Physics 303

Classical Mechanics II

Introduction & Review

A.W. Jackura — William & Mary

Introduction & Review

Classical Mechanics (CM) is concerned with the study of motion of bodies. It serves as the backbone of modern physics, relating simple physical laws to complicated dynamics exhibited by mechanical systems.

This course builds on the foundations you learned in Phys 208. We will apply Newton's laws & Lagrangian mechanics to systems of particles, rigid bodies, & continuous media. We will examine the behavior of systems in non-inertial reference frames, & explore the consequences of chaotic motion from nonlinear systems. Advanced mathematical techniques such as perturbation theory to approximate the dynamics of complicated systems.

Finally, we will introduce another formulation of mechanics, Hamiltonian Mechanics. Hamiltonian Mechanics provides a systematic framework to examine the phase space of dynamic variables, which impacts how one formulates Quantum theories.

Newtonian Mechanics

CM is built upon Newton's laws of motion.
Here we first consider a structureless body (a particle) which exists in 3D space (e.g. (x,y,z)) & time (t). In CM, there is a universal time for which all observers agree upon.

$\boxed{\text{NI}}$ $\exists$ some inertial reference frame such that if there are no external forces, $\vec{F} = \vec{0}$, acting on a body, then

$$\dot{\vec{v}} = \vec{0}$$

object's velocity

Therefore the object moves in a straight line,

reference time

$$\vec{r}(t) = \vec{r}_0 + \vec{v}\,(t - t_0)$$

position wrt some chosen inertial frame

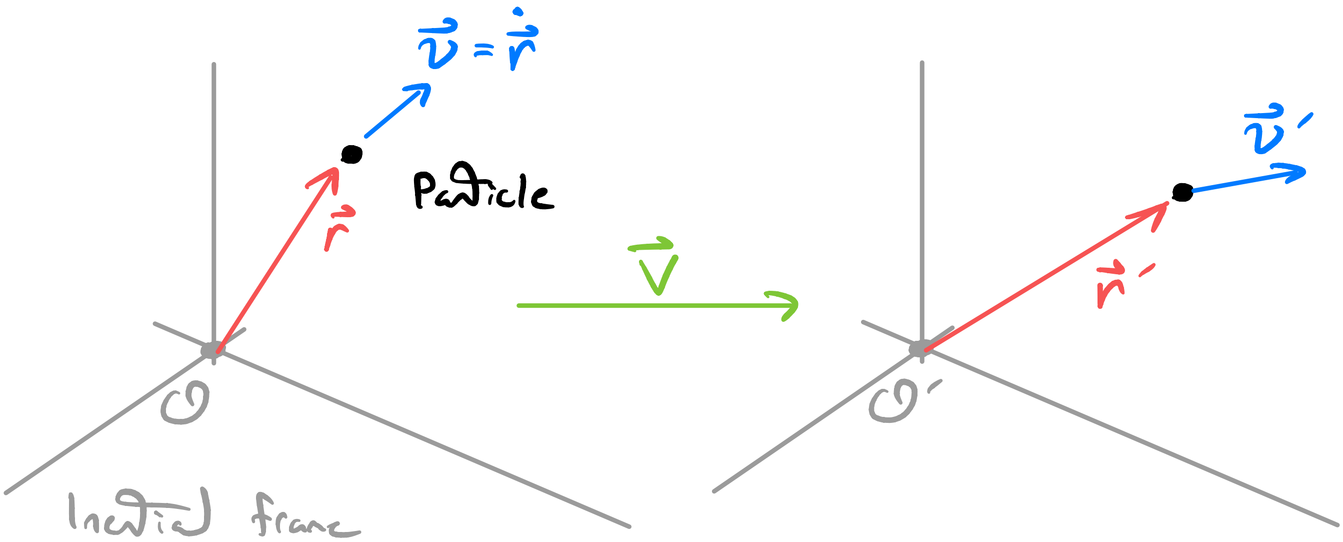
initial position, $\vec{r}(t_0) = \vec{r}_0$

$$\boxed{\vec{v} = \dot{\vec{r}}}$$

We specify an inertial frame with some convenient choice of coordinate system $\mathcal{O}$. The path of the particles motion in space is the particles trajectory.

