Physics 303

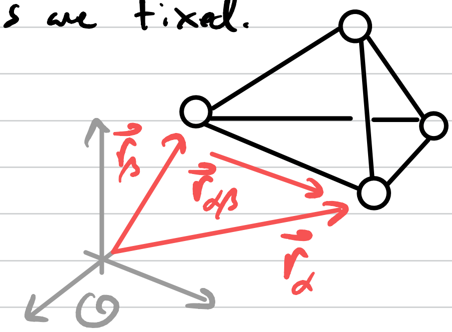
Classical Mechanics II

Rigid Body Motion

A.W. Jackura —— William & Mary

Rigid Bodies

A rigid body is an abstract notion of a collection of particles / objects that move together in such a way to maintain their shape, i.e., their relative positions are fixed.



$$\vec{r}_{\alpha\beta} = \vec{r}_\alpha - \vec{r}_\beta$$

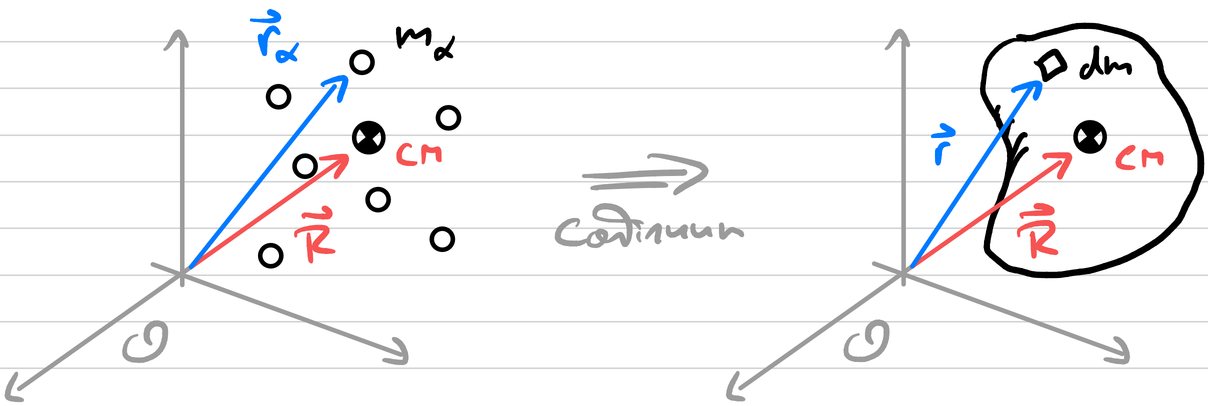
$$\Rightarrow \ |\vec{r}_{\alpha\beta}| = \text{constant}$$

This is an idealization, as atoms and molecules vibrate meaning no object is completely rigid. However, this is a good starting point to build on.

Since the distances between particles are fixed, the system is highly constrained. For N particles, there are 3N coordinates needed. But, since the distances between particles is fixed, the rigid body only needs 6 degrees of freedom

- 3 to specify CM
- 3 to specify orientation

Consider system of N particles $\alpha = 1, \dots, N$ with masses m_α and positions $\vec{r}_\alpha$ measured w.r.t. O



The CM is $\boxed{\vec{R} = \frac{1}{M}\sum_{\alpha=1}^{N} m_\alpha \vec{r}_\alpha}$, $\boxed{M = \sum_\alpha m_\alpha}$

If the particles are small and numerous in a small volume, we can define a density $\rho(\vec{r})$ as

$$\Delta m = \rho(\vec{r})\, \Delta x \Delta y \Delta z$$

then we can consider the rigid body of a continuous distribution of mass $\boxed{M = \int dm = \int \rho(\vec{r})\, dV}$

and CM $\boxed{\vec{R} = \frac{1}{M}\int \vec{r}\, dm}$

We will switch between a discrete and continuous picture as needed.

Momentum & Angular Momentum

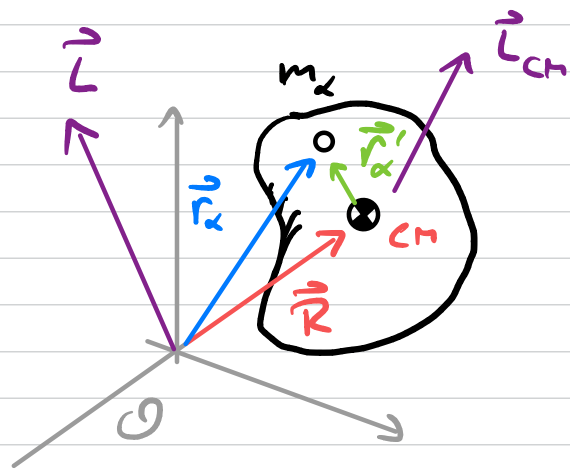
The total momentum of the system is

$$\vec P = \sum_\alpha \vec p_\alpha = \sum_\alpha m_\alpha \dot{\vec r}_\alpha = M\dot{\vec R}$$

If the system is exposed to an external force $\vec F^{ext}$, then NII for the CM is

$$\boxed{\dot{\vec P} = \vec F^{ext} = M\ddot{\vec R}}$$

Next we consider angular momentum.
Let $\vec L$ be the angular momentum of the system wrt O.
We want to split $\vec L$ into an $\vec L_{CM}$, the angular momentum of the body about the CM, and the $\vec L_{orb}$, the angular momentum of the CM.



The angular momentum of α about O is

$$\vec l_\alpha = \vec r_\alpha \times \vec p_\alpha = \vec r_\alpha \times m_\alpha \dot{\vec r}_\alpha$$

So the total angular momentum is $\vec L = \sum_\alpha \vec l_\alpha$

So, $$\vec{L} = \sum_\alpha \vec{l}_\alpha = \sum_\alpha \vec{r}_\alpha \times m_\alpha \dot{\vec{r}}_\alpha$$

Now, let $\vec{r}_\alpha'$ be location of α w.r.t. CM

$$\vec{r}_\alpha = \vec{R} + \vec{r}_\alpha'$$

So, find

$$\vec{L} = \sum_\alpha (\vec{R} + \vec{r}_\alpha') \times m_\alpha (\dot{\vec{R}} + \dot{\vec{r}}_\alpha')$$

$$= \sum_\alpha \vec{R} \times m_\alpha \dot{\vec{R}} + \sum_\alpha \vec{R} \times m_\alpha \dot{\vec{r}}_\alpha'$$

$$+ \sum_\alpha \vec{r}_\alpha' \times m_\alpha \dot{\vec{R}} + \sum_\alpha \vec{r}_\alpha' \times m_\alpha \dot{\vec{r}}_\alpha'$$

