Physics 303

Classical Mechanics II

Coupled Oscillators

A.W. Jackura — William & Mary

Coupled Oscillators

We have seen countless times that small disturbances on some physical system lead to oscillitory behavior. This is because if a physical system is in some equilibrium state, then a small disturbance is of the form

$$U(r_0+\epsilon) \simeq U(r_0) + \cancelto{0}{\epsilon \frac{\partial U}{\partial r}\Big|_{r_0}} + \frac{1}{2}\epsilon^2 \frac{\partial^2 U}{\partial r^2}\Big|_{r_0} + \cdots$$

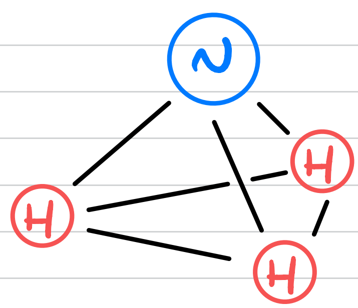
disturbance

equilibrium

SHO, $U \sim \text{const} + \frac{1}{2}k\epsilon^2$

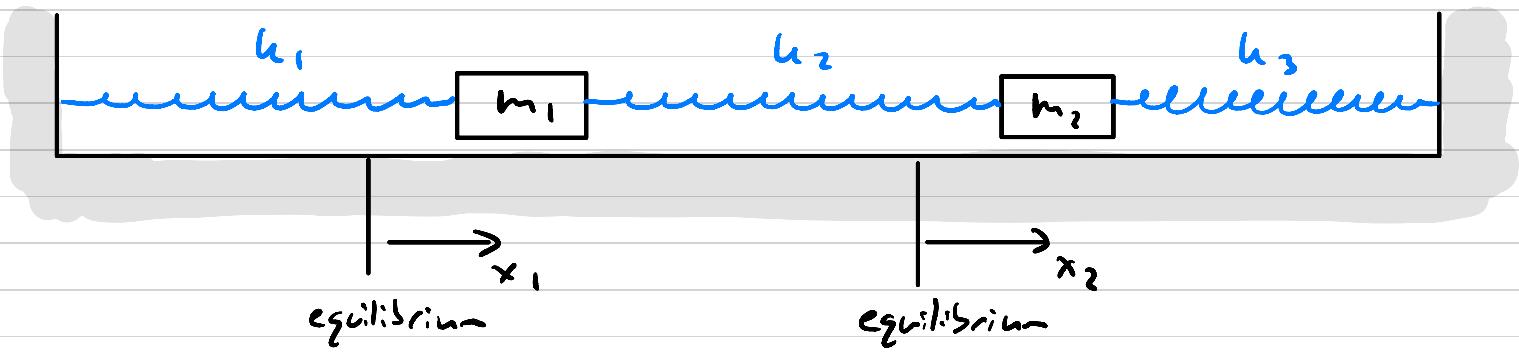
Most (every) mechanical systems are coupled, meaning multiple mechanical processes are mutually interacting. Consider the ammonia molecule, NH_3.

In describing its motion, we first would consider it a rigid body with 6 degrees of freedom.

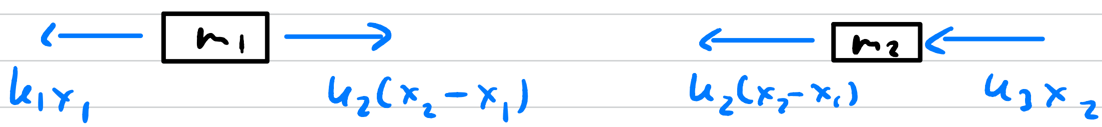


But, atoms vibrate, & thus there are more degrees of freedom. We will now investigate the vibrational motion of coupled systems.

Consider the example of two oscillators coupled by three springs. Assume masses m_1 & m_2 have displacements x_1 & x_2, respectively.



NII gives, $\vec{F}_{spring} = -k(\vec{x} - \vec{x}_0)$



$$m_1 \ddot{x}_1 = -k_1 x_1 + k_2(x_2 - x_1)$$
$$= -(k_1 + k_2) x_1 + k_2 x_2$$
$$m_2 \ddot{x}_2 = -k_2(x_2 - x_1) - k_3 x_2$$
$$= k_2 x_1 - (k_2 + k_3) x_2$$

Can write as

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = - \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\Rightarrow \quad \mathbb{M} \cdot \ddot{\mathbb{X}} = - \mathbb{K} \cdot \mathbb{X}$$

The coupled system of equations looks like a "simple spring system" with a mass matrix $\mathbb{M}$ & spring constant matrix $\mathbb{K}$. Notice that these matrices are symmetric.

