Physics 303

Classical Mechanics II

Continuum Mechanics

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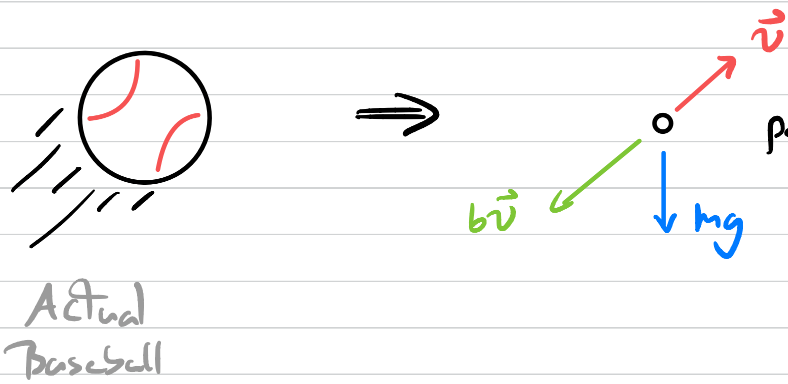
Continuum Mechanics

Classical Mechanics can be generally divided into three main areas, with increasing complexity

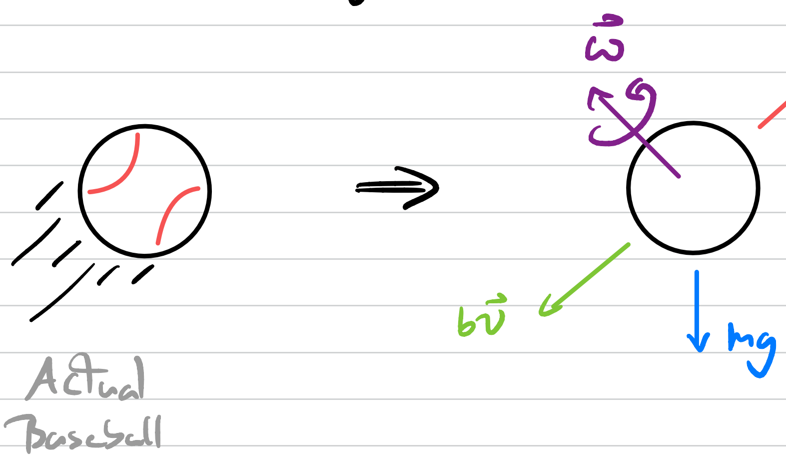
1. Mechanics of point particles

   e.g., flight of baseball



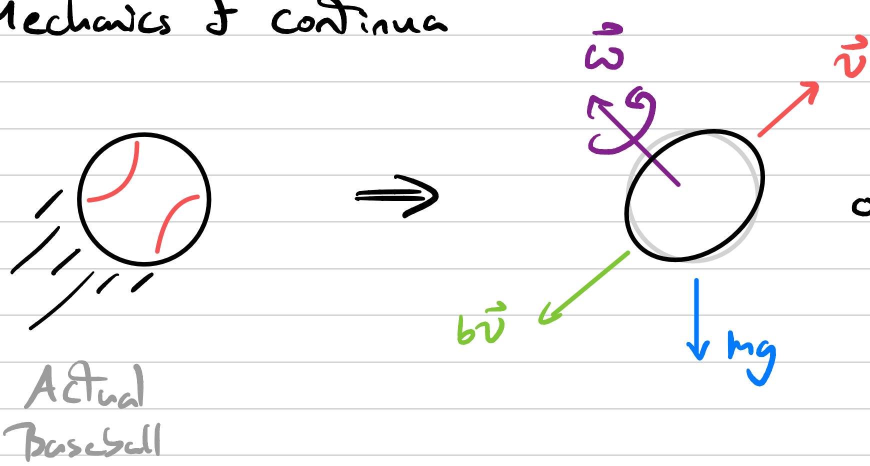
point mass, no structure

2. Mechanics of rigid bodies



extended object, rotational d.o.f.,

3. Mechanics of continua



object can deform

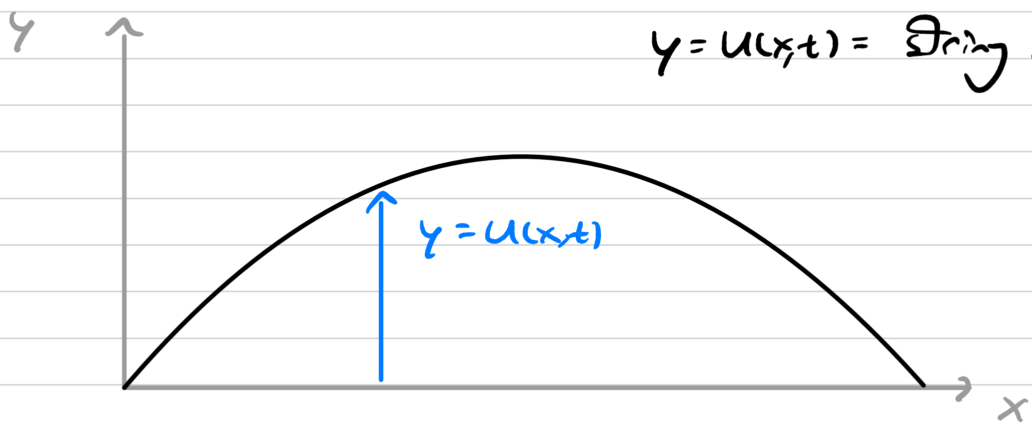
Continuum mechanics can be divided into

- Solid mechanics (our focus)
- Fluid mechanics (see Phys. 302)

In this study, the ordinary differential equations generated from Newton's laws or Euler-Lagrange become partial differential equations.

Wave Motion on a Taut String

As our first example, let's consider the wave motion on an one-dimensional string.



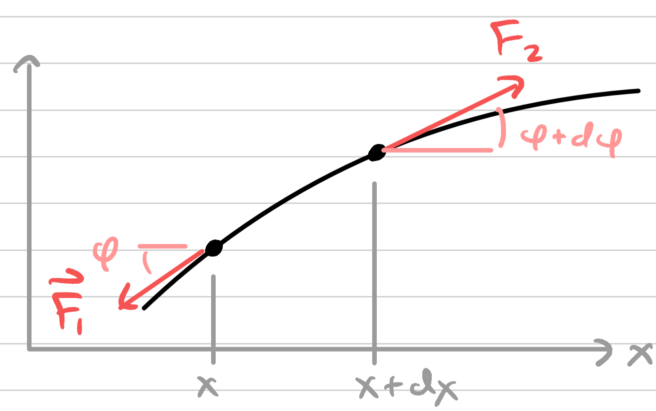
$y = u(x,t) =$ string displacement

In equilibrium, $y = u(x,t) = 0$

Pick a small segment from x to $x+dx$

Assume small displacements

$\varphi,\ \varphi + d\varphi \ll 1$



The net force in x is

$$F_x^{net} = T\cos(\varphi + d\varphi) - T\cos\varphi$$
$$\simeq T\cos\varphi - T d\varphi \sin\varphi - T\cos\varphi$$
$$= -T d\varphi \sin\varphi \simeq -T\varphi d\varphi = \mathcal{O}(\varphi^2)$$

$$F_y^{net} = T\sin(\varphi + d\varphi) - T\sin\varphi$$
$$\simeq T\sin\varphi + T d\varphi \cos\varphi - T\sin\varphi$$
$$= T d\varphi \cos\varphi \simeq T d\varphi$$

Since $\varphi \ll 1 \Rightarrow \sin\varphi \sim \varphi$ & $\cos\varphi \sim 1$

Notice that $\tan\varphi \simeq \varphi \simeq \frac{\partial y}{\partial x} = \frac{\partial u}{\partial x}$

Therefore, $F_y^{net} \simeq T d\varphi \simeq T \frac{\partial \varphi}{\partial x} dx = T \frac{\partial^2 u}{\partial x^2} dx$

NII $\Rightarrow \vec{F} = m\vec{a} \Rightarrow F_y^{net} = dm\, a_y$ ← acceleration in y direction

↳ mass element of string

$$\Rightarrow T \frac{\partial^2 u}{\partial x^2} dx = a_y dm$$
$$= \frac{\partial^2 u}{\partial t^2} (\mu dx)$$

↳ linear mass density

$$\Rightarrow \frac{\partial^2 u}{\partial t^2} = \frac{T}{\mu} \frac{\partial^2 u}{\partial x^2}$$

Define $\boxed{c = \sqrt{\frac{T}{\mu}}}$ as the speed of the wave.

Notice, more taut string has higher speed!

So,

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}}$$ This is the wave equation!