Recall $M = \sum_\alpha m_\alpha$

$$\Rightarrow \vec{L} = \vec{R} \times M \dot{\vec{R}} + \vec{R} \times \sum_\alpha m_\alpha \dot{\vec{r}}_\alpha'$$

$$+ \left(\sum_\alpha m_\alpha \vec{r}_\alpha' \right) \times \dot{\vec{R}} + \sum_\alpha \vec{r}_\alpha' \times m_\alpha \dot{\vec{r}}_\alpha'$$

Now, $\sum_\alpha m_\alpha \vec{r}_\alpha' = \vec{0}$ since this is location of CM relative to CM (of course)

likewise, $$\sum_\alpha m_\alpha \dot{\vec{r}}_\alpha' = \vec{0}$$

So,
$$\vec{L} = \vec{R}\times\vec{P} + \sum_\alpha \vec{r}_\alpha{}' \times m_\alpha \dot{\vec{r}}_\alpha{}'$$

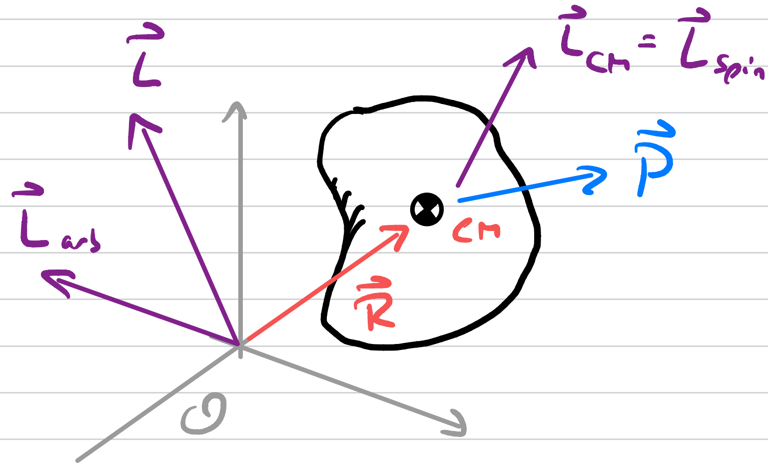
angular momentum of CM relative to O

angular momentum relative to the CM

Define
$$\boxed{\vec{L}_{CM} = \vec{L}_{spin} = \sum_\alpha \vec{r}_\alpha{}' \times m_\alpha \dot{\vec{r}}_\alpha{}'}$$

$$\boxed{\vec{L}_{orb} = \vec{R}\times\vec{P}}$$

$$\Rightarrow \boxed{\vec{L} = \vec{L}_{orb} + \vec{L}_{CM}}$$



This separation is often useful as both are approximately conserved

$$\dot{\vec{L}}_{orb} = \cancel{\dot{\vec{R}}\times\vec{P}}^{\,0} + \vec{R}\times\dot{\vec{P}} = \vec{R}\times\vec{F}^{ext} \quad ; \quad \vec{F}^{ext} = \sum_\alpha \vec{F}_\alpha^{ext}$$

We know $\dot{\vec{L}} = \vec{\Gamma}^{ext}$, the external torque relative to O

So,
$$\dot{\vec{L}}_{CM} = \dot{\vec{L}} - \dot{\vec{L}}_{orb} = \vec{\Gamma}^{ext} - \vec{R}\times\vec{F}^{ext}$$
$$= \sum_\alpha (\vec{r}_\alpha - \vec{R})\times\vec{F}_\alpha^{ext} = \vec{\Gamma}_{CM}^{ext}$$

external torque relative to CM

Kinetic & Potential Energy

The total kinetic energy of N particles is

$$T = \sum_{\alpha=1}^{N} \frac{1}{2} m_\alpha \dot{\vec{r}}_\alpha^{\,2}$$

As before, write $\vec{r}_\alpha = \vec{R} + \vec{r}_\alpha{}'$, $\vec{r}_\alpha{}'$ position relative to CM

$$\Rightarrow \quad \dot{\vec{r}}_\alpha^{\,2} = \dot{\vec{R}}^2 + \dot{\vec{r}}_\alpha{}'^{\,2} + 2\dot{\vec{R}}\cdot\dot{\vec{r}}_\alpha{}'$$

$$\Rightarrow \quad T = \frac{1}{2}\sum_\alpha m_\alpha \dot{\vec{R}}^2 + \frac{1}{2}\sum_\alpha m_\alpha \dot{\vec{r}}_\alpha{}'^{\,2} + \dot{\vec{R}}\cdot\underbrace{\sum_\alpha m_\alpha \dot{\vec{r}}_\alpha{}'}_{\substack{=0 \\ \text{as before}}}$$

$$= \frac{1}{2} M \dot{\vec{R}}^2 + \frac{1}{2}\sum_\alpha m_\alpha \dot{\vec{r}}_\alpha{}'^{\,2}$$

Define KE relative to CM $\boxed{T_{CM} = \frac{1}{2}\sum_\alpha m_\alpha \dot{\vec{r}}_\alpha{}'^{\,2}}$

So,

$$\boxed{T = \frac{1}{2} M \dot{\vec{R}}^2 + T_{CM}}$$

KE of CM

For conservative forces, can write potential energy and decompose as

$$U = U_{ext} + U_{int}$$

$\uparrow$ external PE $\quad$ $\uparrow$ internal PE

Where $U_{int} = \sum_{\alpha<\beta} U_{\alpha\beta}(|\vec{r}_\alpha - \vec{r}_\beta|)$

assuming central forces

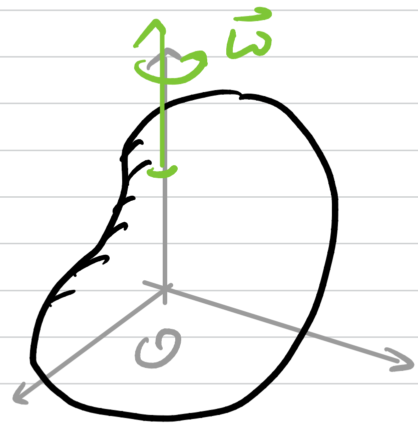
Since $|\vec{r}_{\alpha\beta}| = \text{const.}$,