In general, the solution is some linear combo of sines & cosines. As is usual, let's define a complex number

$$z_1 = a_1 e^{i\omega t} \qquad \Rightarrow \qquad \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{i\omega t}$$
$$z_2 = a_2 e^{i\omega t}$$

$$a_1, a_2 \in \mathbb{C} \qquad \Rightarrow \qquad \mathbb{Z}(t) = \mathbb{A} e^{i\omega t}$$

The solution is then $\mathbb{X}(t) = \mathrm{Re}\, \mathbb{Z}(t)$

or,

$$\boxed{x_j(t) = \mathrm{Re}\, z_j(t)}$$

So, $\mathbb{M}\ddot{\mathbb{X}} = -\omega^2 \mathbb{M} \cdot \mathbb{A} e^{i\omega t} = -\mathbb{K} \cdot \mathbb{A} e^{i\omega t}$

$$\Rightarrow \quad \boxed{(\mathbb{K} - \omega^2 \mathbb{M}) \cdot \mathbb{A} = 0}$$

generalized eigenvalue problem.

If $A \neq 0$, then the solution is given by

$$\boxed{\det(\mathbb{K} - \omega^2 \mathbb{M}) = 0}$$

The eigenvalues for this problem, or eigenfrequencies, are called the normal frequencies for the system. The corresponding solutions are called normal modes

Consider our problem, for $k_1 = k_2 = k_3 = k$ & $m_1 = m_2 = m$. Then,

$$\mathbb{M} = m\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \& \quad \mathbb{K} = k\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

So,

$$\mathbb{K} - \omega^2 \mathbb{M} = \begin{pmatrix} 2k - m\omega^2 & -k \\ -k & 2k - m\omega^2 \end{pmatrix}$$

$$\Rightarrow \det(\mathbb{K} - \omega^2 \mathbb{M}) = \det\begin{pmatrix} 2k - m\omega^2 & -k \\ -k & 2k - m\omega^2 \end{pmatrix}$$

$$= (2k - m\omega^2)^2 - k^2$$

$$= (k - m\omega^2)(3k - m\omega^2)$$

$$= 0$$

So, the two normal mode frequencies are

$$\omega_1 = \sqrt{\frac{k}{m}} \quad \& \quad \omega_2 = \sqrt{\frac{3k}{m}}$$

Let's look at the corresponding normal modes (eigenvectors)

<u>1st Mode</u> $\omega_1 = \sqrt{\frac{k}{m}}$, so $\mathbb{K} - \omega^2 \mathbb{M} = k \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

So,

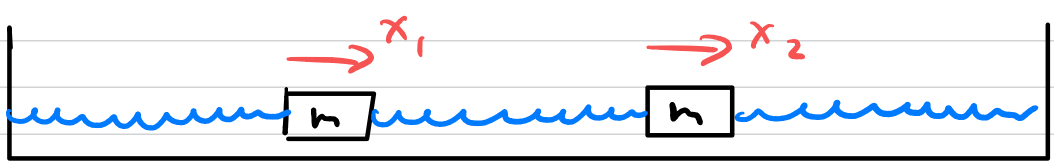
$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0 \Rightarrow \begin{matrix} a_1 - a_2 = 0 \\ -a_1 + a_2 = 0 \end{matrix}$$

$\Rightarrow a_1 = a_2$. Let $a_1 = a_2 = A e^{-i\delta}$, $A, \delta \in \mathbb{R}$

$$\Rightarrow \mathbb{Z}(t) = A \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{i(\omega_1 t - \delta)}$$

Since $\mathbb{X} = \mathrm{Re}\,\mathbb{Z}$, we find

$$\begin{cases} x_1(t) = A\cos(\omega_1 t - \delta) \\ x_2(t) = A\cos(\omega_1 t - \delta) \end{cases}$$



2nd Mode

$\omega_2 = \sqrt{\frac{3k}{m}}$, so $\mathbb{K} - \omega^2 m = k\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

So,

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0 \quad \Rightarrow \quad \begin{matrix} a_1 + a_2 = 0 \\ a_1 + a_2 = 0 \end{matrix}$$

$$\Rightarrow a_1 = -a_2 \quad . \quad \text{Let} \quad a_1 = -a_2 = Ae^{-i\delta}$$

$$\Rightarrow \mathbb{Z}(t) = A\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{i(\omega_2 t - \delta)}$$

So, $\mathbb{X} = \mathrm{Re}\,\mathbb{Z} \Rightarrow \begin{cases} x_1 = A\cos(\omega_2 t - \delta) \\ x_2 = -A\cos(\omega_2 t - \delta) \end{cases}$

A general solution will be a linear combo of both normal modes,

$$\mathbb{X}_1(t) = A_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}\cos(\omega_1 t - \delta_1)$$

$$\mathbb{X}_2(t) = A_2\begin{pmatrix} 1 \\ -1 \end{pmatrix}\cos(\omega_2 t - \delta_2)$$

$$\Rightarrow \boxed{\mathbb{X}(t) = A_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}\cos(\omega_1 t - \delta_1) + A_2\begin{pmatrix} 1 \\ -1 \end{pmatrix}\cos(\omega_2 t - \delta_2)}$$

Normal Coordinates

We see that x_1 & x_2 are complicated functions of time. Even the individual normal modes are not "simple" as both carts move. It is possible to introduce an alternative set of coordinates, called normal coordinates, to separate the two normal modes. These variables are less physical, but are advantageous to study each normal mode in isolation.