General solution to Wave Equation

Introduce two variables $\xi = x - ct$ & $\eta = x + ct$

$$\Rightarrow \quad x = \frac{1}{2}(\xi + \eta), \quad t = \frac{1}{2c}(\eta - \xi)$$

So,

$$\frac{\partial}{\partial \xi} = \frac{\partial x}{\partial \xi}\frac{\partial}{\partial x} + \frac{\partial t}{\partial \xi}\frac{\partial}{\partial t} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{1}{c}\frac{\partial}{\partial t}\right)$$

$$\frac{\partial}{\partial \eta} = \frac{\partial x}{\partial \eta}\frac{\partial}{\partial x} + \frac{\partial t}{\partial \eta}\frac{\partial}{\partial t} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{1}{c}\frac{\partial}{\partial t}\right)$$

$$\Rightarrow \quad \frac{\partial}{\partial \xi}\frac{\partial}{\partial \eta} = \frac{-1}{4c^2}\left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right)$$

So, wave eqn. $\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$

becomes $$\boxed{\frac{\partial}{\partial \xi}\frac{\partial u}{\partial \eta} = 0}$$

To solve this eqn., let $h = \frac{\partial u}{\partial \eta}$

$\Rightarrow \quad \frac{\partial h}{\partial \xi} = 0 \quad \Rightarrow$ h is independent of ξ, but it can depend on η,
$\Rightarrow h = h(\eta)$

So, for a given ξ,

$$\frac{\partial u}{\partial \eta} = h(\eta) \quad \Rightarrow \quad u = \int d\eta\, h(\eta) + \text{const.}$$

Since $h \neq h(\xi) \Rightarrow \int d\eta\, h(\eta) = g(\eta)$

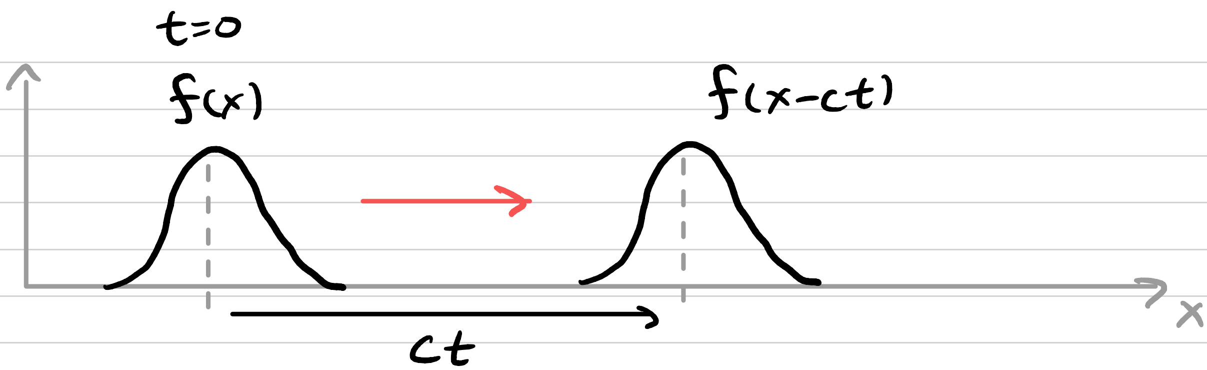
Also, the const is for a given $\xi \Rightarrow$ const $\rightarrow f(\xi)$

So, general solution is

$$u(\xi, \eta) = f(\xi) + g(\eta)$$

or,

$$\boxed{u(x,t) = \underbrace{f(x-ct)}_{\text{wave moving to right}} + \underbrace{g(x+ct)}_{\text{wave moving to left}}}$$



Consider solution $u(x,t) = f(x-ct)$

At $t=0$, $f(x)$ has a maximum @ $x=0$

At t, $f(x-ct)$ has a maximum @ $x-ct=0$

$\Rightarrow x = ct$

A special example is the standing wave

Consider $f(x-ct) = A\sin(kx-\omega t)$

where $\omega = kc$ & A, k are arbitrary constants.

A is called the amplitude,

k is wave number $\Rightarrow \lambda = \frac{2\pi}{k}$ is wave length

ω is angular frequency $\Rightarrow \tau = \frac{2\pi}{\omega}$ is period

If $g(x+ct) = A\sin(kx+\omega t)$

then,

$$u(x,t) = A\sin(kx-\omega t) + A\sin(kx+\omega t)$$

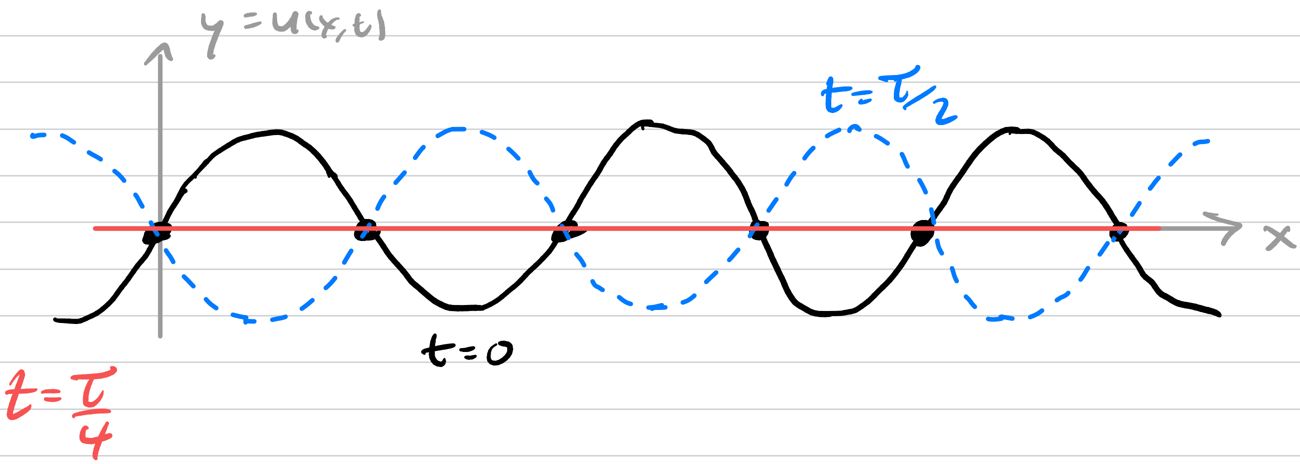
$$= 2A\sin kx \cos\omega t$$

Notice that the wave does not travel, it merely oscillates up and down.

$$u(x,t) = \underbrace{[2A \sin kx]}_{\text{amplitude}} \underbrace{\cos \omega t}_{\text{oscillatory time dependence}}$$

Notice that the zeros of the amplitude are fixed

$$\Rightarrow \quad kx = n\pi \quad \Rightarrow \quad x = \frac{n\pi}{k} \quad \text{are nodes}$$



We will see that these standing waves are the continuum analogue of normal modes in coupled oscillators.

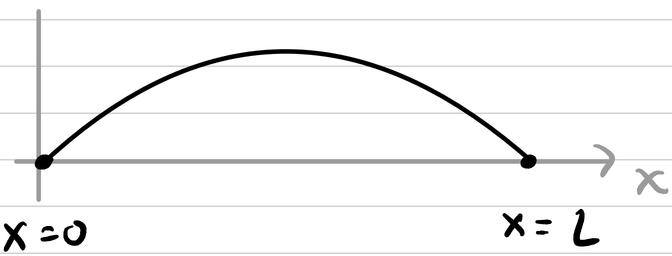
Boundary Conditions on Finite String

The wave equations requires Initial and/or Boundary conditions to completely specify a solution.

Consider a wave on a finite string subject to Dirichlet BCs

$$\boxed{u(0,t) = u(L,t) = 0}$$

for all t.