physical laws are invariant (i.e., the same) under change of inertial frames. Under such a Galilean transformation, $\vec{V} = \text{const}$,

$$\boxed{\begin{aligned} \vec{r} &\to \vec{r}\,' = \vec{r} + \vec{V}(t - t_0) \\ \vec{v} &\to \vec{v}\,' = \vec{v} + \vec{V} \end{aligned}}$$

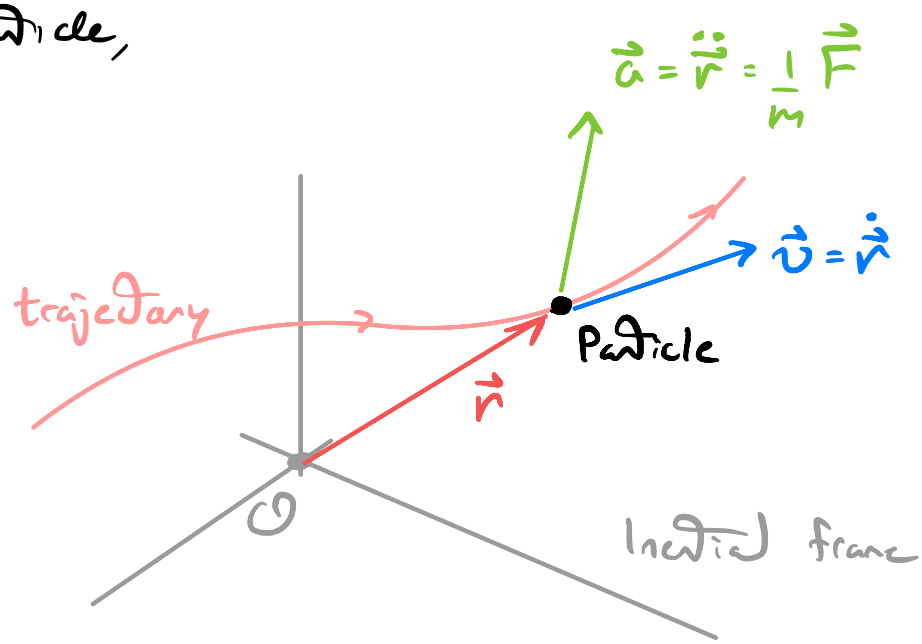


$\boxed{\text{NII}}$ Upon action of an external force $\vec{F}$, a body experiences an acceleration in the direction of $\vec{F}$,

$$\boxed{\vec{a} = \frac{1}{m}\vec{F}} \qquad \boxed{\vec{a} = \dot{\vec{v}}}$$

↳ m = particle's mass

The action of the force changes the trajectory of the particle,



In general, the force can depend on the position and velocity of the particle at a given time. This leads to a differential equation for the particle's position as a function of time,