Let $\xi_1 = \frac{1}{2}(x_1 + x_2)$ & $\xi_2 = \frac{1}{2}(x_1 - x_2)$

The variables ξ_1, ξ_2 label the configuration of the system. With these coordinates,

$$\Rightarrow \begin{pmatrix} \ddot{\xi}_1 \\ \ddot{\xi}_2 \end{pmatrix} = -\frac{k}{m}\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$$

Decoupled!

So, solution is

$$\begin{cases} \xi_1 = A\cos(\omega_1 t - \delta) \\ \xi_2 = 0 \end{cases}$$

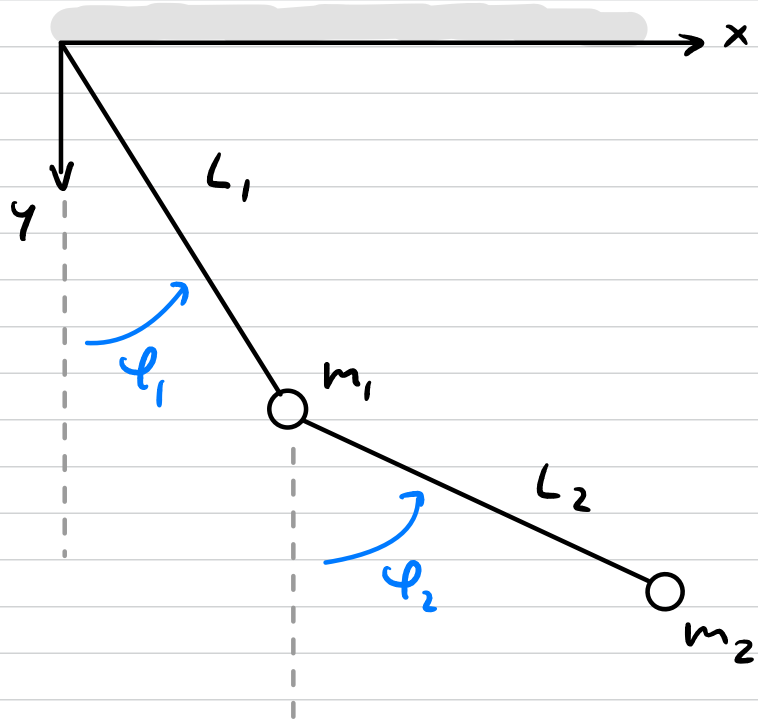
1st mode

$$\begin{cases} \xi_1 = 0 \\ \xi_2 = A\cos(\omega_2 t - \delta) \end{cases}$$

2nd mode

Double Pendulum

Consider the double pendulum



Let's construct the Lagrangian.

$$\mathcal{L} = T + U$$

So, $T = T_1 + T_2$

$$= \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$$

In terms of φ_1 & φ_2,

$$\begin{cases} x_1 = L_1 \sin\varphi_1 \\ y_1 = L_1 \cos\varphi_1 \end{cases} \qquad \begin{cases} x_2 = L_1 \sin\varphi_1 + L_2 \sin\varphi_2 \\ y_2 = L_1 \cos\varphi_1 + L_2 \cos\varphi_2 \end{cases}$$

$$\Rightarrow \dot{x}_1^2 + \dot{y}_1^2 = (L_1 \dot{\varphi}_1 \cos\varphi_1)^2 + (-L_1 \dot{\varphi}_1 \sin\varphi_1)^2$$
$$= L_1^2 \dot{\varphi}_1^2$$

$$\dot{x}_2^2 + \dot{y}_2^2 = (L_1 \dot{\varphi}_1 \cos\varphi_1 + L_2 \dot{\varphi}_2 \cos\varphi_2)^2 + (-L_1 \dot{\varphi} \sin\varphi_1 - L_2 \dot{\varphi}_2 \sin\varphi_2)^2$$
$$= L_1^2 \dot{\varphi}_1^2 + L_2^2 \dot{\varphi}_2^2 + 2 L_1 L_2 \dot{\varphi}_1 \dot{\varphi}_2 [\cos\varphi_1 \cos\varphi_2 + \sin\varphi_1 \sin\varphi_2]$$
$$= L_1^2 \dot{\varphi}_1^2 + L_2^2 \dot{\varphi}_2^2 + 2 L_1 L_2 \dot{\varphi}_1 \dot{\varphi}_2 \cos(\varphi_1 - \varphi_2)$$

The potential energy is

$$U = U_1 + U_2$$
$$= -m_1 g y_1 - m_2 g y_2$$
$$= -m_1 g L_1 \cos\varphi_1 - m_2 g (L_1 \cos\varphi_1 + L_2 \cos\varphi_2)$$
$$= -(m_1 + m_2) g L_1 \cos\varphi_1 - m_2 g L_2 \cos\varphi_2$$