We want a solution to

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Try $u(x,t) = X(x) \cos(\omega t - \delta)$

"Separation of Variables"

$$\Rightarrow \quad -\omega^2 X(x) \cos(\omega t - \delta) = c^2 \frac{d^2 X}{dx^2} \cos(\omega t - \delta)$$

$$\Rightarrow \quad \frac{d^2 X(x)}{dx^2} = -k^2 X(x) \qquad \text{w/} \quad k = \frac{\omega}{c}$$

Solution is $X(x) = A \sin kx + B \cos kx$

Now, this solution is subject to Dirichlet BCs

$$\Rightarrow \quad X(0) = X(L) = 0$$

$X(0)=0 \Rightarrow \quad 0 = A \cdot 0 + B(1) \Rightarrow B = 0$

$X(L)=0 \Rightarrow \quad 0 = A \sin(kL)$

$A=0$ is trivial solution, so find $kL = n\pi$, $n \in \mathbb{N}$

$$\Rightarrow \boxed{k_n = \frac{n\pi}{L} \quad , n = 1, 2, \ldots}$$

The wave vector is quantized!

$$\Rightarrow \omega_n = \frac{n\pi}{L} c$$

So,

$$u(x,t) = \sum_n A_n \sin(k_n x) \cos(\omega_n t - \delta_n)$$

↳ an infinite # of standing waves

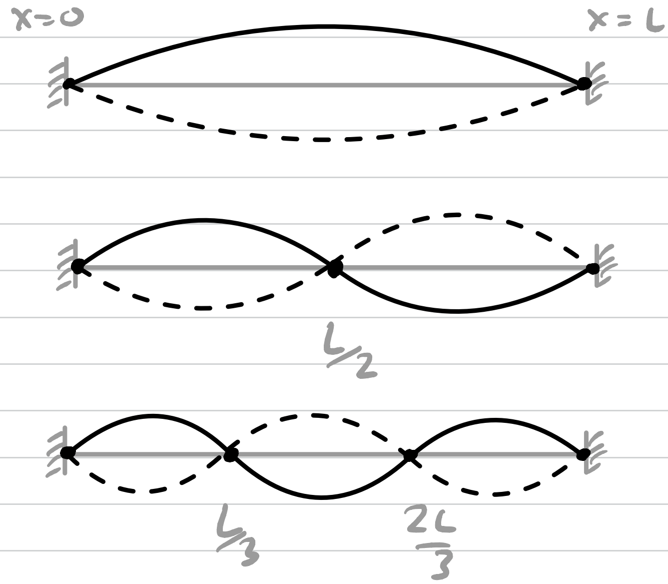
$\Rightarrow$ Normal modes

$\sin\left(\frac{\pi}{L}x\right)$ $n=1$

$\sin\left(\frac{2\pi}{L}x\right)$ $n=2$

$\sin\left(\frac{3\pi}{L}x\right)$ $n=3$

$\vdots$



Coefficients A_n & δ_n are fixed by initial configuration

Let $A_n \cos(\omega_n t - \delta_n) = B_n \cos \omega_n t + C_n \sin \omega_n t$

$$\Rightarrow \quad u(x,t) = \sum_n \sin k_n x \left(B_n \cos \omega_n t + C_n \sin \omega_n t \right)$$

At $t=0$, we are given $u(x,0)$ & $\dot{u}(x,0)$

We find $u(x,0) = \sum_n B_n \sin k_n x$

$$\dot{u}(x,0) = \sum_n C_n \omega_n \sin k_n x$$

To get B_n & C_n, use Fourier's trick from Fourier Series

$$u(x,0) = \sum_n B_n \sin\left(\frac{n\pi x}{L}\right)$$

$$\Rightarrow \int_0^L dx\, u(x,0) \sin\left(\frac{m\pi x}{L}\right) = \sum_n B_n \int_0^L dx \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right)$$

Can show that

$$\int_0^L dx \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) = \frac{L}{2}\delta_{nm}$$

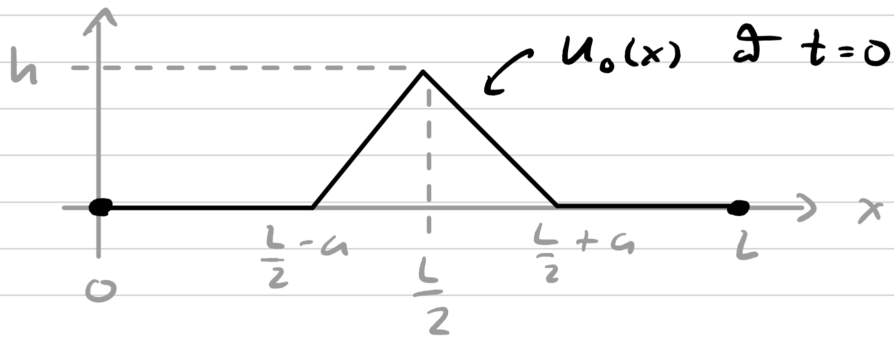
$$\Rightarrow \quad B_n = \frac{2}{L}\int_0^L dx\, u(x,0) \sin\left(\frac{n\pi x}{L}\right)$$

Similarly, for C_n find $C_n = \frac{2}{L}\frac{1}{\omega_n}\int_0^L dx\, \dot{u}(x,0) \sin\left(\frac{n\pi x}{L}\right)$

Let's look at a particular example.

Example: Triangular wave on finite string

for example,
$h = L = a = 1$



So,

$$u_0(x) = u(x,0) = \begin{cases} 0 & 0 \le x < L/2 - a \\ \frac{h}{a}\left(x - \left(\frac{L}{2} - a\right)\right) & L/2 - a \le x < L/2 \\ \frac{h}{a}\left(\frac{L}{2} + a - x\right) & L/2 \le x < L/2 + a \\ 0 & \frac{L}{2} + a \le x \le L \end{cases}$$

in our example,

$$u_0(x) = u(x,0) = \begin{cases} 0 & 0 \le x < 3 \\ (x-3) & 3 \le x < 4 \\ (5-x) & 4 \le x < 5 \\ 0 & 5 \le x < 8 \end{cases}$$

We are also given $\dot{u}(x,0) = 0 \quad \forall\ x \in [0, L]$

Notice that the wave is symmetric about $x = \frac{L}{2}$ $(=4)$

Define $x' = x - \frac{L}{2}$, so $u_0(x') = u_0(-x')$

$$\Rightarrow B_n = \frac{2}{L} \int_0^L dx\, u_0(x) \sin\left(\frac{n\pi x}{L}\right)$$

check sign

$$= \frac{2}{L} \int_{-L/2}^{L/2} dx'\, u_0(x') \sin\left(\frac{n\pi}{L} x' - \frac{n\pi}{2}\right)$$

Since $u_0(x')$ is even

$$\Rightarrow B_{2n} = 0 \quad \forall\, n \in \mathbb{N}$$

Why?

$\sin(\alpha - n\pi) = (-1)^n \sin(\alpha)$

$$B_{2n} = \frac{2}{L}\int_{-L/2}^{L/2} dx'\, u_0(x') \sin\left(\frac{2\pi}{L} n x' - n\pi\right)$$

$$= (-1)^n \frac{2}{L}\int_{-L/2}^{L/2} dx'\, u_0(x') \sin\left(\frac{2\pi n}{L} x'\right) = 0$$

even even odd

So, look @ odd modes

$$B_{2n+1} = \frac{2}{L}\int_{-L/2}^{L/2} dx'\, u_0(x') \sin\left(\frac{(2n+1)\pi}{L} x' - \frac{(2n+1)\pi}{2}\right)$$

$$= (-1)^n \frac{2}{L}\int_{-L/2}^{L/2} dx'\, u_0(x') \sin\left(\frac{(2n+1)\pi x'}{L} - \frac{\pi}{2}\right)$$

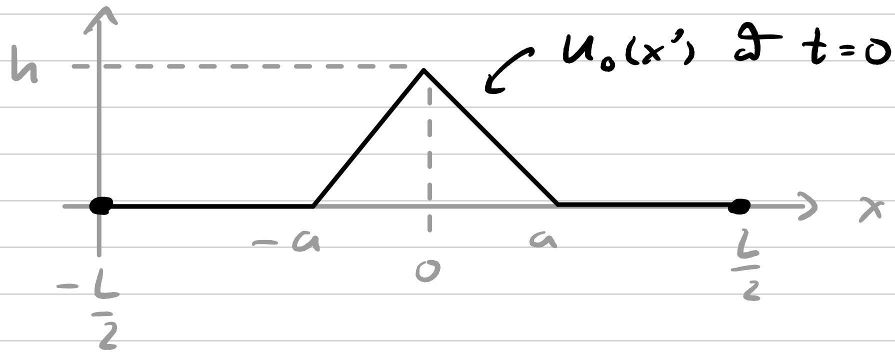
$\hookrightarrow \sin(\alpha - \pi/2) = -\cos\alpha$

$$= (-1)^{n+1} \frac{2}{L}\int_{-L/2}^{L/2} dx'\, u_0(x') \cos\left(\frac{(2n+1)\pi}{L} x'\right)$$

even even even

$$\Rightarrow B_{2n+1} \neq 0$$

To evaluate further, $u_0(x')$ is given by $x = x' + \frac{L}{2}$



$$u_0(x') = u_0(x',0) = \begin{cases} 0 & 0 \le x' < -a \\ h(x'+a)/a & -a \le x' < 0 \\ h(-x'+a)/a & 0 \le x' < a \\ 0 & a \le x' < L \end{cases}$$