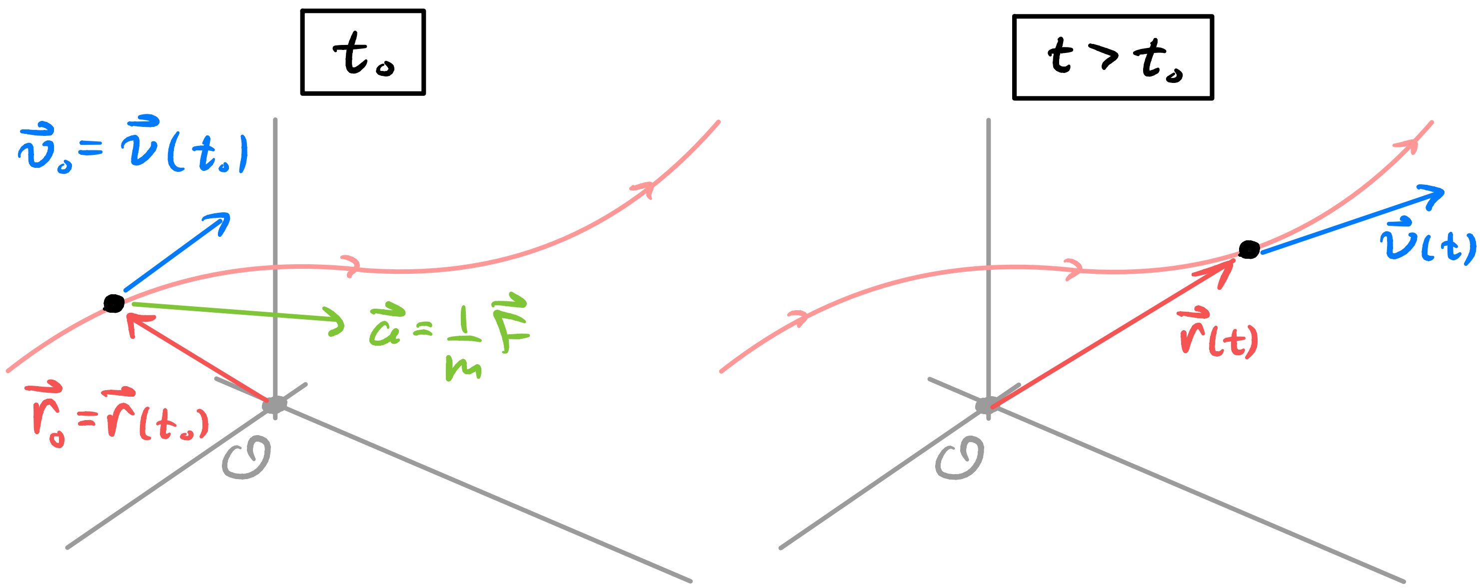
$$\ddot{\vec{r}} = \frac{1}{m}\vec{F}(\vec{r}, \dot{\vec{r}}, t)$$

To solve this, we require not only the force function, but also some initial conditions to fully specify the trajectory. This is usually the position and velocity at some reference time,

$$\vec{r}_0 = \vec{r}(t_0) \quad \text{and} \quad \vec{v}_0 = \vec{v}(t_0)$$

Newton's Principal of Determinacy

The initial state of a mechanical system, i.e., the totality of positions & velocities of its points at some moment in time, uniquely determines the all of its motion.



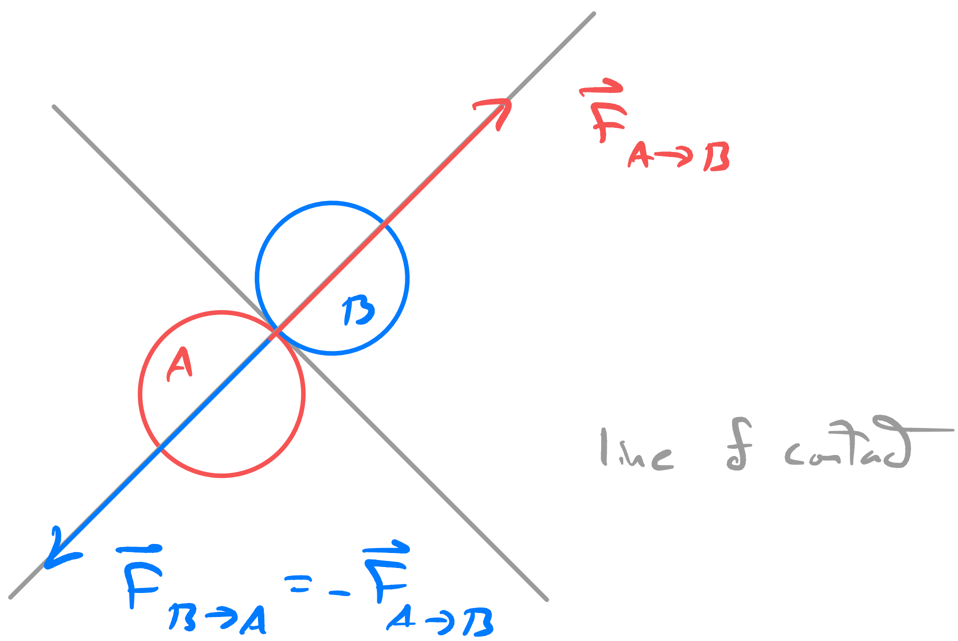
that is, the motion is completely determine via the integrals of motion constrained via $\vec{r}_0$ & $\vec{v}_0$,

$$\vec{v}(t) - \vec{v}_0 = \int_{\vec{v}_0}^{\vec{v}} d\vec{v}\,' = \frac{1}{m}\int_{t_0}^{t} dt'\, \vec{F}(\vec{r}(t'), \dot{\vec{r}}(t'), t')$$

$$\vec{r}(t) - \vec{r}_0 = \int_{\vec{r}_0}^{\vec{r}} d\vec{r}\,' = \int_{t_0}^{t} dt'\, \vec{v}(t')$$