So, $\mathcal{L} = T - U$

$$= \frac{1}{2}(m_1+m_2) L_1^2 \dot{\varphi}_1^2 + \frac{1}{2} m_2 L_2^2 \dot{\varphi}_2^2$$
$$+ m_2 L_1 L_2 \dot{\varphi}_1 \dot{\varphi}_2 \cos(\varphi_1 - \varphi_2)$$
$$+ (m_1+m_2) g L_1 \cos\varphi_1 + m_2 g L_2 \cos\varphi_2$$

For small oscillations, $\cos\varphi \simeq 1 - \frac{1}{2}\varphi^2 + \mathcal{O}(\varphi^4)$

$$\Rightarrow \mathcal{L}\Big|_{SMA} = \frac{1}{2}(m_1+m_2) L_1^2 \dot{\varphi}_1^2 + \frac{1}{2} m_2 L_2^2 \dot{\varphi}_2^2 + m_2 L_1 L_2 \dot{\varphi}_1 \dot{\varphi}_2$$
$$- \frac{1}{2}(m_1+m_2) g L_1 \varphi_1^2 - \frac{1}{2} m_2 g L_2 \varphi_2^2$$

EL EOM

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\varphi}_1} = \frac{\partial \mathcal{L}}{\partial \varphi_1} \quad \& \quad \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\varphi}_2} = \frac{\partial \mathcal{L}}{\partial \varphi_2}$$

$$\Rightarrow \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\varphi}_1} = (m_1+m_2)L_1^2\ddot{\varphi}_1 + m_2 L_1 L_2 \ddot{\varphi}_2$$

$$\frac{\partial \mathcal{L}}{\partial \varphi_1} = -(m_1+m_2) g L_1 \varphi_1$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\varphi}_2} = m_2 L_2^2 \ddot{\varphi}_2 + m_2 L_1 L_2 \ddot{\varphi}_1$$

$$\frac{\partial \mathcal{L}}{\partial \varphi_2} = -m_2 g L_2 \varphi_2$$

$$\Rightarrow \begin{cases} (m_1+m_2)L_1^2\ddot{\varphi}_1 + m_2 L_1 L_2 \ddot{\varphi}_2 = -(m_1+m_2) g L_1 \varphi_1 \\ m_2 L_1 L_2 \ddot{\varphi}_1 + m_2 L_2^2 \ddot{\varphi}_2 = -m_2 g L_2 \varphi_2 \end{cases}$$

Define $\mathbb{M} = \begin{pmatrix} (m_1+m_2)L_1^2 & m_2 L_1 L_2 \\ m_2 L_1 L_2 & m_2 L_2^2 \end{pmatrix}$

$$\mathbb{K} = \begin{pmatrix} (m_1+m_2) g L_1 & 0 \\ 0 & m_2 g L_2 \end{pmatrix}$$

$$\Rightarrow \mathbb{M}\ddot{\Phi} = -\mathbb{K}\Phi \quad , \quad \Phi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$$

Again, solution is $\Phi = \mathrm{Re}\,\mathbb{Z}$ with $\mathbb{Z} = \mathbb{A} e^{i\omega t}$

with

$$\det(\mathbb{K} - \omega^2 \mathbb{M}) = 0$$

Let's consider case where $l_1 = l_2 = l$, $m_1 = m_2 = m$

$$\Rightarrow \quad \mathbb{M} = mL^2 \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\mathbb{K} = mL^2 \begin{pmatrix} 2\omega_0^2 & 0 \\ 0 & \omega_0^2 \end{pmatrix}$$

w/ $\omega_0^2 = \frac{g}{L}$ frequency of single pendulum

So,

$$\mathbb{K} - \omega^2 \mathbb{M} = mL^2 \begin{pmatrix} 2(\omega_0^2 - \omega^2) & -\omega^2 \\ -\omega^2 & (\omega_0^2 - \omega^2) \end{pmatrix}$$

&

$$\det(\mathbb{K} - \omega^2 \mathbb{M}) = 0$$

$$\Rightarrow \quad 2(\omega_0^2 - \omega^2)^2 - \omega^4 = \omega^4 - 4\omega_0^2\omega^2 + 2\omega_0^4 = 0$$

Solutions are

$$\omega^2 = (2 \pm \sqrt{2})\,\omega_0^2$$

So, normal frequencies are

$$\omega_1^2 = (2 - \sqrt{2})\,\omega_0^2 \quad \text{and} \quad \omega_2^2 = (2 + \sqrt{2})\,\omega_0^2$$

$$\Rightarrow \quad \omega_1 \simeq 0.77\,\omega_0 \quad \& \quad \omega_2 \simeq 1.85\,\omega_0$$

Let's find normal modes

1st Mode $\omega_1^2 = (2-\sqrt{2})\omega_0^2$

$$\Rightarrow \mathbb{K} - \omega_1^2 \mathbb{M} = mL^2\omega_0^2(\sqrt{2}-1)\begin{pmatrix} 2 & -\sqrt{2} \\ -\sqrt{2} & 1 \end{pmatrix}$$