So,

$$B_{2n+1} = (-1)^{n+1} \frac{2}{L} \int_{-L/2}^{L/2} dx' \, u_0(x') \cos\left(\frac{(2n+1)\pi}{L} x' \right)$$

$$= (-1)^{n+1} \frac{4}{L} \cdot \frac{h}{a} \int_0^a dx' (a - x') \cos\left(\frac{(2n+1)\pi}{L} x' \right)$$

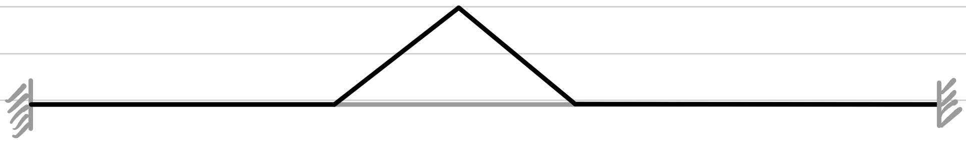
$$= (-1)^{n+1} \frac{4h}{La} \cdot \frac{L^2}{(2n+1)^2\pi^2} \left[1 - \cos\left((2n+1)\pi \frac{a}{L} \right) \right]$$

$$= \frac{4hL}{a} \frac{(-1)^{n+1\,*}}{(2n+1)^2\pi^2} \left[1 - \cos\left((2n+1)\pi \frac{a}{L} \right) \right]$$

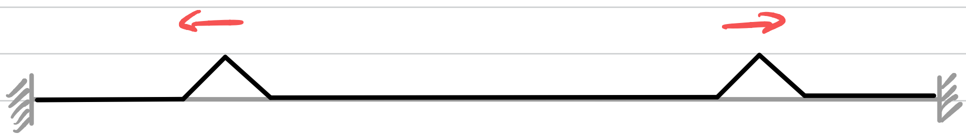
* $(-1)^n$ from wrong sign on $\cos\alpha = \sin(\alpha + \frac{\pi}{2})$

Let's look at the time evolution for the fundamental frequency $\omega_1 = \frac{\pi}{L} c \Rightarrow \tau = \frac{2\pi}{\omega_1}$

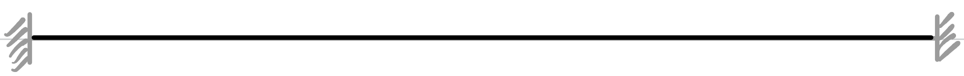
$t = 0$



$t = \frac{\tau}{8}$



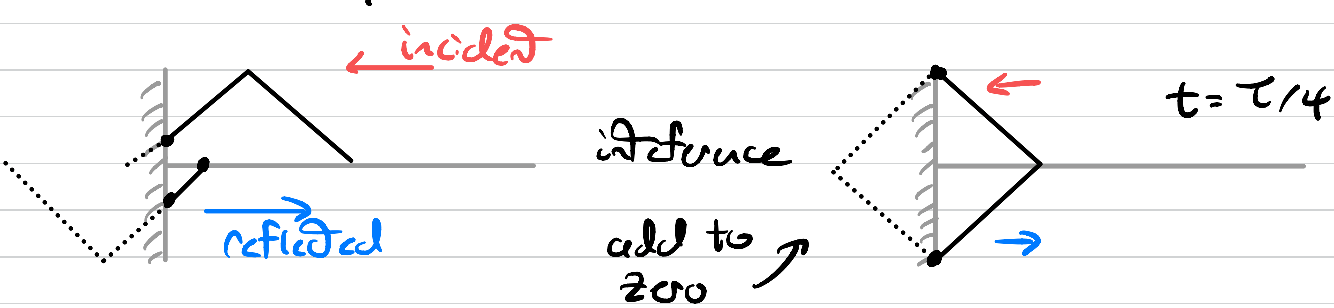
$t = \frac{\tau}{4}$



$t = \frac{3\tau}{8}$

$t = \frac{\tau}{2}$

At the boundary



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Wave Equation in 3D

We can generalize the 1D wave eqn, $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$, to 3D in the expected way.

Let $p(\vec{r},t) = p(x,y,z,t)$ denote some disturbance of a 3D system (e.g., pressure in sound wave through air), then the wave equation is

$$\frac{\partial^2 p}{\partial t^2} = c^2 \left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} \right)$$

c = speed of wave

For sound in air, $c^2 = \frac{BM}{\rho_0}$

$BM \rightarrow$ Bulk modulus (see later)

$\rho_0 \rightarrow$ equilibrium density

Define the vector operator

$$\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

So that

$$\boxed{\vec{\nabla}^2 = \vec{\nabla} \cdot \vec{\nabla} = \left(\frac{\partial}{\partial x} \right)^2 + \left(\frac{\partial}{\partial y} \right)^2 + \left(\frac{\partial}{\partial z} \right)^2}$$

Laplacian

Therefore, the 3D wave eqn. is

$$\boxed{\frac{\partial^2 p}{\partial t^2} = c^2 \vec{\nabla}^2 p}$$

Plane wave Solution

If the wave front is propagating in the $\hat{n}$ direction, then

$$p(\vec{r},t) = f(\hat{n}\cdot\vec{r} - ct)$$

Verify: $\vec{\nabla} p = \frac{\partial f}{\partial(\hat{n}\cdot\vec{r})}\hat{n} = -\frac{1}{c}\frac{\partial f}{\partial t}\hat{n}$

$\frac{\partial f(x-y)}{\partial x} = -\frac{\partial f(x-y)}{\partial y}$

$$\Rightarrow \vec{\nabla}^2 p = -\frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla} f)\,\hat{n}$$

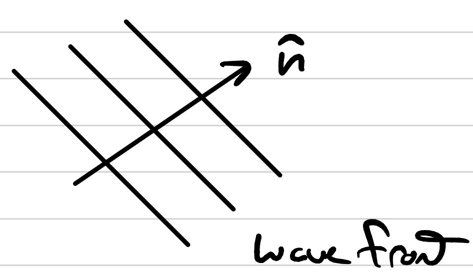
$$= -\frac{1}{c}\frac{\partial}{\partial t}\left(-\frac{1}{c}\frac{\partial f}{\partial t}\hat{n}\right)\cdot\hat{n}$$

using tricks as before

$$= \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}$$

If wave is in free space, no Boundary conditions, then

$$p(\vec{r},t) \propto \cos(k(\hat{n}\cdot\vec{r} - ct))$$



Spherical Wave Solutions

Another important example is spherical wave solutions, i.e., a disturbance traveling radially outward.

$$p = p(r,t)$$

Can show $\quad \vec{\nabla}^2 p = \frac{1}{r}\frac{\partial^2 (rp)}{\partial r^2}$

So, wave eqn. is

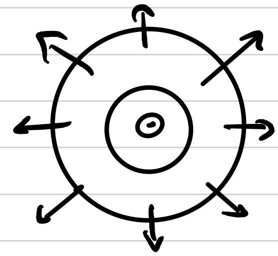
$$\frac{\partial^2 p}{\partial t^2} = c^2 \frac{1}{r}\frac{\partial^2}{\partial r^2}(rp)$$

Can see that $\quad \frac{\partial^2 (rp)}{\partial t^2} = c^2 \frac{\partial^2 (rp)}{\partial r^2}$

has a solution $\quad rp(r,t) = f(r-ct) + g(r+ct)$

If disturbance is radially outward, $g=0$

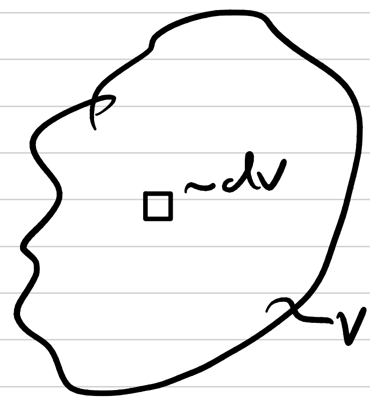
$$\Rightarrow \quad p(r,t) = \frac{1}{r} f(r-ct)$$



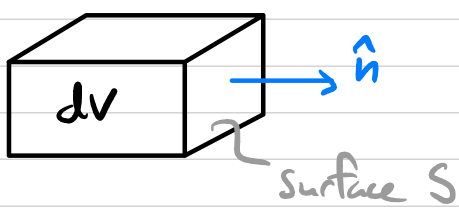
Volume & Surface Forces

We now aim to construct the Equations of motion of a continuous 3D system.