NIII If body A imparts a force $F_{A\to B}$ onto body B, then body B imparts a force $F_{B\to A}$ onto A with equal magnitude and opposite direction,

$$\boxed{\vec{F}_{B\to A} = -\vec{F}_{A\to B}}$$



NIII is a consequence of something deeper, Conservation of Momentum, $\boxed{\vec{p} = m\vec{v}}$

Conservation Laws

While working with Newton's laws is possible, many problems, especially those involving systems of many particles, are complicated to actually apply the laws of motion. It turns out we can generate equivalent laws of motion manipulating Newton's laws to exploit quantities susceptible to conservation laws.

$\boxed{\text{Momentum}}$ Consider NII written in terms of the particles momentum, $\vec{p}$,

$$\boxed{\dot{\vec{p}} = \vec{F}} \quad , \quad \boxed{\vec{p} = m\vec{v}}$$

We can construct a 1st integral of motion by

$$\vec{p}_f - \vec{p}_i = \int_{\vec{p}_i}^{\vec{p}_f} d\vec{p} = \underbrace{\int_{t_i}^{t_f} dt\, \vec{F}}_{\text{the impulse}}$$

If there are no external forces on the system, then momentum is conserved! $\boxed{\vec{p}_f = \vec{p}_i}$

Conservation of momentum proves very effective in systems of many particles, such as those involved in collisions, where we do not care about the internal forces.

Angular Momentum Define the angular momentum $\vec{l}$ as

$$\boxed{\vec{l} = \vec{r} \times \vec{p}}$$

The rate of change in time

$$\dot{\vec{l}} = \dot{\vec{r}} \times \vec{p} + \vec{r} \times \dot{\vec{p}} \qquad \dot{\vec{p}} = \vec{F}$$

$$\dot{\vec{p}} = m\dot{\vec{r}} \Rightarrow \dot{\vec{r}} \times \dot{\vec{r}} = 0$$

$$= \vec{r} \times \vec{F}$$

Define the torque $\boxed{\vec{\Gamma} \equiv \vec{r} \times \vec{F}}$.

Thus, we have $\boxed{\dot{\vec{l}} = \vec{\Gamma}}$

Notice that if a system experiences no external torques, then angular momentum is conserved!

Unlike $\vec{p}$, $\vec{l}$ depends on the choice of coordinate system. Let $\vec{r} \to \vec{r}_0 + \vec{r} = \vec{r}\,'$

$$\Rightarrow \quad \vec{l} \to \vec{l}\,' = \vec{r}_0 \times \vec{p} + \vec{r} \times \vec{p} \neq \vec{l}.$$

Energy

Consider a force $\vec{F}(\vec{r})$ acting on a particle. The <u>work performed on the particle</u> as it moves from $\vec{r}_A$ to $\vec{r}_B$ is

$$\boxed{W_{A\to B} = \int_{\vec{r}_A}^{\vec{r}_B} d\vec{r} \cdot \vec{F}(\vec{r})}$$

Note that work is in general dependent on the path the particle takes.

From NII, $\vec{F} = m \dfrac{d\vec{v}}{dt}$

But, $d\vec{r} \cdot \dfrac{d\vec{v}}{dt} = \vec{v} \cdot d\vec{v} = \dfrac{1}{2} d(v^2)$

$$\Rightarrow \int_{\vec{r}_A}^{\vec{r}_B} d\vec{r} \cdot \vec{F}(\vec{r}) = \frac{1}{2} m \int_{v_A}^{v_B} d(v^2) = \frac{1}{2} m v_B^2 - \frac{1}{2} m v_B^2$$

Defining the <u>kinetic energy</u> $\boxed{T = \frac{1}{2} m v^2}$, we arrive at <u>Work-Energy theorem</u>

$$\boxed{T_B - T_A = W_{A\to B}}$$

If the force is conservative, $\vec{\nabla} \times \vec{F} = \vec{0}$,

then we can write $\boxed{\vec{F} = -\vec{\nabla} U}$

where $U(\vec{r})$ is the potential energy.