Therefore, $(\mathbb{K}-\omega_1^2\mathbb{M})\mathbb{A} = 0 \Rightarrow$

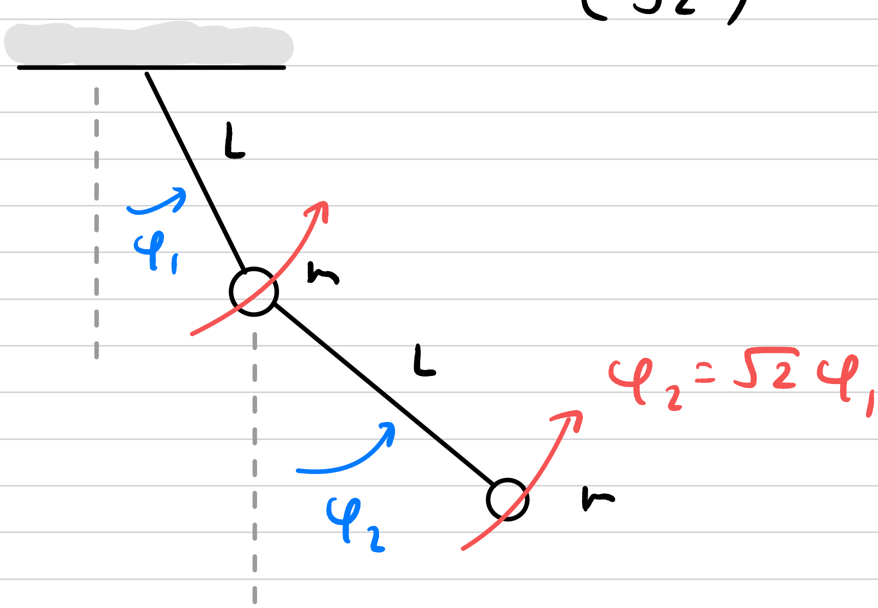
$$2a_1 - \sqrt{2}a_2 = 0$$
$$-\sqrt{2}a_1 + a_2 = 0$$

$$\Rightarrow a_2 = \sqrt{2}a_1.$$

Let $a_1 = A_1 e^{-i\delta_1}$

$$\Rightarrow \Phi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \mathrm{Re}\left(\mathbb{A}e^{i\omega_1 t}\right)$$

$$= A_1 \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} \cos(\omega_1 t - \delta_1)$$



2nd Mode

$$\omega_2 = (2+\sqrt{2})\,\omega_0^2$$

$$\Rightarrow \mathbb{K} - \omega^2 \mathbb{M} = -mL^2\omega_0^2(\sqrt{2}+1)\begin{pmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix}$$

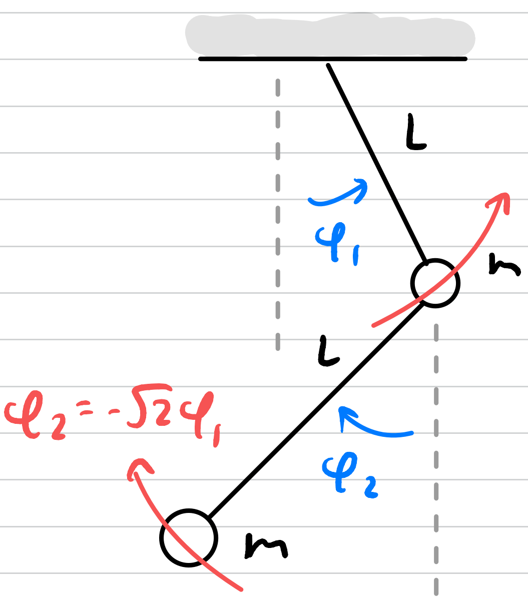
Therefore,

$$\begin{aligned} 2a_1 + \sqrt{2}\,a_2 &= 0 \\ \sqrt{2}\,a_1 + a_2 &= 0 \end{aligned} \quad \Rightarrow \quad a_2 = -\sqrt{2}\,a_1$$

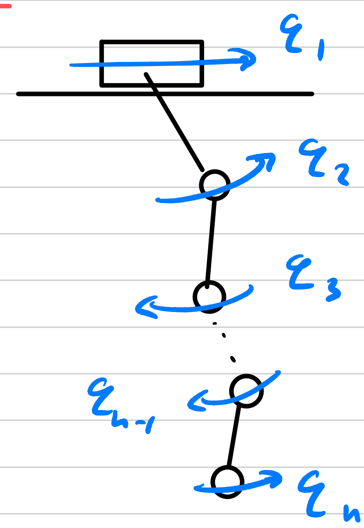
Let $a_1 = A_2 e^{-i\delta_2}$

$$\Rightarrow \phi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \mathrm{Re}\left(A\,e^{-i\omega_2 t}\right)$$

$$= A_2 \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} \cos(\omega_2 t - \delta_2)$$



General Case for Small Oscillations



Let some mechanical system consist of n degrees of freedom specified by generalized coordinates $q_1, \ldots, q_n$; $\vec{q} = (q_1, \ldots, q_n)$

The KE is generally

$$T = \sum_{\alpha} \frac{1}{2} m_\alpha \dot{\vec{r}}_\alpha^{\,2}$$

where $\vec{r}_\alpha = \vec{r}_\alpha(q_1, \ldots, q_n)$

Changing coordinates, T will have the form

$$T = \frac{1}{2} \sum_{j,k} A_{jk}(\vec{q})\, \dot{q}_j \dot{q}_k$$

For small oscillations, we choose q's such that $\vec{q} = \vec{0}$ is equilibrium point.