⇒ Apply NII to small mass element



Consider small element



Surface area is specified by normal vector $\hat{n}$ pointed "outward"

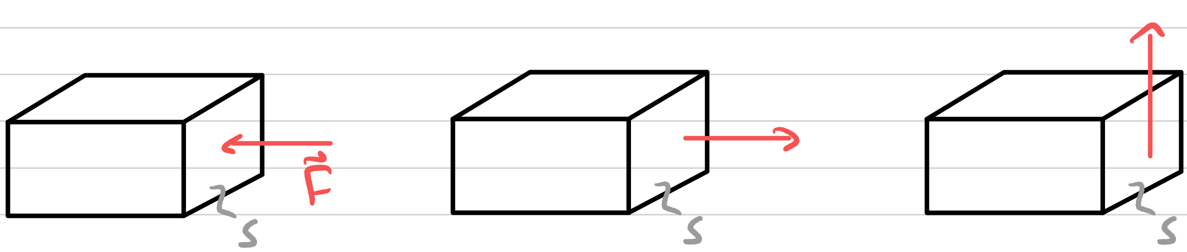
Two types of forces on dV

- Volume forces ($F \propto dV$)

  e.g., gravity $\vec{F} = \rho g \, dV$

  mass density

- surface forces ($F \propto dA$)



pressure | tension | shear

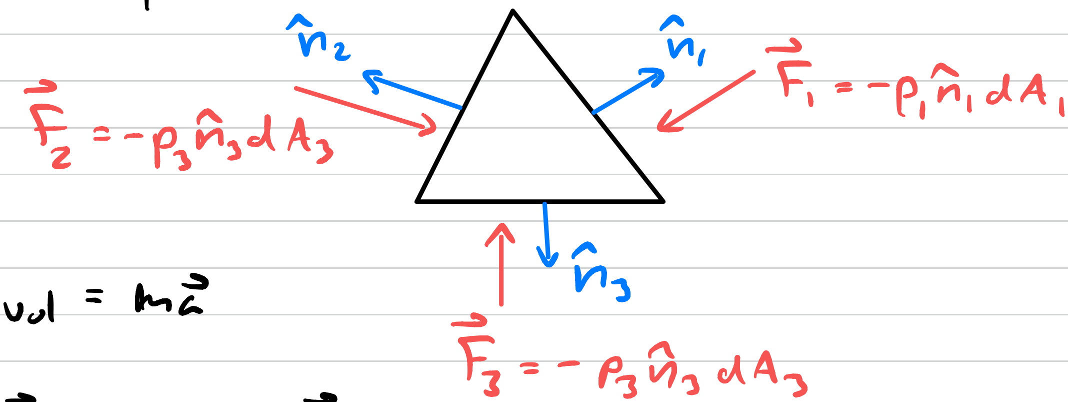
$\vec{F} = -p\hat{n}\, dA$

Ideal fluids have <u>no</u> shear modulus

(Real fluids have small shear modulus $\Rightarrow$ viscocity)

Isotropic Pressure of Fluids

Let S_1 & S_2 be two surfaces w/ normal vectors $\hat{n}_1$ & $\hat{n}_2$. Construct third surface S_3 w/ $\hat{n}_3$ to form isosceles prism



NII gives

$$\vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \vec{F}_{vol} = m\vec{a}$$

$$\Rightarrow \underbrace{\vec{F}_1 + \vec{F}_2 + \vec{F}_3}_{\text{surface forces}} = \underbrace{m\vec{a} - \vec{F}_{vol}}_{\text{volume forces}}$$

Shrink size by λ factor

$$\Rightarrow \lambda^2 (\vec{F}_1 + \vec{F}_2 + \vec{F}_3) = \lambda^3 (m\vec{a} - \vec{F}_{vol})$$

as $\lambda \to 0 \Rightarrow \vec{F}_1 + \vec{F}_2 + \vec{F}_3 = \lambda (m\vec{a} - \vec{F}_{vol}) = \vec{0}$

Since isosceles, $\hat{n}_3 \cdot \vec{F}_1 \parallel \hat{n}_3 \parallel \vec{F}_2 \Rightarrow F_1 = F_2 \Rightarrow p_1 = p_2$

$\Rightarrow$ isotropic pressure (Direct result of no shear modulus)

Stress & Strain

Stress is the ratio of surface force F to the applied area

examples:

$$\text{stress} = \frac{F}{A} = \text{pressure } P \text{ for static fluid}$$

$$\text{stress} = \frac{T}{A} \quad \text{for wire in tension}$$

$$\text{stress} = \frac{\text{shearing force}}{A} = \text{shear stress}$$

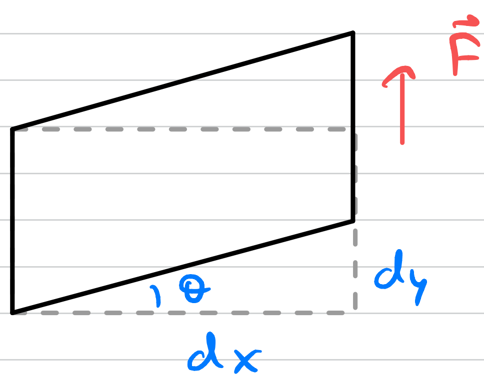
Strain is the deformation of object as the result of stress. (fractional deformation)

examples:

$$\text{strain} = \frac{dV}{V} \quad \text{for static fluid}$$

$$\text{strain} = \frac{dl}{l} \quad \text{for wire in tension}$$

$$\text{strain} = \frac{dy}{dx} \quad \text{for shear}$$



Stress & strain are related by properties of matter. For stresses in a medium which is not too large, expect strain to be linear to stress

$$\text{stress} \propto \text{strain}$$

The proportionality factor is called the Elastic modulus

For a stretched wire,

$$\frac{dF}{A} = YM \frac{dl}{l}$$

↳ Young's modulus

For hydrostatic pressure,

$$dp = -BM \frac{dV}{V}$$

↳ Bulk modulus

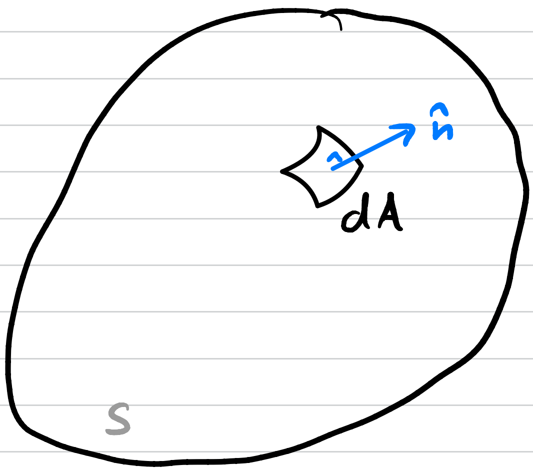
For shearing forces,

$$\frac{F}{A} = SM \frac{dy}{dx}$$

↳ Shear modulus

The Stress Tensor

Here we will make the concept of stress more rigorous. Consider a surface force on a small area dA of a closed surface S of some continuous medium.



Define oriented vector

$$d\vec{A} = \hat{n}\, dA$$

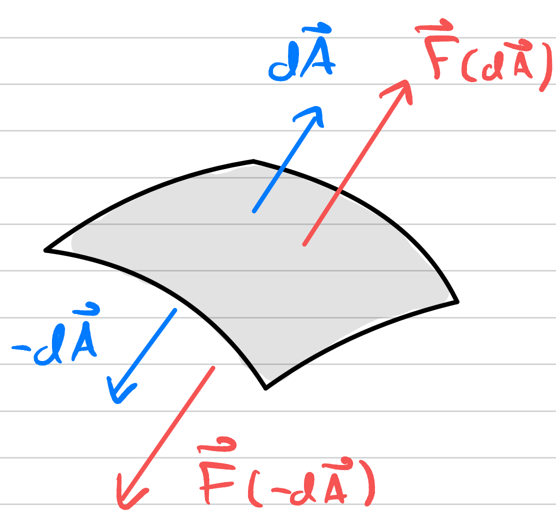
The surface force acting on the area $d\vec{A}$ is $\vec{F}(d\vec{A})$.