$\Rightarrow U_{int} = \text{const.}$

$\therefore U_{int}$ is irrelevant for rigid body dynamics

Rotation about a Fixed Axis

Here we consider the rotation of a rigid body about some fixed axis. Since the axis is fixed, let us define it as the z-axis



$$\Rightarrow \vec{\omega} = (0, 0, \omega)$$

If the body consist of N particles, then

$$\vec{L} = \sum_{\alpha} \vec{l}_{\alpha}$$

$$= \sum_{\alpha} \vec{r}_{\alpha} \times m_{\alpha} \vec{v}_{\alpha}$$

Since the axis of rotation is fixed, $\vec{v}_{\alpha} = \vec{\omega} \times \vec{r}_{\alpha}$, so, with $\vec{r}_{\alpha} = (x_{\alpha}, y_{\alpha}, z_{\alpha})$

$$\Rightarrow \vec{v}_{\alpha} = (-\omega y_{\alpha}, \omega x_{\alpha}, 0)$$

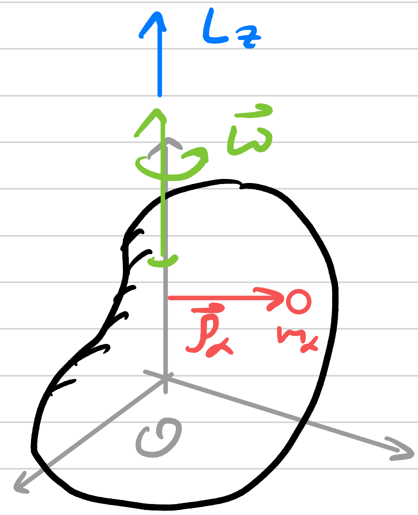
$$\therefore \quad \vec{l}_{\alpha} = m_{\alpha} \vec{r}_{\alpha} \times \vec{v}_{\alpha}$$

$$= m_{\alpha} \omega (-z_{\alpha} x_{\alpha}, -z_{\alpha} y_{\alpha}, x_{\alpha}^2 + y_{\alpha}^2)$$

Thus, $\vec{L} = \sum_{\alpha} \vec{l}_{\alpha}$ is total angular momentum.

Let's examine components of $\vec{L}$.

The z-component is



$$L_z = \sum_\alpha m_\alpha (x_\alpha^2 + y_\alpha^2)\,\omega$$

Notice that

$$\rho_\alpha^2 = x_\alpha^2 + y_\alpha^2$$

with ρ_α being the distance to any point from the z-axis.

Thus, $$L_z = \sum_\alpha m_\alpha \rho_\alpha^2\, \omega \equiv I_z\, \omega$$

where $$\boxed{I_z = \sum_\alpha m_\alpha \rho_\alpha^2}$$

is the moment of inertia about the z-axis.

The kinetic energy is then

$$T = \frac{1}{2}\sum_\alpha m_\alpha v_\alpha^2 .$$

Since $v_\alpha = \rho_\alpha \omega$ for a rotation about fixed z-axis,

$$\Rightarrow T = \frac{1}{2}\sum_\alpha m_\alpha \rho_\alpha^2\, \omega^2 = \frac{1}{2} I_z\, \omega^2 .$$

These should be familiar results from Phys 101.

Notice though that there is non-zero components for L_x & L_y,

$$L_x = -\sum_\alpha m_\alpha x_\alpha z_\alpha \omega$$

$$L_y = -\sum_\alpha m_\alpha y_\alpha z_\alpha \omega$$

we define the products of inertia about the z-axis as

$$L_x = I_{xz}\omega \quad , \quad L_y = I_{yz}\omega$$

with $$I_{xz} = -\sum_\alpha m_\alpha x_\alpha z_\alpha$$

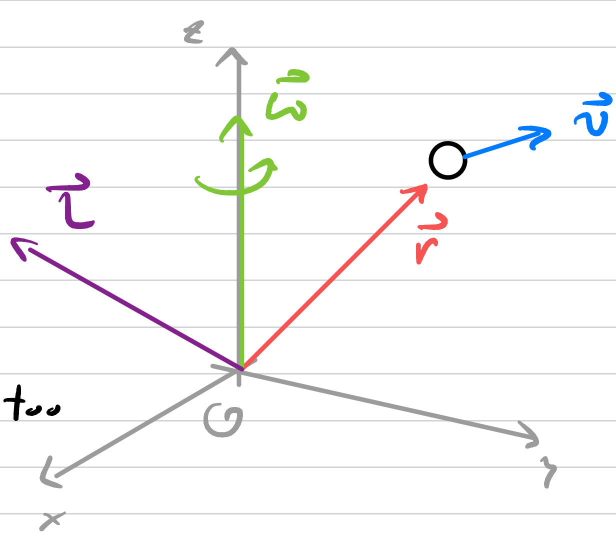
$$I_{yz} = -\sum_\alpha m_\alpha y_\alpha z_\alpha$$

Obviously, $\vec{L}$ is not parallel to $\vec{\omega}$!

$$\vec{L} = (I_{xz}\omega, I_{yz}\omega, I_{zz}\omega)$$

with $I_{zz} \equiv I_z$. Consider a single point particle,

$$\vec{L} = \vec{r} \times m\vec{v}$$



If $\vec{v} \parallel -\hat{x}$

$\vec{r}$ lies in yz plane,

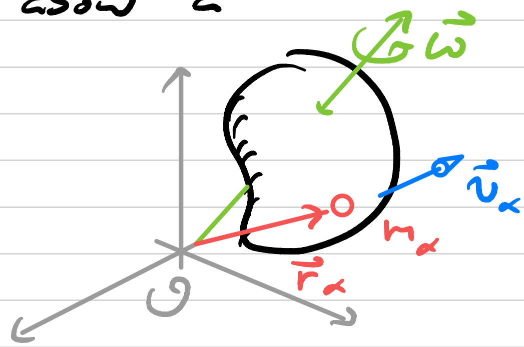
find $\vec{L}$ lies in yz plane too

The Inertia Tensor

We saw that $\vec{L} \neq I\vec{\omega}$ with I being a number.

In general, I is a 3x3 symmetric tensor.

Let's see by consider a rotation about a general fixed axis $\vec{\omega}$.



$$\vec{L} = \sum_\alpha \vec{r}_\alpha \times m_\alpha \vec{v}_\alpha$$

$$= \sum_\alpha m_\alpha \vec{r}_\alpha \times (\vec{\omega} \times \vec{r}_\alpha)$$

Recall identity $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$

$$\Rightarrow \vec{L} = \sum_\alpha m_\alpha \left[\vec{r}_\alpha^{\,2} \vec{\omega} - (\vec{r}_\alpha \cdot \vec{\omega}) \vec{r}_\alpha \right]$$

Let's look at i^{th}-component,

$$L_i = \sum_\alpha m_\alpha \left[\vec{r}_\alpha^{\,2} \omega_i - \left(\sum_j r_{\alpha,j} \omega_j \right) r_{\alpha,i} \right]$$

$$= \sum_j \left[\sum_\alpha m_\alpha \left(\vec{r}_\alpha^{\,2} \delta_{ij} - r_{\alpha,i} r_{\alpha,j} \right) \right] \omega_j$$

$$\equiv \sum_j I_{ij} \omega_j$$

We define the Inertia tensor as $\mathbb{I}$ with matrix elements

$$\boxed{I_{ij} = \sum_\alpha m_\alpha \left(\vec{r}_\alpha^{\,2} \delta_{ij} - r_{\alpha,i} r_{\alpha,j} \right)}$$

In terms of a continuous distribution,

$$\boxed{I_{ij} = \int dm \left(\vec{r}\,\delta_{ij} - r_i r_j \right)}$$

By inspection, $\mathbb{I}$ is symmetric, $\mathbb{I}^T = \mathbb{I}$ or $I_{ij} = I_{ji}$.