So, for oscillations about $\vec{q} = \vec{0}$,

$$T \simeq \frac{1}{2} \sum_{j,k} M_{jk}\, \dot{q}_j \dot{q}_k$$

where $M_{jk} \equiv A_{jk}(\vec{0})$.

Now, the potential energy is $U(\vec{q})$.

Taylor expanding,

$$U(\vec{q}) = U(\vec{0}) + \sum_j \frac{\partial U}{\partial q_j} q_j + \frac{1}{2} \sum_{j,k} \frac{\partial^2 U}{\partial q_j \partial q_k} q_j q_k + \cdots$$

Small oscillations about equilibrium, $\dfrac{\partial U}{\partial q_j} = 0$

$$\Rightarrow \quad U(\vec{q}) \simeq \underbrace{U(0)}_{\text{constant}} + \frac{1}{2} \sum_{j,k} K_{jk} q_j q_k$$

where $K_{jk} \equiv \left. \dfrac{\partial^2 U}{\partial q_j \partial q_k} \right|_{\vec{q}=\vec{0}}$

So,

$$\boxed{\begin{aligned} \mathcal{L} &= T - U \\ &= \frac{1}{2} \sum_{j,k} M_{jk} q_j q_k - \frac{1}{2} \sum_{j,k} K_{jk} q_j q_k \end{aligned}}$$

The EL EOM are

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_j} = \frac{\partial \mathcal{L}}{\partial q_j} \quad \text{for } j = 1, \ldots, n$$

So,

$$\frac{\partial L}{\partial \dot{q}_j} = \frac{\partial}{\partial \dot{q}_j}\left(\frac{1}{2}\sum_{l,k} M_{lk}\,\dot{q}_l\,\dot{q}_k\right)$$

$$= \frac{1}{2}\sum_{l,k} M_{lk}\,\frac{\partial}{\partial \dot{q}_j}\left(\dot{q}_l\,\dot{q}_k\right)$$

$$= \frac{1}{2}\sum_{l,k} M_{lk}\left[\frac{\partial \dot{q}_l}{\partial \dot{q}_j}\cdot \dot{q}_k + \dot{q}_l \cdot \frac{\partial \dot{q}_k}{\partial \dot{q}_j}\right]$$

Note that $\dfrac{\partial x_j}{\partial x_k} = \delta_{jk}$

$$\Rightarrow \frac{\partial L}{\partial \dot{q}_j} = \frac{1}{2}\sum_{l,k} M_{lk}\left[\delta_{lj}\,\dot{q}_k + \dot{q}_l\,\delta_{jk}\right]$$

$$= \frac{1}{2}\sum_{k} M_{jk}\,\dot{q}_k + \frac{1}{2}\sum_{l} M_{lj}\,\dot{q}_j$$

l is dummy index
replace by k

Recall, $M_{jk} = M_{kj}$

$$\Rightarrow \frac{\partial L}{\partial \dot{q}_j} = \sum_k M_{jk}\,\dot{q}_k$$

So, $$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = \sum_k M_{jk}\,\ddot{q}_k$$

Likewise, $$\frac{\partial L}{\partial \dot{q}_j} = -\sum_k K_{jk}\,q_k$$

So, EOM are

$$\boxed{\sum_k M_{jk}\ddot{q}_k = -\sum_k K_{jk} q_k \qquad , \; j = 1, \dots, n}$$

In matrix form

$$\boxed{\mathbb{M}\ddot{\mathbb{Q}} = -\mathbb{K}\mathbb{Q}} \quad , \quad \mathbb{Q} = \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix}$$

Solution is $\mathbb{Q}(t) = \mathrm{Re}\,\mathbb{Z}(t)$, $\mathbb{Z}(t) = \mathbb{A}e^{i\omega t}$, $\mathbb{A} \in \mathbb{C}^n$

with

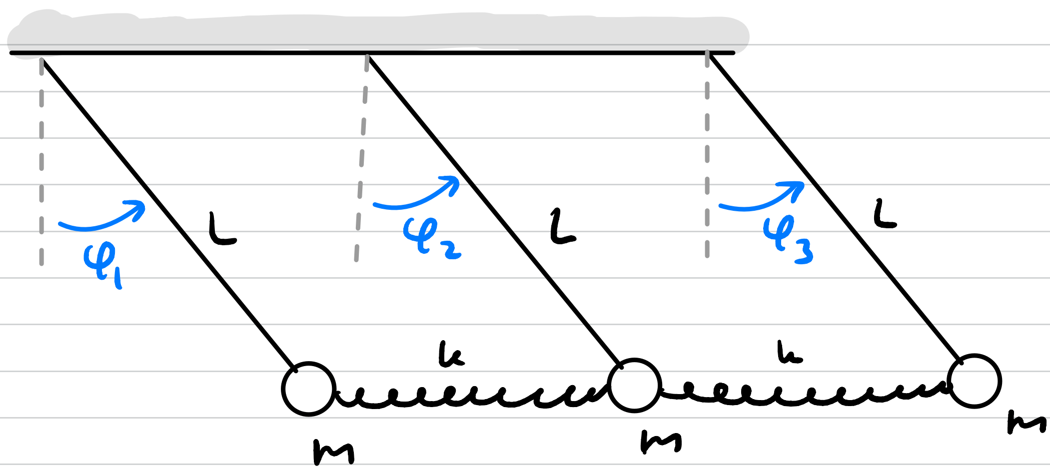
$$(\mathbb{K} - \omega^2 \mathbb{M})\mathbb{A} = 0$$