Gravity is an example of a conservative force,

$$U_{grav} = -\frac{G m_1 m_2}{r}$$

$$\Rightarrow \quad \vec{F}_{grav} = -\vec{\nabla} U_{grav}$$
$$= -\frac{G m_1 m_2}{r^2} \hat{r}$$

We can define the total mechanical energy E

$$E = T + \underbrace{\sum_{j=1}^{n} U_j}_{\text{PEs from conservative forces}}$$

so that

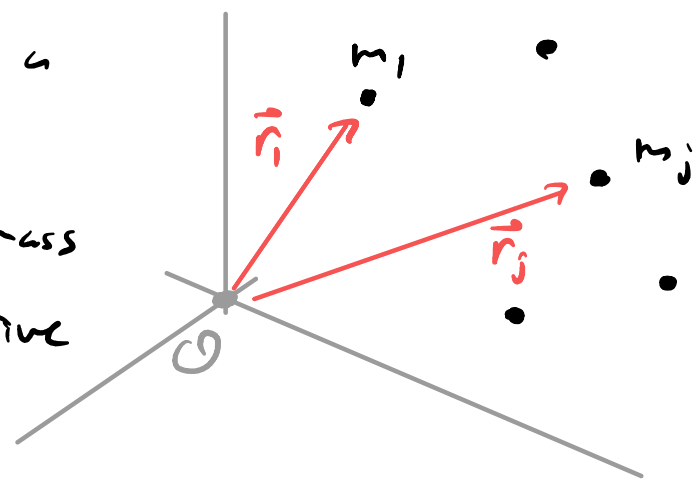
$$\boxed{\Delta E = W_{NC}}$$

W_{NC} → Work from non-conservative forces.

If $\vec{F}_{NC} = \vec{0}$, then $\Delta E = 0$ and energy is conserved!

System of particles

All of these notions extend to systems of particles. We generally break up motion of a system of particles into the center-of-mass motion and the relative motion.



Consider N particles with masses m_j, $j=1,\dots,N$, and positions $\vec{r}_j$, $j=1,\dots,N$, as defined by an inertial system $\mathcal{O}$.

The center-of-mass (CM) is defined as

$$\boxed{\vec{R} = \frac{1}{M}\sum_{j=1}^{N} m_j \vec{r}_j}$$

where $M = \sum_{j=1}^{N} m_j$ is the total mass of the system.

Let's apply NII to particle j,

$$\dot{\vec{p}}_j = \vec{F}_j^{(ext)} + \sum_{k\neq j} \vec{F}_{kj}$$

some external force

forces due to other particles in the system.

Let us examine the motion of the CM,

$$M\ddot{\vec{R}} = \sum_j m_j \ddot{\vec{r}}_j = \sum_j \dot{\vec{p}}_j = \sum_j \vec{F}_j^{(ext)} + \sum_j \sum_{k \neq j} \vec{F}_{kj}$$

Note that the last term can be written as

$$\sum_j \sum_{k \neq j} \vec{F}_{kj} = \frac{1}{2} \sum_{j,k} \left(\vec{F}_{kj} + \vec{F}_{jk} \right)$$

where we have defined $\vec{F}_{jj} = \vec{0}$ to reflect the absence of self forces. By N III, $\vec{F}_{kj} = -\vec{F}_{jk}$, so identically

$$\sum_j \sum_{k \neq j} \vec{F}_{kj} = \vec{0}$$

Therefore, the CM has an EOM $\boxed{M\ddot{\vec{R}} = \sum_j \vec{F}^{(ext)} = \vec{F}^{(ext)}}$

The CM moves as an effective particle of mass M.