It characterizes an objects resistance to change in rotational motion

$$\boxed{\vec{L} = \mathbb{I} \cdot \vec{\omega} \quad \text{or} \quad L_i = \sum_j I_{ij}\,\omega_j}$$

The Cartesian components are

$$\mathbb{I} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$

with $I_{xx} \equiv I_x = \sum_\alpha m_\alpha \left(\vec{r}_\alpha^{\,2} - x_\alpha^2 \right) = \sum_\alpha m_\alpha \left(y_\alpha^2 + z_\alpha^2 \right)$

$$I_{xy} = -\sum_\alpha m_\alpha x_\alpha y_\alpha$$

etc...

Explicitly, $\vec{L} = \mathbb{I} \cdot \vec{\omega}$ is

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

The inertia tensor is a 3×3 symmetric matrix which must transform as $\mathbb{I}' = R \, \mathbb{I} \, R^T$

or $I'_{ij} = \sum_{m,n} R_{im} R_{jn} I_{mn}$

rotation matrix

this is what makes $\mathbb{I}$ a tensor

To see this, note that $\vec{L}$ & $\vec{\omega}$ are physical vectors which must transform as $\vec{L}' = R \cdot \vec{L}$, $\vec{\omega}' = R \cdot \vec{\omega}$ under a rotation R.

$$\therefore \quad \vec{L}' = R \cdot \vec{L} = R \cdot \mathbb{I} \cdot \vec{\omega}$$

insert $\mathbb{1} = R^{-1} \cdot R$
$= R^T \cdot R$ since R is orthogonal

$$= R \cdot \mathbb{I} \cdot R^T R \vec{\omega}$$
$$= (R \cdot \mathbb{I} \cdot R^T) \cdot \vec{\omega}'$$
$$= \mathbb{I}' \cdot \vec{\omega}'$$

$\Rightarrow$ Require $\mathbb{I}' = R \cdot \mathbb{I} \cdot R^T$ if $\vec{L}, \vec{\omega}$ are physical vectors.

The kinetic energy is

$$T = \frac{1}{2}\sum_\alpha' m_\alpha \dot{\vec{r}}_\alpha^{\,2} = \frac{1}{2}\sum_\alpha' m_\alpha (\vec{\omega}\times\vec{r}_\alpha)^2$$

$$= \frac{1}{2}\sum_\alpha' m_\alpha \sum_i' (\vec{\omega}\times\vec{r}_\alpha)_i (\vec{\omega}\times\vec{r}_\alpha)_i$$

$$= \frac{1}{2}\sum_\alpha' m_\alpha \sum_i' \sum_{j,k}' \epsilon_{ijk}\,\omega_j\, r_{\alpha,k} \sum_{l,m}' \epsilon_{ilm}\,\omega_l\, r_{\alpha,m}$$

Note the relation $\sum_i' \epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{kl}\delta_{jm}$

$$\Rightarrow T = \frac{1}{2}\sum_\alpha' m_\alpha \sum_{j,k}' \sum_{l,m}' (\delta_{jl}\delta_{km} - \delta_{kl}\delta_{jm})\,\omega_j\omega_l\, r_{\alpha,k} r_{\alpha,m}$$

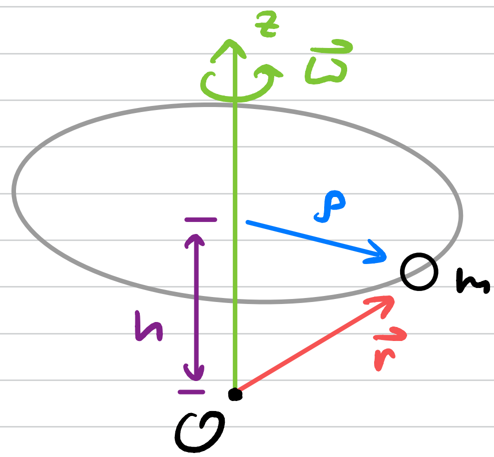
$$= \frac{1}{2}\sum_{i,j}' \omega_i \left[\sum_\alpha' m_\alpha (\vec{r}_\alpha^{\,2}\delta_{ij} - r_{\alpha,i} r_{\alpha,j})\right]\omega_j$$

$$= \frac{1}{2}\sum_{i,j}' \omega_i I_{ij}\,\omega_j$$

$$\Rightarrow \boxed{T = \frac{1}{2}\vec{\omega}^T\cdot \mathbb{I}\cdot\vec{\omega} = \frac{1}{2}\vec{\omega}\cdot\vec{L}}$$

Example – Consider a point-particle with mass m rotating around z-axis at a constant radius ρ, height h above origin, & angular velocity $\vec{\omega}$.

Compute the elements of the inertia tensor.



$$\vec{\omega} = (0, 0, \omega)$$

& position

$$\vec{r}(t) = \rho\cos\omega t\,\hat{x} + \rho\sin\omega t\,\hat{y} + h\,\hat{z}$$

Since the rotation is about z, there are only 3 – non-zero components, I_z, $I_{xz} = I_{zx}$, $I_{yz} = I_{zy}$

$$I_z = m(x^2+y^2) = m\rho^2$$

$$I_{xz} = -mxz = -mh\rho\cos\omega t$$

$$I_{yz} = -myz = -mh\rho\sin\omega t$$

So, $\vec{L} = \mathbb{I}\cdot\vec{\omega} = I_{xz}\omega\,\hat{x} + I_{yz}\omega\,\hat{y} + I_z\omega\,\hat{z}$

$$= -mh\rho\omega(\cos\omega t\,\hat{x} + \sin\omega t\,\hat{y}) + m\rho^2\omega\,\hat{z}$$

Exercise: Compare $\vec{L}$ to $\vec{L} = \vec{r}\times m\vec{v}$.

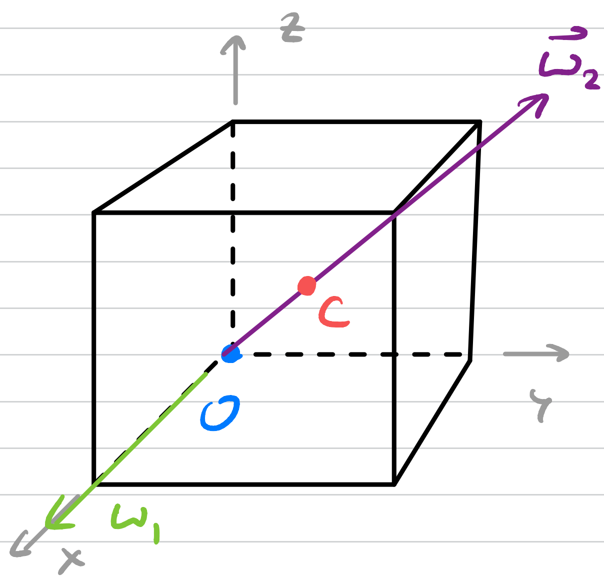
Notice that if $h=0$, i.e., the origin is in the plane of rotation

$$\Rightarrow \vec{L}_{h=0} = I_z \vec{\omega}$$

which is the result from Phys 101. $\blacksquare$

Example – Compute the inertia tensor of a solid cube of mass M and side length a about (a) the corner, (b) the center.
Compute $\vec{L}$ for both cases given
$\vec{\omega}_1 = \omega(1,0,0)$ & $\vec{\omega}_2 = \frac{\omega}{\sqrt{3}}(1,1,1)$.