with characteristic eqn

$$\boxed{\det(\mathbb{K} - \omega^2 \mathbb{M}) = 0}$$

Three Coupled Pendulums

As a final example, consider three identical pendulums coupled with two identical springs



The KE is simply $T = \frac{1}{2} m L^2 (\dot{\varphi}_1^2 + \dot{\varphi}_2^2 + \dot{\varphi}_3^2)$

The PE has two parts, $U = U_{grav} + U_{spring}$

$$U_{grav} = \frac{1}{2} mgL(\varphi_1^2 + \varphi_2^2 + \varphi_3^2)$$

& $$U_{spring} = \frac{1}{2} k L^2 \left[(\varphi_2 - \varphi_1)^2 + (\varphi_3 - \varphi_2)^2 \right]$$

$$= \frac{1}{2} k L^2 (\varphi_1^2 + 2\varphi_2^2 + \varphi_3^2 - 2\varphi_1\varphi_2 - 2\varphi_2\varphi_3)$$

Define $\omega_0^2 = \frac{g}{L}$ as natural frequency of single pendulum

$\omega_s^2 = \frac{k}{m}$ as natural frequency of single spring

So,
$$\mathcal{L} = T - U$$
$$= \frac{1}{2} mL^2(\dot\varphi_1^2 + \dot\varphi_2^2 + \dot\varphi_3^2)$$
$$- \frac{1}{2} mL^2\left[\omega_0^2(\varphi_1^2 + \varphi_2^2 + \varphi_3^2)\right]$$
$$- \frac{1}{2} mL^2\left[\omega_s^2(\varphi_1^2 + 2\varphi_2^2 + \varphi_3^2 - 2\varphi_1\varphi_2 - 2\varphi_2\varphi_3)\right]$$

mL^2 is a constant factor, so $\bar{\mathcal{L}} \equiv \mathcal{L}/mL^2$

In other words, work in "natural units" where $mL^2 = 1$

$$\Rightarrow \bar{\mathcal{L}} = \frac{1}{2}(\dot\varphi_1^2 + \dot\varphi_2^2 + \dot\varphi_3^2)$$
$$- \frac{1}{2}(\omega_0^2 + \omega_s^2)(\varphi_1^2 + \varphi_3^2) - \frac{1}{2}(\omega_0^2 + 2\omega_s^2)\varphi_2^2$$
$$+ \omega_s^2(\varphi_1\varphi_2 + \varphi_2\varphi_3)$$

From the general formalism, we know

$$\mathbb{M} = mL^2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

&

$$\mathbb{K} = mL^2\begin{pmatrix} \omega_0^2 + \omega_s^2 & -\omega_s^2 & 0 \\ -\omega_s^2 & \omega_0^2 + 2\omega_s^2 & -\omega_s^2 \\ 0 & -\omega_s^2 & \omega_0^2 + \omega_s^2 \end{pmatrix}$$

The characteristic frequencies are

$$\det(\mathbb{K} - \omega^2 \mathbb{M}) = 0$$

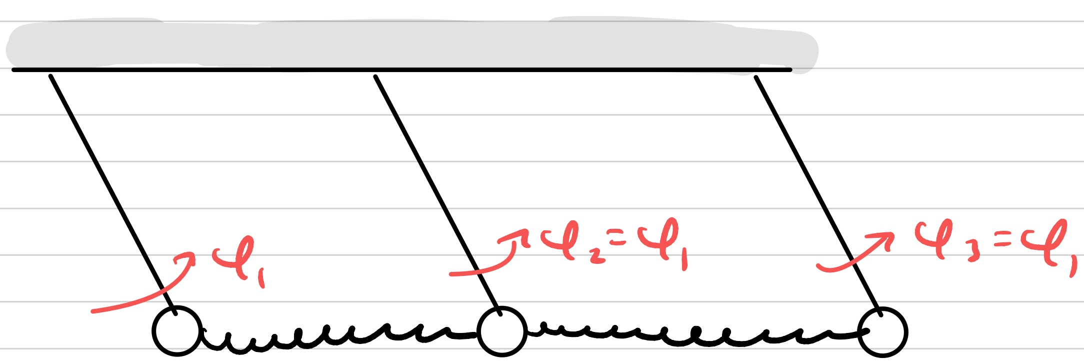
$$\Rightarrow (\omega_0^2 - \omega^2)(\omega_0^2 + \omega_s^2 - \omega^2)(\omega_0^2 + 3\omega_s^2 - \omega^2) = 0$$

w/ normal frequencies

$$\omega_1^2 = \omega_0^2, \quad \omega_2^2 = \omega_0^2 + \omega_s^2, \quad \omega_3^2 = \omega_0^2 + 3\omega_s^2$$