It is a linear function of $d\vec{A}$, i.e.,

$$\vec{F}(\lambda_1 d\vec{A}_1 + \lambda_2 d\vec{A}_2) = \lambda_1 \vec{F}(d\vec{A}_1) + \lambda_2 \vec{F}(d\vec{A}_2)$$

Proof: First note that as long as $d\vec{A}$ small,

$$\vec{F}(\lambda d\vec{A}) = \lambda \vec{F}(d\vec{A})$$

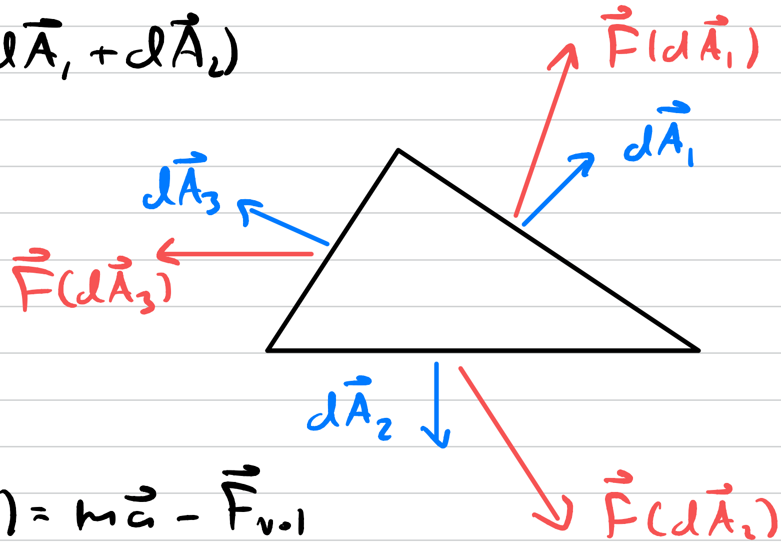


likewise, by N III

$$\vec{F}(-d\vec{A}) = -\vec{F}(d\vec{A})$$

Next, consider two elements $d\vec{A}_1$ & $d\vec{A}_2$

Find a third $d\vec{A}_3 = -(d\vec{A}_1 + d\vec{A}_2)$



So, NII gives

$$\vec{F}(d\vec{A}_1) + \vec{F}(d\vec{A}_2) + \vec{F}(d\vec{A}_3) = m\vec{a} - \vec{F}_{vol}$$

As before, if surface size $\rightarrow 0$, then

$$\vec{F}(d\vec{A}_1) + \vec{F}(d\vec{A}_2) + \vec{F}(d\vec{A}_3) = \vec{0}$$

$$\Rightarrow \quad \vec{F}(-d\vec{A}_3) = -\vec{F}(d\vec{A}_3) = \vec{F}(d\vec{A}_1) + \vec{F}(d\vec{A}_2)$$

But, $d\vec{A}_3 = -(d\vec{A}_1 + d\vec{A}_2)$

$$\Rightarrow \quad \vec{F}(d\vec{A}_1 + d\vec{A}_2) = \vec{F}(d\vec{A}_1) + \vec{F}(d\vec{A}_2)$$

Combine with $\vec{F}(\lambda d\vec{A}) = \lambda \vec{F}(d\vec{A})$, and yield

$$\vec{F}(\lambda_1 d\vec{A}_1 + \lambda_2 d\vec{A}_2) = \lambda_1 \vec{F}(d\vec{A}_1) + \lambda_2 \vec{F}(d\vec{A}_2) \quad \blacksquare$$

So, $\vec{F}(d\vec{A})$ is linear in $d\vec{A}$.

For a fluid, $F(d\vec{A}) = -p\,d\vec{A}$. But, the most general relation is

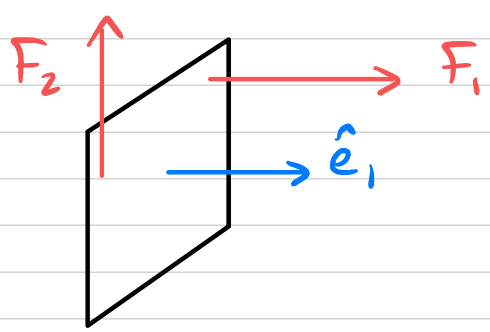
$$\boxed{F_j(d\vec{A}) = \sum_{k=1}^{3} \sigma_{jk}\, dA_k \qquad (j,k = 1,2,3 = x,y,z)}$$

This defines the 3x3 stress tensor Σ, with elements σ_{jk}. In matrix form,

$$\vec{F}(d\vec{A}) = \Sigma \cdot d\vec{A}$$

Notice – $\vec{F}$ does not need to be $\parallel$ or $\perp$ to surface

Consider the surface element $d\vec{A} = dA\,\hat{e}_1$



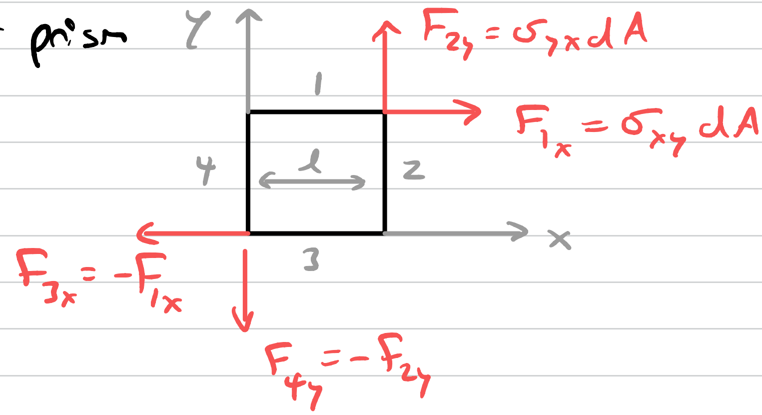
$$F_j = \sum_k \sigma_{jk}\, dA\, e_{1,k}$$

$$= \sigma_{j1}\, dA$$

so, $F_1 = \sigma_{11}\, dA$ normal force

$$\left.\begin{aligned} F_2 &= \sigma_{21}\, dA \\ F_3 &= \sigma_{31}\, dA \end{aligned}\right\} \text{shearing forces}$$

The stress tensor is symmetric. Consider the square element of prism



The torque on that element is

$$\Gamma_z = F_{2y}\ell - F_{1x}\ell$$

$$= (\sigma_{yx} - \sigma_{xy})\ell\, dA$$

$$= \frac{dL_z}{dt}$$

Now, rescale 3D prism by λ

$$\Rightarrow \Gamma_z \to \lambda^3 \Gamma_z$$

but, $L_z \propto I \propto m\ell^2 \propto \rho \ell^4 \to \lambda^4 L_z$

$\Rightarrow$ as $\lambda \to 0 \Rightarrow \Gamma_z \to 0 \Rightarrow \sigma_{xy} = \sigma_{yx}$

Similar arguments hold for other edges

$\Rightarrow \boxed{\sigma_{jk} = \sigma_{kj}} \Rightarrow \Sigma$ is symmetric!

So, Σ has 6 independent components

Simple example – hydrostatic fluid

In this case, $\vec{F}(d\vec{A}) = -p\, d\vec{A}$

$\hookrightarrow$ constant

$$\Rightarrow \quad \sigma_{jk} = -p\,\delta_{jk}$$

$$\Rightarrow \quad \Sigma = -p\,\mathbb{1}$$

$$= \begin{pmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{pmatrix}$$

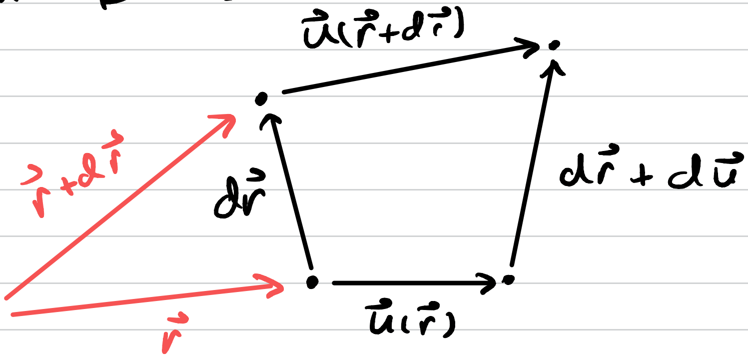
Strain Tensor

Consider a small volume originally at position $\vec{r}$, but its new position is shifted to $\vec{r} + \vec{u}(\vec{r})$.
A uniform $\vec{u}(\vec{r}) = \vec{u}_0$ is just an overall translation, but not a distortion. A generic shift is

$$du_i = \sum_j \frac{\partial u_i}{\partial r_j} dr_j$$

Define the derivative matrix $\mathbb{D}$ as

$$D_{ij} = \frac{\partial u_i}{\partial r_j}$$



such that

$$\mathbb{D} = \begin{pmatrix} \frac{\partial u_x}{\partial r_x} & \cdots & \frac{\partial u_x}{\partial r_z} \\ \vdots & \ddots & \vdots \\ \frac{\partial u_z}{\partial r_x} & \cdots & \frac{\partial u_z}{\partial r_z} \end{pmatrix}$$

However, a rotation will cause a non-vanishing D_{ij}, but it's not a distortion either!