For a continuous distribution



$$I_{ij} = \int (\vec{r}^{\,2}\delta_{ij} - r_i r_j)\,\rho\, dV$$

where $\rho = \frac{M}{a^3}$

- Corner (point O) $\Rightarrow (0,0,0)$
- Center (point C) $\Rightarrow (\frac{a}{2}, \frac{a}{2}, \frac{a}{2})$

(a) for point O,

$$I_x(O) = \int_0^a dx \int_0^a dy \int_0^a dz \, \overset{\text{constant}}{\rho} \, (y^2 + z^2)$$

$$= \rho \left(\int_0^a dx\right)\left(\int_0^a dy\, y^2\right)\left(\int_0^a dz\right)$$

$$+ \rho \left(\int_0^a dx\right)\left(\int_0^a dy\right)\left(\int_0^a dz\, z^2\right)$$

$$= \rho \cdot a \cdot \frac{a^3}{3} \cdot a + \rho \cdot a \cdot a \cdot \frac{a^3}{3}$$

$$= \frac{2}{3} \rho a^5 = \frac{2}{3}\left(\frac{M}{a^3}\right) a^5$$

$$\Rightarrow \quad I_x(O) = \frac{2}{3} M a^2$$

By inspection, find $I_x(O) = I_y(O) = I_z(O) = \frac{2}{3} M a^2$

The product of inertia is

$$I_{xy}(O) = -\rho \int_0^a dx \int_0^a dy \int_0^a dz \cdot xy$$

$$= -\left(\frac{M}{a^3}\right) \frac{a^2}{2} \cdot \frac{a^2}{2} \cdot a = -\frac{1}{4} M a^2$$

By inspection, find all products of inertia are equal

So,

$$\mathbb{I}(0) = Ma^2 \begin{pmatrix} 2/3 & -1/4 & -1/4 \\ -1/4 & 2/3 & -1/4 \\ -1/4 & -1/4 & 2/3 \end{pmatrix}$$

(b) for point C,

$$I_x(C) = \rho \int_{-a/2}^{a/2} dx \int_{-a/2}^{a/2} dy \int_{-a/2}^{a/2} dz \, (y^2 + z^2)$$

$$= \frac{M}{a^3} \cdot a \cdot a \cdot \frac{1}{3} \cdot 2 \left[\left(\frac{a}{2}\right)^3 + \left(\frac{a}{2}\right)^3 \right]$$

$$= \frac{1}{6} M a^2$$

Likewise, $I_y(C) = I_z(C) = \frac{1}{6} M a^2$

$$I_{xy}(C) = \rho \int_{-a/2}^{a/2} dx \int_{-a/2}^{a/2} dy \int_{-a/2}^{a/2} dz \; xy = 0$$

↳ odd integrand over even interval

$\Rightarrow$ All $I_{ij} = 0$ for $i \neq j$

$$\Rightarrow \quad \mathbb{I}(C) = \frac{1}{6} M a^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \frac{1}{6} M a^2 \mathbb{1} .$$

The angular momenta are

$$\vec{L}_1(O) = \mathbb{I}(O)\cdot\vec{\omega}_1$$

$$= Ma^2\omega \begin{pmatrix} 2/3 & -1/4 & -1/4 \\ -1/4 & 2/3 & -1/4 \\ -1/4 & -1/4 & 2/3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \frac{1}{12} Ma^2\omega\,(8,-3,-3)$$

$$\vec{L}_2(O) = \mathbb{I}(O)\cdot\vec{\omega}_2$$

$$= \frac{1}{\sqrt{3}} Ma^2\omega \begin{pmatrix} 2/3 & -1/4 & -1/4 \\ -1/4 & 2/3 & -1/4 \\ -1/4 & -1/4 & 2/3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \frac{1}{6\sqrt{3}} Ma^2\omega\,(1,1,1) = \frac{1}{6} Ma^2\vec{\omega}_2$$

For point C,

$$\vec{L}_1(C) = \mathbb{I}(C)\cdot\vec{\omega}_1 = \frac{1}{6} Ma^2\,\mathbb{1}\cdot\vec{\omega}_1 = \frac{1}{6} Ma^2\vec{\omega}_1$$

$$\vec{L}_2(C) = \mathbb{I}(C)\cdot\vec{\omega}_2 = \frac{1}{6} Ma^2\,\mathbb{1}\cdot\vec{\omega}_2 = \frac{1}{6} Ma^2\vec{\omega}_2$$

$\blacksquare$

The previous example shows something interesting,
For a particular choice of origin and/or axis of rotation, the relation $\vec{L} = \mathbb{I} \cdot \vec{\omega}$ simplifies such that $\vec{L} \parallel \vec{\omega}$.

This particular set of axis are called principal axes. The moment of inertia about the principal axes is called the principal moments, and generally the moment of inertia is

$$\mathbb{I} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

so that $\vec{L} = \lambda \vec{\omega}$.

Principal axes are associated with some symmetry axis.

Theorem: Existence of Principal Axes

For any rigid body and point O, $\exists$ three perpendicular axes through O s.t. $\mathbb{I}$ is diagonal.

$$\Rightarrow \mathbb{I}\vec{\omega} = \lambda \vec{\omega} \quad \text{and} \quad \vec{L} \parallel \vec{\omega}$$

To prove this, we need to recall some Linear Algebra...

Diagonalizing a Real-Symmetric Matrix

Let us remind ourself of some aspects of linear algebra, namely eigensystems & solutions.

Consider a real, symmetric $n \times n$ matrix $\mathbb{A}$.
We'd like to solve the eigenvalue equation

$$\mathbb{A}\vec{v} = \lambda\vec{v} \qquad , \quad \lambda = \text{eigenvalue}$$
$$\vec{v} = \text{eigenvector}$$

$(\mathbb{I}\vec{\omega} = \lambda\vec{\omega})$

number

This is equivalent to $(\mathbb{A} - \lambda\mathbb{1})\vec{v} = 0$

From linear algebra, we know that this has a nontrivial solution $(\vec{v} \neq \vec{0})$ iff $\det(\mathbb{A} - \lambda\mathbb{1}) = 0$.

characteristic eqn.

This is a polynomial of degree n. In general, it has n complex solutions.