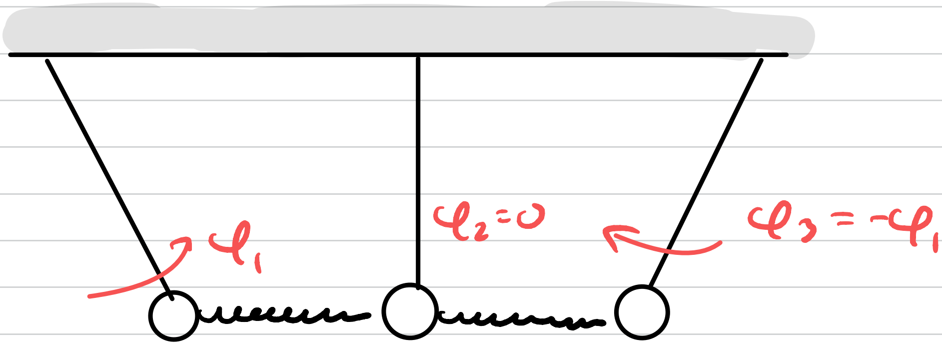
The first mode corresponds to $a_1 = a_2 = a_3 = A_1 e^{-i\delta_1}$

$$\Rightarrow \varphi_1 = \varphi_2 = \varphi_3 = A_1 \cos(\omega_1 t - \delta_1)$$



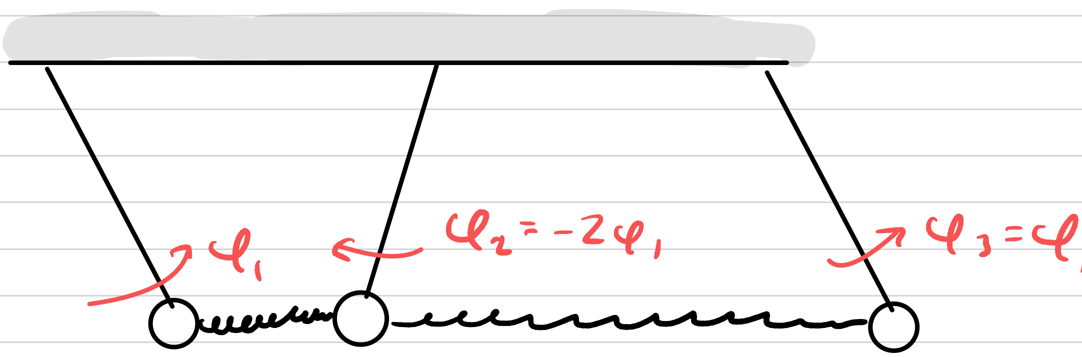
The second mode is $a_1 = -a_3 = A_2 e^{-i\delta_2}$, $a_2 = 0$

$$\Rightarrow \quad \varphi_2 = 0, \quad \varphi_1 = -\varphi_3 = A_2 \cos(\omega_2 t - \delta_2)$$



The third mode is $a_1 = -\frac{1}{2} a_2 = a_3 = A_3 e^{-i\delta_3}$

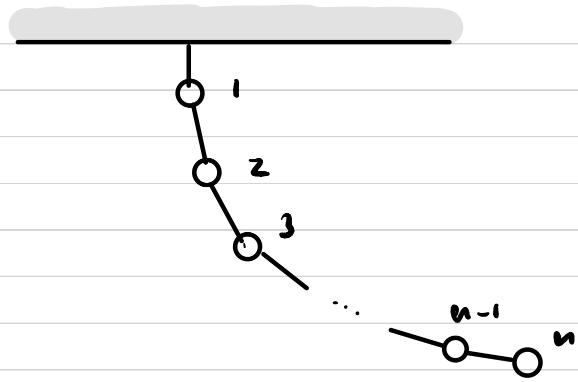
$$\Rightarrow \quad \varphi_1 = -\frac{1}{2}\varphi_2 = \varphi_3 = A_3 \cos(\omega_3 t - \delta_3)$$



Towards Continua

The main advantage of studying coupled systems is to lead us to the ideas of continuum mechanics. Consider an n-body pendulum.

Each bob has mass Δm & length $\Delta \ell$. If the total length is L, then



$$L = \sum_{j=1}^{n} \Delta \ell = n \Delta \ell$$

For fixed L, as $n \to \infty$, $\Delta \ell \to 0 \Rightarrow L = \int_0^L d\ell$

With the total mass $M = \int_0^m dm$ being fixed.

This is nothing but a swinging rope. In fact, one can understand the wave motion of such a system much in the same way in terms of normal modes.