For a rotation,

$$d\vec{u}(\vec{r}) = \vec{v}\,dt = \vec{\omega}\,dt \times d\vec{r} = d\vec{\theta} \times \vec{r}$$

So,

$$\mathbb{D} = \begin{pmatrix} 0 & \theta_3 & -\theta_2 \\ -\theta_3 & 0 & \theta_1 \\ \theta_2 & -\theta_1 & 0 \end{pmatrix}$$

Which is antisymmetric. We decompose D_{ij} into symmetric & antisymmetric parts

$$D_{ij} = \frac{1}{2}(D_{ij} + D_{ji}) + \frac{1}{2}(D_{ij} - D_{ji})$$

$$= \epsilon_{ij} + A_{ij}$$

A_{ij} represents rigid rotation

ϵ_{ij} are the elements of the strain tensor $\mathbb{E}$.
The strain tensor is symmetric ($\epsilon_{ij} = \epsilon_{ji}$) and represents deformations of a continuous media.

Qualitatively, the strain measures the fractional change in an objects size, $\epsilon \sim \frac{\Delta L}{L}$

Examples of Strain:

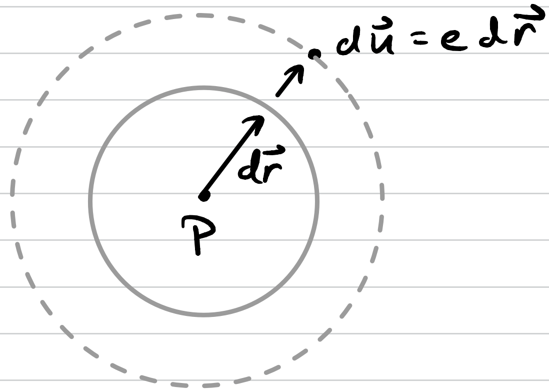
Dilatation (or Dilation)

$$\varepsilon_{ij} = e\,\delta_{ij} \Rightarrow \mathbb{E} = e\,\mathbb{1}$$

This is the trace-part of $\mathbb{E}$

$$e = \frac{1}{3}\mathrm{Tr}(\mathbb{E})$$

represents spherical strain, or dilatation, of medium (Expansion & contraction)



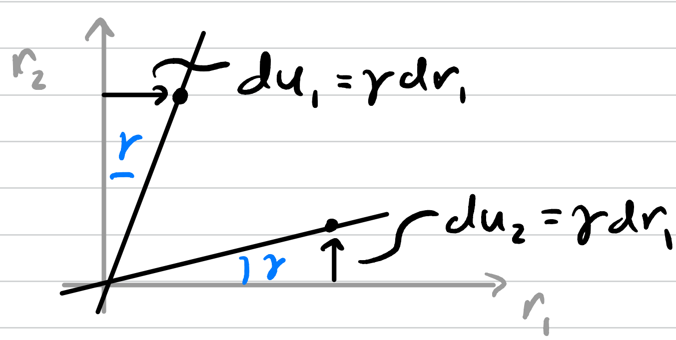
Each component stretches by a factor $(1+e)$

so, $V \to (1+e)^3 V \simeq (1+3e)V$ for $e \ll 1$

$$\Rightarrow \frac{dV}{V} = 3e \quad \text{for } e \ll 1.$$

Shearing Strain

$$\mathbb{E} = \begin{pmatrix} 0 & \gamma & 0 \\ \gamma & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \gamma \ll 1$$



also,

$$\mathbb{E} = \begin{pmatrix} 0 & 0 & \gamma \\ 0 & 0 & 0 \\ \gamma & 0 & 0 \end{pmatrix} \quad \mathbb{E} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \gamma \\ 0 & \gamma & 0 \end{pmatrix}$$

We can decompose a general strain tensor into a piece corresponding to pure stretching & a piece to pure shear.

If the diagonal elements of $\mathbb{E}$ are $\epsilon_{11}, \epsilon_{22}, \epsilon_{33}$,

$$\mathbb{E} = \begin{pmatrix} \epsilon_{11} & & \\ & \epsilon_{22} & \\ & & \epsilon_{33} \end{pmatrix}$$

If $\epsilon_{11} = \epsilon_{22} = \epsilon_{33} = e$, then $\mathbb{E} = e\mathbb{1}$

For a generic strain tensor, define an average e via

$$e = \frac{1}{3}\left(\epsilon_{11} + \epsilon_{22} + \epsilon_{33}\right)$$

Recall the definition of the trace of a matrix

$$\mathrm{tr}[\mathbb{M}] = \sum_{j=1}^{n} m_{jj} = m_{11} + m_{22} + \cdots + m_{nn}$$

$$\Rightarrow \boxed{e = \frac{1}{3}\mathrm{tr}[\mathbb{E}]}$$

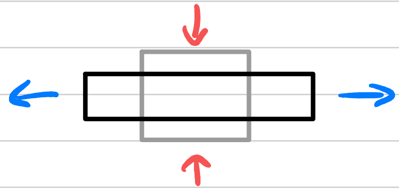
Can separate this trace from $\mathbb{E}$ as

$$\boxed{\mathbb{E} = e\mathbb{1} + \underbrace{\mathbb{E}'}_{\text{traceless part}}}$$

with $\underbrace{e = \frac{1}{3}\mathrm{tr}[\mathbb{E}]}_{\text{spherical part}}$

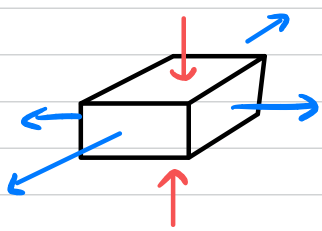
The traceless part is called strain deviator. It contains shearing & non-dilatation deformations

e.g., if $\mathbb{E}' = \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & -\epsilon & 0 \\ 0 & 0 & 0 \end{pmatrix}$



if

$$\mathbb{E}' = \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & -2\epsilon \end{pmatrix}$$



Hooke's Law

We now want to construct a relation between the stress Σ & strain $\mathbb{E}$. Such relations are called constitutive equations, which relate two physical quantities via some material property.

Here we assume that a system experiences small deformations and stresses, such that the constitutive equation is linear. The most general relation (Generalized Hooke's Law) is

$$\sigma_{jk} = \sum_{lm} C_{jklm} \, \epsilon_{lm}$$

$\hookrightarrow$ elasticity tensor

The elasticity tensor is a $3\times3\times3\times3$ object
$\Rightarrow$ 81 components!
However, since σ_{ij} & ϵ_{ij} are symmetric
$\Rightarrow$ only 21 independent components.

Further symmetries of materials will reduce this further.

Here, we focus on an isotropic medium, that is the system is rotationally invariant.
We will see this reduces Hooke's law to 2 independent components.

For isotropic media $\Rightarrow$ $\Sigma(\mathbb{E}_R) = \Sigma_R(\mathbb{E})$

Recall that the strain tensor is decomposed as

$$\mathbb{E} = e\mathbb{1} + \mathbb{E}'$$

Since $e\mathbb{1}$ is spherically symmetric, each term transforms separately under rotations

$$\Rightarrow \boxed{\Sigma = \alpha e \mathbb{1} + \beta \mathbb{E}'}$$

where α, β are coefficients. It is often convenient to express in terms of $\mathbb{E} = e\mathbb{1} + \mathbb{E}'$,

$$\Rightarrow \boxed{\Sigma = (\alpha - \beta) e \mathbb{1} + \beta \mathbb{E}}$$

Any solid respecting Hooke's law is an elastic solid.

Note that we can solve for $\mathbb{E} = \mathbb{E}(\Sigma)$. Take the trace,

$$\mathrm{tr}[\Sigma] = 3\alpha e + \cancelto{0}{\mathrm{tr}[\mathbb{E}']}$$
$$= 3\alpha e$$

$$\Rightarrow \quad e = \frac{1}{3\alpha}\Sigma$$

So,
$$\boxed{\begin{aligned} \mathbb{E} &= \frac{1}{\beta}\left[\Sigma - (\alpha-\beta) e \mathbb{1}\right] \\ &= \frac{1}{\beta}\Sigma - \left(\frac{\alpha-\beta}{3\alpha\beta}\right)\mathrm{tr}[\Sigma]\,\mathbb{1} \end{aligned}}$$

Q: What is the physical meaning of α & β?