The total momentum of the system is

$$\vec{P} = \sum_{j=1}^{N} \vec{p}_j = \frac{d}{dt} \sum_{j=1}^{N} m_j \vec{r}_j = M\dot{\vec{R}} \equiv M\vec{V}$$

velocity of CM

The angular momentum of the system is

$$\vec{L} = \sum_j \vec{r}_j \times \vec{p}_j$$

Taking the time derivative,

$$\dot{\vec{L}} = \sum_j \vec{r}_j \times \dot{\vec{p}}_j = \sum_j \vec{r}_j \times \left(\vec{F}_j^{(ext)} + \sum_{k \neq j} \vec{F}_{kj} \right)$$

Now, the last term

$$\sum_{j,k} \vec{r}_j \times \vec{F}_{kj} = \frac{1}{2} \sum_{j,k} (\vec{r}_j - \vec{r}_k) \times \vec{F}_{kj}$$

When the force $\vec{F}_{kj} \parallel \vec{r}_j - \vec{r}_k$, this is zero. Examples where this is not true is the magnetic force, where one needs to include EM field momenta.

So,

$$\dot{\vec{L}} = \sum_j \vec{r}_j \times \vec{F}_j^{(ext)} = \Gamma^{(ext)}.$$

Multiparticle Energy laws follow similarly, e.g.,

$$U = \frac{1}{2} \sum_{j,k} U_{kj} + \sum_j U_j^{(ext)}.$$

Lagrangian Mechanics

Conservation laws play central roles in physics. The Newtonian formulation of mechanics often clouds the nature of conservation laws. Noether showed that conservation laws are a direct consequence of symmetries of the mechanical system under the action of some transformation.

Conservation Laws	$\longleftrightarrow$	Symmetry
Energy	$\longleftrightarrow$	time translation
Momentum	$\longleftrightarrow$	space translation
Angular Momentum	$\longleftrightarrow$	Rotations

An alternative formulation of mechanics, called Lagrangian Mechanics, allows one to study the symmetries of mechanical systems. It offers an equivalent formulation to Newton's laws.

One defines the Lagrangian $\mathcal{L}$

$$\boxed{\mathcal{L} = T - U}$$

The Lagrangian encodes all the dynamical information of a mechanical system. To access the equations of motion, we construct the action S

$$\boxed{S = \int_{t_i}^{t_f} dt\, L}$$

By varying the action $\delta S = 0$, we can generate the Euler-Lagrange equations, which are equivalent to NII. Consider a system with generalized coordinates $q_1, \ldots, q_n$, which can be positions, angles, etc. By minimizing the action, which effectively produces the optimal trajectory for the system, we arrive at

$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i} \quad , \; i = 1, \ldots, n}$$

For example, consider the motion of a single particle in some potential well $U(\vec{r})$.

The KE of the particle is

$$T = \frac{1}{2} m \dot{\vec{r}}^{\,2} = \frac{1}{2} m \sum_{j=1}^{3} \dot{r}_j^{\,2}$$

So,

$$\begin{aligned} \mathcal{L} &= T - U \\ &= \frac{1}{2} m \dot{\vec{r}}^{\,2} - U(\vec{r}) \end{aligned}$$

The Euler-Lagrange equations yield

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \dot{r}_j} &= \frac{\partial}{\partial \dot{r}_j} \left(\frac{1}{2} m \sum_{k=1}^{3} \dot{r}_k^{\,2} \right) \\ &= m \sum_{k=1}^{3} \dot{r}_k \delta_{kj} = m \dot{r}_j \end{aligned}$$

and

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{r}_j} = m \ddot{r}_j$$

Finally,

$$\frac{\partial \mathcal{L}}{\partial r_j} = -\frac{\partial}{\partial r_j} U(\vec{r})$$

Recall $\vec{F} = -\vec{\nabla} U(\vec{r})$, so $F_j = -\frac{\partial}{\partial r_j} U$

Therefore,

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{r}_j} = \frac{\partial U}{\partial r_j} \Rightarrow m \ddot{r}_j = -\frac{\partial}{\partial r_j} U = F_j$$

$$\Rightarrow \boxed{m \ddot{\vec{r}} = \vec{F}}$$