For each solution λ_α, $\det(\mathbb{A} - \lambda_\alpha\mathbb{1}) = 0$, so $\exists$ a null vector $\vec{v}_\alpha$ of $\mathbb{A} - \lambda_\alpha\mathbb{1}$, i.e.,

$$\mathbb{A}\vec{v}_\alpha = \lambda_\alpha\vec{v}_\alpha \qquad (1)$$

and $\vec{v}_\alpha$ an eigenvector.

In general, $\vec{v}_\alpha \in \mathbb{C}^n$. Let's act (1) on the left by $\vec{v}_\alpha^{\,+} \equiv (v_\alpha^T)^*$ ($+ \equiv$ conjugate transpose)

$$\vec{v}_\alpha^{\,+} \mathbb{A} \vec{v}_\alpha = \lambda_\alpha \vec{v}_\alpha^{\,+} \vec{v}_\alpha$$

Note that $\vec{v}_\alpha^{\,+} \vec{v}_\alpha \equiv |\vec{v}_\alpha|^2 \in \mathbb{R}$

$$\Rightarrow \quad \lambda_\alpha = \frac{\vec{v}_\alpha^{\,+} \mathbb{A} \vec{v}_\alpha}{|\vec{v}_\alpha|^2}$$

Recall that, in general, $\lambda_\alpha \in \mathbb{C}$, so it is a 1×1 matrix and is symmetric, $\lambda_\alpha^T = \lambda_\alpha$

Similarly, $|\vec{v}_\alpha|^2 \in \mathbb{R} \Rightarrow (|\vec{v}_\alpha|^2)^T = |\vec{v}_\alpha|^2$

So, take transpose,

$$\lambda_\alpha = \lambda_\alpha^T = \frac{(\vec{v}_\alpha^{\,+} \mathbb{A} \vec{v}_\alpha)^T}{|\vec{v}_\alpha|^2}$$

Recall $(ABC)^T = C^T B^T A^T$

$$\Rightarrow (\vec{v}_\alpha^{\,+} \mathbb{A} \vec{v}_\alpha)^T = \vec{v}_\alpha^{\,T} \mathbb{A}^T \vec{v}_\alpha^{\,*}$$

$$= v_\alpha^{+*} \mathbb{A}^T \vec{v}_\alpha^{\,*}$$

Now, $\mathbb{A}$ is real and symmetric $\Rightarrow \mathbb{A}^T = \mathbb{A} = \mathbb{A}^*$

So,

$$\lambda_\alpha = \frac{\vec{V}_\alpha^{+*} \mathbb{A}^* \vec{V}_\alpha^*}{|\vec{V}_\alpha|^2}$$

$$= \left(\frac{\vec{V}_\alpha^{+} \mathbb{A} \vec{V}_\alpha}{|\vec{V}_\alpha|^2} \right)^* = \lambda_\alpha^*$$

$$\therefore \quad \lambda_\alpha = \lambda_\alpha^* \Rightarrow \boxed{\lambda_\alpha \in \mathbb{R}}$$

for real, symmetric matrix $\mathbb{A}$

Notice also

$$\mathbb{A}^* \vec{V}_\alpha^* = \lambda_\alpha^* \vec{V}_\alpha^* \Rightarrow \mathbb{A} \vec{V}_\alpha^* = \lambda_\alpha \vec{V}_\alpha^*$$

So, $\vec{V}_\alpha^*$ is also an eigenvector w/ same eigenvalue as $\vec{V}_\alpha$. $\Rightarrow$ Can take $\vec{V}_\alpha + \vec{V}_\alpha^*$, this must also be an eigenvector with eigenvalue λ_α.

But, $\vec{V}_\alpha + \vec{V}_\alpha^* = 2\mathrm{Re}(\vec{V}_\alpha) \in \mathbb{R}^n$.

$\Rightarrow$ Through suitable manipulations, all eigenvectors can be chosen to be real.

We may also normalize the eigenvectors

$$\vec{V}_\alpha \to \frac{\vec{V}_\alpha}{\sqrt{\vec{V}_\alpha^T \vec{V}_\alpha}}, \quad \text{so that } \vec{V}_\alpha^T V_\alpha = 1.$$

From now on, assume $\vec{v}_\alpha$ is normalized.

Finally, consider two eigenvalues $\lambda_\alpha \neq \lambda_\beta$.

Then,

$$\mathbb{A}\,\vec{v}_\alpha = \lambda_\alpha \vec{v}_\alpha$$

$$\& \quad \mathbb{A}\,\vec{v}_\beta = \lambda_\beta \vec{v}_\beta \quad \Rightarrow \quad \vec{v}_\beta^{\,T} \mathbb{A} = \lambda_\beta \vec{v}_\beta^{\,T}$$

take transpose

Act second eqn. on $\vec{v}_\alpha$

$$\Rightarrow \vec{v}_\beta^{\,T} \underbrace{\mathbb{A}\,\vec{v}_\alpha}_{} = \lambda_\beta \vec{v}_\beta \cdot \vec{v}_\alpha$$

$$\shortparallel$$

$$\vec{v}_\beta^{\,T} \cdot (\lambda_\alpha \vec{v}_\alpha) = \lambda_\alpha \vec{v}_\beta^{\,T} \cdot \vec{v}_\alpha$$

$$\therefore \quad (\lambda_\alpha - \lambda_\beta)\, \vec{v}_\beta \cdot \vec{v}_\alpha = 0 \quad \Rightarrow \quad \vec{v}_\beta^{\,T} \cdot \vec{v}_\alpha = 0$$

We conclude

$$\boxed{\vec{v}_\alpha^{\,T} \cdot \vec{v}_\beta = \delta_{\alpha\beta}} \qquad \text{(orthonormality)}$$

With all this, we can now show that $\mathbb{A}$ can be diagonalized as $\mathbb{A} = \mathbb{V}\,\mathbb{D}\,\mathbb{V}^T$ where $\mathbb{D}$ is diagonal matrix with the eigenvalues on the diagonal and $\mathbb{V}$ is an orthogonal matrix formed by placing $\vec{v}_\alpha$ at column α in the same order as λ_α in $\mathbb{D}$.

Proof. Since $\vec{v}_\alpha$ are orthonormal, they form a complete basis & we just need to show

$$\mathbb{A}\vec{v}_\alpha = (\mathbb{V}\mathbb{D}\mathbb{V}^T)\vec{v}_\alpha$$

In the usual basis of $\mathbb{A}$,
$$\vec{e}_1 = (1,0,0,\ldots,0)$$
$$\vec{e}_2 = (0,1,0,\ldots,0)$$
$$\vdots$$
$$\vec{e}_\alpha = (0,0,\ldots,\underset{\substack{\uparrow \\ \alpha^{\text{th}} \text{ element}}}{1},\ldots,0)$$

We can write the β-element of the α-basis vector $(\vec{e}_\alpha)_\beta = \delta_{\alpha\beta}$.

