^2 \dot{\psi} = l = b|\vec{p}|$$

$$\Rightarrow \quad |\Delta\vec{p}| = \int_{-\psi_0}^{\psi_0} d\psi \quad \frac{\gamma \cos\psi}{r^2} \cdot \frac{m r^2}{b|\vec{p}|}$$

$$= \frac{\gamma m}{b|\vec{p}|} \left[\sin\psi \right]_{-\psi_0}^{\psi_0}$$

$$= \frac{2\gamma m}{b|\vec{p}|} \sin\psi_0$$

Now, from $\theta = \pi - 2\psi_0 \Rightarrow \sin\psi_0 = \cos\theta/2$

$$\Rightarrow \quad |\Delta\vec{p}| = \frac{2\gamma m}{b|\vec{p}|} \cos\frac{\theta}{2} \qquad (2)$$

Equate (1) & (2) $\Rightarrow \quad 2|\vec{p}| \sin\frac{\theta}{2} = \frac{2\gamma m}{b|\vec{p}|} \cos\frac{\theta}{2}$

Solve for b, using $|\vec{p}| = m v$

$$\Rightarrow \quad b = \frac{\gamma m}{|\vec{p}|^2} \frac{\cos\theta/2}{\sin\theta/2} = \frac{\gamma}{m v^2} \cot\frac{\theta}{2}$$

So,

$$\frac{db}{d\theta} = \frac{\gamma}{mv^2}\frac{d}{d\theta}\cot\theta/2 = \frac{-\gamma}{2mv^2}\frac{1}{\sin^2\theta/2}$$

Therefore, the differential cross-section is

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

$$= \frac{\gamma}{mv^2}\frac{\cot\theta/2}{\sin\theta}\cdot\frac{\gamma}{2mv^2}\frac{1}{\sin^2\theta/2}$$

$$= \frac{\gamma^2}{m^2v^4}\frac{\cos\theta/2}{2\sin\theta/2\cos\theta/2}\cdot\frac{1}{\sin\theta/2}\cdot\frac{1}{2}\frac{1}{\sin^2\theta/2}$$

$$= \frac{\gamma^2}{4m^2v^4}\frac{1}{\sin^4\theta/2}$$

Recall that $\gamma = kqQ$.

The total energy of incident particle is

$$E = \frac{1}{2}mv^2 \Rightarrow m^2v^4 = 4E^2$$

$$\Rightarrow \boxed{\frac{d\sigma}{d\Omega} = \left(\frac{kqQ}{4E\sin^2\theta/2}\right)^2}$$

Rutherford Scattering formula