A: Look at 2 cases involving Bulk & Shear moduli

Bulk Modulus

Consider a system with no shear & isotropic pressure,

$$\Sigma = -p\mathbb{1} = \begin{pmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{pmatrix}$$

$$\Rightarrow \mathrm{tr}[\Sigma] = -3p$$

So,

$$\mathbb{E} = \frac{1}{\beta}(-p\mathbb{1}) - \left(\frac{\alpha-\beta}{3\alpha\beta}\right)(-3p)\mathbb{1}$$

$$= -\frac{p}{\alpha}\mathbb{1}$$

$$\Rightarrow \quad e = -\frac{p}{\alpha}$$

Recall that $\frac{dV}{V} = 3e \Rightarrow \frac{dV}{V} = -\frac{3p}{\alpha}$

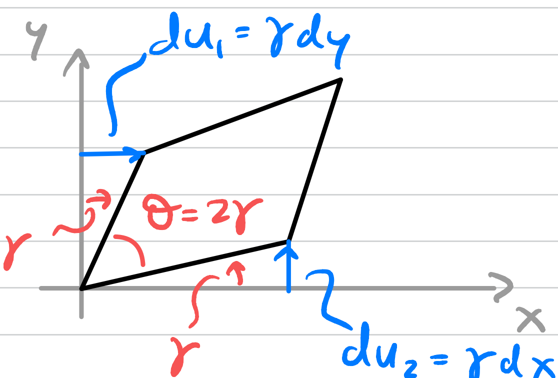
But, we defined the Bulk modulus as $p = -BM\frac{dV}{V}$

So, $\boxed{\alpha = 3BM}$

Shear Modulus

Recall the shear modulus is defined as

$$\frac{F}{A} = SM\frac{dy}{dx} = SM\,\theta$$



if $e=0 \Rightarrow \Sigma' = \beta\,\mathbb{E}$

For $\Sigma' = \begin{pmatrix} 0 & \sigma_{12} & 0 \\ \sigma_{12} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \beta\begin{pmatrix} 0 & \epsilon_{12} & 0 \\ \epsilon_{12} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$$\Rightarrow \frac{F}{A} = \sigma_{12} = \beta\epsilon_{12} = \beta\gamma = \beta\frac{\theta}{2} = SM\theta$$

$$\Rightarrow \boxed{\beta = 2SM}$$

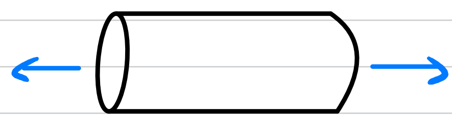
So,

$$\Sigma' = (\alpha - \beta)\frac{1}{3}\mathrm{tr}[\mathbb{E}]\,\mathbb{1} + \beta\,\mathbb{E}$$
$$= \left(BM - \frac{2}{3}SM\right)\mathrm{tr}[\mathbb{E}] + 2SM\,\mathbb{E}$$

Young's Modulus

The Young's modulus is defined via $\frac{dF}{A} = YM\frac{dl}{l}$

Here, $\Sigma' = \begin{pmatrix} \sigma_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$



$$\Rightarrow \mathbb{E} = \frac{1}{3\alpha\beta}\left[3\alpha\Sigma' + (\beta - \alpha)\mathrm{tr}[\Sigma']\,\mathbb{1}\right]$$

$$= \frac{\sigma_{11}}{3\alpha\beta}\begin{pmatrix} 2\alpha+\beta & 0 & 0 \\ 0 & \beta-\alpha & 0 \\ 0 & 0 & \beta-\alpha \end{pmatrix}$$

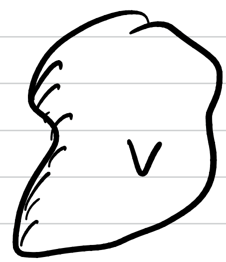
So, $\frac{dF}{A} = \sigma_{11}$, $\frac{dl}{l} = \epsilon_{11} = \frac{\sigma_{11}}{3\alpha\beta}(2\alpha+\beta)$

$$\Rightarrow YM = \frac{\sigma_{11}}{\epsilon_{11}} = \frac{3\alpha\beta}{2\alpha+\beta} = \frac{9BM\cdot SM}{3BM + SM}$$

Equation of Motion for Elastic solid

NII gives for volume

$$\int \rho\, dV \frac{\partial^2 \vec{u}}{\partial t^2} = \vec{F}_{vol} + \vec{F}_{sur}$$



w/ $\vec{F}_{vol} = \int \rho \vec{g}\, dV$

$\vec{F}_{sur} = \int \Sigma \cdot d\vec{A}$

Recall the divergence theorem : $\int \vec{c} \cdot d\vec{A} = \int \vec{\nabla} \cdot \vec{c}\, dV$

$\Rightarrow \quad \vec{F}_{sur} = \int \vec{\nabla} \cdot \Sigma\, dV \qquad \vec{\nabla} \cdot \vec{c} = \frac{\partial c_1}{\partial x} + \frac{\partial c_2}{\partial y} + \frac{\partial c_3}{\partial z}$

So, for volume element $\quad \rho \frac{\partial^2 \vec{u}}{\partial t^2} = \rho \vec{g} + \vec{\nabla} \cdot \Sigma$

For the j^{th} component, $\quad \rho \frac{\partial^2 u_j}{\partial t^2} = \rho g_j + \frac{\partial}{\partial r_k} \sigma_{kj}$

Now, Hooke's law

$$\Sigma = (\alpha - \beta) e \mathbb{1} + \beta \mathbb{E}$$

or,

$$\sigma_{jk} = (\alpha - \beta) e \delta_{jk} + \beta \epsilon_{jk}$$

$$= (\alpha - \beta) \sum_{l} \epsilon_{ll} \delta_{jk} + \frac{\beta}{2}\left(\frac{\partial u_j}{\partial r_k} + \frac{\partial u_k}{\partial r_j}\right)$$

$$= (\alpha - \beta) \frac{1}{3} (\vec{\nabla} \cdot \vec{u}) \delta_{jk} + \frac{\beta}{2}\left(\frac{\partial u_j}{\partial r_k} + \frac{\partial u_k}{\partial r_j}\right)$$

Therefore,

$$\sigma_{jk} = \frac{1}{3}(\alpha-\beta)\delta_{jk}(\vec{\nabla}\cdot\vec{u}) + \frac{1}{2}\beta\left(\frac{\partial u_j}{\partial r_k} + \frac{\partial u_k}{\partial r_j}\right)$$

So, $(\vec{\nabla}\cdot\Sigma)_j = \frac{\partial}{\partial r_i}\sigma_{ij}$

$$= \frac{1}{3}(\alpha-\beta)\delta_{ij}\frac{\partial}{\partial r_i}(\vec{\nabla}\cdot\vec{u}) + \frac{1}{2}\beta\left[\underbrace{\frac{\partial}{\partial r_i}\frac{\partial}{\partial r_j}u_i}_{=\frac{\partial}{\partial r_j}(\vec{\nabla}\cdot\vec{u})} + \frac{\partial^2}{\partial r_i^2}u_j\right]$$

So,

$$\vec{\nabla}\cdot\Sigma = \left(\frac{\alpha}{3}+\frac{\beta}{6}\right)\vec{\nabla}(\vec{\nabla}\cdot\vec{u}) + \frac{\beta}{2}\vec{\nabla}^2\vec{u}$$

$$= \left(B M + \frac{SM}{3}\right)\vec{\nabla}(\vec{\nabla}\cdot\vec{u}) + SM\,\vec{\nabla}^2\vec{u}$$

$$\Rightarrow \boxed{\rho\frac{\partial^2\vec{u}}{\partial t^2} = \rho\vec{g} + \left(BM + \frac{SM}{3}\right)\vec{\nabla}(\vec{\nabla}\cdot\vec{u}) + SM\,\vec{\nabla}^2\vec{u}}$$

This is the Navier or Navier-Cauchy Equation

Wave Equation in Elastic Solids

Let $\vec{g} = \vec{0}$. Consider two cases...

Longitudinal Disturbance

Let $\vec{u} = (u_x(x,t), 0, 0)$

So,
$$\vec{\nabla}\cdot\vec{u} = \frac{\partial u_x}{\partial x} \Rightarrow \vec{\nabla}(\vec{\nabla}\cdot\vec{u}) = \frac{\partial^2}{\partial x^2} u_x \,\hat{e}_x$$
$$\vec{\nabla}^2\vec{u} = \frac{\partial^2}{\partial x^2} u_x \,\hat{e}_x$$

$$\Rightarrow \rho \frac{\partial^2 u_x}{\partial t^2} = \left(BM + \frac{4}{3}SM\right)\frac{\partial^2 u_x}{\partial x^2}$$

So, speed of longitudinal wave $c_L = \sqrt{\frac{BM + \frac{4}{3}SM}{\rho}}$

Transverse Disturbance

Let $\vec{u} = (0, u_y(x,t), 0)$

So, $\vec{\nabla}\cdot\vec{u} = 0$

$$\Rightarrow \rho \frac{\partial^2 u_y}{\partial t^2} = SM\,\vec{\nabla}^2 u_y$$

So, speed of transverse wave $c_T = \sqrt{\frac{SM}{\rho}}$

For fluids, $SM \simeq 0 \Rightarrow$ only longitudinal waves!