With this basis, we can expand the eigenvector as

$$\vec{v}_\alpha = \sum_\beta v_{\alpha,\beta}\,\vec{e}_\beta$$

Then, $\mathbb{V}_{\alpha\beta} = v_{\alpha,\beta}$ is an orthogonal matrix $\mathbb{V}^{-1} = \mathbb{V}^T$.
To see this, $\mathbb{V}\vec{e}_\alpha = \vec{v}_\alpha$, also

$$(\mathbb{V}^T\vec{v}_\alpha)_\beta = \sum_\gamma (\mathbb{V}^T)_{\beta\gamma}(\vec{v}_\alpha)_\gamma$$
$$= \sum_\gamma (\mathbb{V})_{\gamma\beta}(\vec{v}_\alpha)_\gamma = \sum_\gamma v_{\beta,\gamma}\,v_{\alpha,\gamma}$$
$$= \vec{v}_\beta^{\,T}\cdot\vec{v}_\alpha = \delta_{\alpha\beta}$$

$$\Rightarrow \mathbb{V}^T \vec{v}_\alpha = \vec{e}_\alpha \Rightarrow \mathbb{V}^T \mathbb{V} \vec{e}_\alpha = \vec{e}_\alpha$$

$$\Rightarrow \mathbb{V}^T \mathbb{V} = \mathbb{1} \quad \forall \alpha$$

Finally,

$$(\mathbb{V} \mathbb{D} \mathbb{V}^T) \vec{v}_\alpha = \mathbb{V} \mathbb{D} \vec{e}_\alpha = \lambda_\alpha \mathbb{V} \vec{e}_\alpha = \lambda_\alpha \vec{v}_\alpha \equiv \mathbb{A} \vec{v}_\alpha$$

$$\therefore \quad \boxed{\mathbb{A} = \mathbb{V} \mathbb{D} \mathbb{V}^T}$$

Principle Axes & Principal Moments

For the inertia tensor, a real symmetric matrix, we can diagonalize it, i.e., choose a set of axes, such that $\vec{L} \parallel \vec{\omega}$.

The diagonal elements are given by the eigenvalues of $\mathbb{I}$, i.e., the solutions to

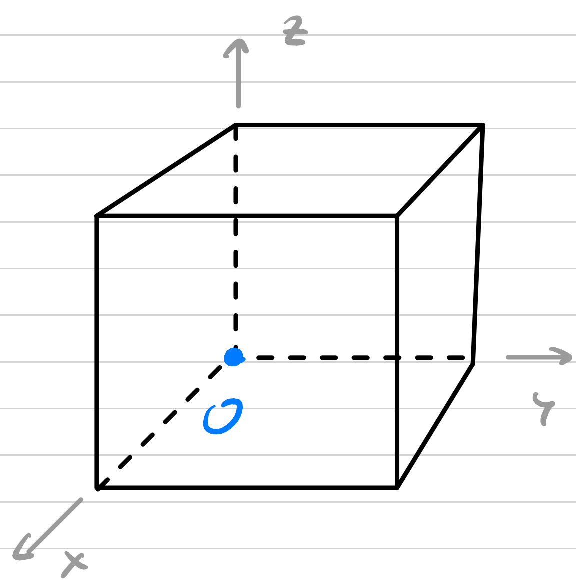
$$\boxed{\det(\mathbb{I} - \lambda \mathbb{1}) = 0}$$

Example - Principal axes of cube about corner?

Recall from previous example

$$\mathbb{I}_0 = Ma^2 \begin{pmatrix} 2/3 & -1/4 & -1/4 \\ -1/4 & 2/3 & -1/4 \\ -1/4 & -1/4 & 2/3 \end{pmatrix}$$

$$= \mu \begin{pmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{pmatrix}$$



with $\mu = \frac{1}{12} Ma^2$

So, we want to solve $\det(\mathbb{I}_0 - \lambda \mathbb{1}) = 0$

$$\Rightarrow \quad \det \begin{bmatrix} 8\mu - \lambda & -3\mu & -3\mu \\ -3\mu & 8\mu - \lambda & -3\mu \\ -3\mu & -3\mu & 8\mu - \lambda \end{bmatrix} = 0$$

To solve, note the following:

- Replacing a column (row) of a matrix with the sum of that column (row) & a multiple of another column (row) does NOT change det.

- Multiplying a column (row) by number, multiplies det by same number

- $$\det \begin{bmatrix} a_1 & \cdot & \cdot \\ 0 & a_2 & \cdot \\ 0 & 0 & a_3 \end{bmatrix} = \det \begin{bmatrix} a_1 & 0 & 0 \\ \cdot & a_2 & 0 \\ \cdot & \cdot & a_3 \end{bmatrix} = a_1 a_2 a_3$$

So, take

$$0 = \det \begin{bmatrix} 8\mu - \lambda & -3\mu & -3\mu \\ -3\mu & 8\mu - \lambda & -3\mu \\ -3\mu & -3\mu & 8\mu - \lambda \end{bmatrix}$$

$-$

take col 1 $\rightarrow$ col 1 $-$ col 2

$$\Rightarrow \quad 0 = \det \begin{bmatrix} 11\mu - \lambda & -3\mu & -3\mu \\ -11\mu + \lambda & 8\mu - \lambda & -3\mu \\ 0 & -3\mu & 8\mu - \lambda \end{bmatrix}$$

row2 → row 2 + row 1

$$= \det \begin{bmatrix} 11\mu - \lambda & -3\mu & -3\mu \\ 0 & 5\mu - \lambda & -6\mu \\ 0 & -3\mu & 8\mu - \lambda \end{bmatrix}$$

col 2 → col 2 − col 3

$$= \det \begin{bmatrix} 11\mu - \lambda & 0 & -3\mu \\ 0 & 11\mu - \lambda & -6\mu \\ 0 & -11\mu + \lambda & 8\mu - \lambda \end{bmatrix}$$

row 3 → row3 + row 2

$$= \det \begin{bmatrix} 11\mu - \lambda & 0 & -3\mu \\ 0 & 11\mu - \lambda & -6\mu \\ 0 & 0 & 2\mu - \lambda \end{bmatrix}$$

So, find

$$\det[\cdots] = (11\mu - \lambda)^2 (2\mu - \lambda) = 0$$

$$\Rightarrow \begin{cases} \lambda_1 = 2\mu \\ \lambda_2 = \lambda_3 = 11\mu \end{cases}$$

So, the principal moments are

$$\mathbb{I}_0' = \mathbb{D} = \mu\begin{pmatrix} 2 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{pmatrix}$$

What about the axes?
Want $\hat{e}_1, \hat{e}_2, \hat{e}_3$

First, find $\vec{\omega}_1 = \omega\hat{e}_1$ associated w/ λ_1
Solve $(\mathbb{I}_0 - \lambda_1 \mathbb{1})\vec{\omega}_1 = 0$

$$\Rightarrow \quad \mu\begin{pmatrix} 6 & -3 & -3 \\ -3 & 6 & -3 \\ -3 & -3 & 6 \end{pmatrix}\begin{pmatrix} \omega_{1,x} \\ \omega_{1,y} \\ \omega_{1,z} \end{pmatrix} = 0$$

$$\Rightarrow \quad 2\omega_{1,x} - \omega_{1,y} - \omega_{1,z} = 0 \qquad (a)$$
$$-\omega_{1,x} + 2\omega_{1,y} - \omega_{1,z} = 0 \qquad (b)$$
$$-\omega_{1,x} - \omega_{1,y} + 2\omega_{1,z} = 0 \qquad (c)$$

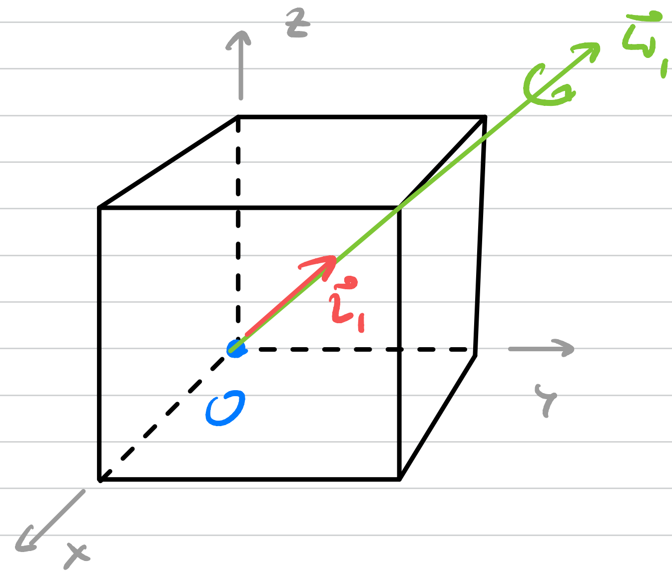
Take $(a)-(b) \Rightarrow \omega_{1,x} = \omega_{1,y}$
From $(a) \Rightarrow \omega_{1,x} = \omega_{1,y} = \omega_{1,z}$

So, $\vec{\omega} \parallel (1,1,1) \Rightarrow$ Normalize to find
$$\hat{e}_1 = \frac{1}{\sqrt{3}}(1,1,1)$$

This means that if $\vec{\omega}_1 = \omega \hat{e}_1$

that $\vec{L}_1 = \mathbb{I}\,\vec{\omega}_1 = \omega\,\mathbb{I}\,\hat{e}_1 = \omega \lambda_1 \hat{e}_1 = \lambda_1 \vec{\omega}_1$

$$\Rightarrow \vec{L}_1 = \lambda_1 \vec{\omega}_1$$



For $\vec{\omega}_2$ & $\vec{\omega}_3$, $\lambda_2 = \lambda_3$

solve

$$(\mathbb{I}_0 - \lambda_2 \mathbb{1})\,\vec{\omega} = 0$$

$$\Rightarrow \mu \begin{bmatrix} -3 & -3 & -3 \\ -3 & -3 & -3 \\ -3 & -3 & -3 \end{bmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = 0$$

$$\Rightarrow \quad \omega_x + \omega_y + \omega_z = 0$$

Notice that this is equal to $\vec{\omega} \cdot \hat{e}_1 = 0$

$\Rightarrow$ $\vec{\omega}$ needs to be orthogonal to $\hat{e}_1$

$\Rightarrow$ $\hat{e}_2$ & $\hat{e}_3$ need to be orthogonal to $\hat{e}_1$

Two such solutions are

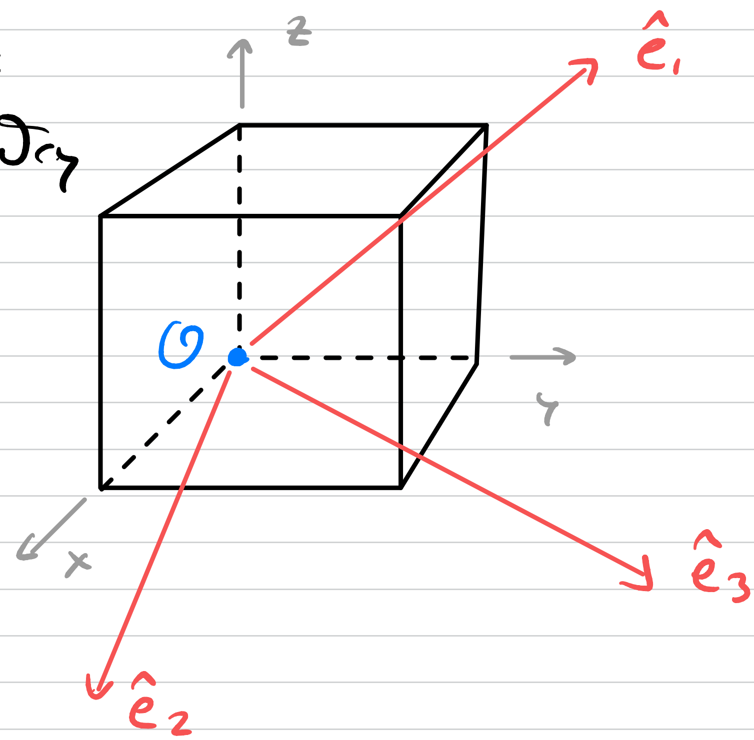
$$\hat{e}_2 = \frac{1}{\sqrt{2}}(1,0,-1), \quad \hat{e}_3 = \frac{1}{\sqrt{6}}(-1,2,-1)$$

(verify)

So, principal axes (eigenvectors) are

$$\hat{e}_1 = \frac{1}{\sqrt{3}}(1,1,1)\,, \quad \hat{e}_2 = \frac{1}{\sqrt{2}}(1,0,-1)\,, \quad \hat{e}_3 = \frac{1}{\sqrt{6}}(-1,2,-1)$$

The principal axes correspond to symmetry of the cube about the body diagonal with center at O.



Can decompose $\mathbb{I}_0 = \mathbb{V}\,\mathbb{D}_0\,\mathbb{V}^T$

with
$$\mathbb{D}_0 = \frac{1}{12} M a^2 \begin{pmatrix} 2 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{pmatrix}$$

$$\mathbb{V} = \frac{1}{\sqrt{6}} \begin{array}{c} \begin{pmatrix} \sqrt{2} & \sqrt{3} & -1 \\ \sqrt{2} & 0 & 2 \\ \sqrt{2} & -\sqrt{3} & -1 \end{pmatrix} \\ \begin{matrix} \uparrow & \quad\uparrow & \quad\uparrow \\ \hat{e}_1 & \quad\hat{e}_2 & \quad\hat{e}_3 \end{matrix} \end{array}$$

$\blacksquare$

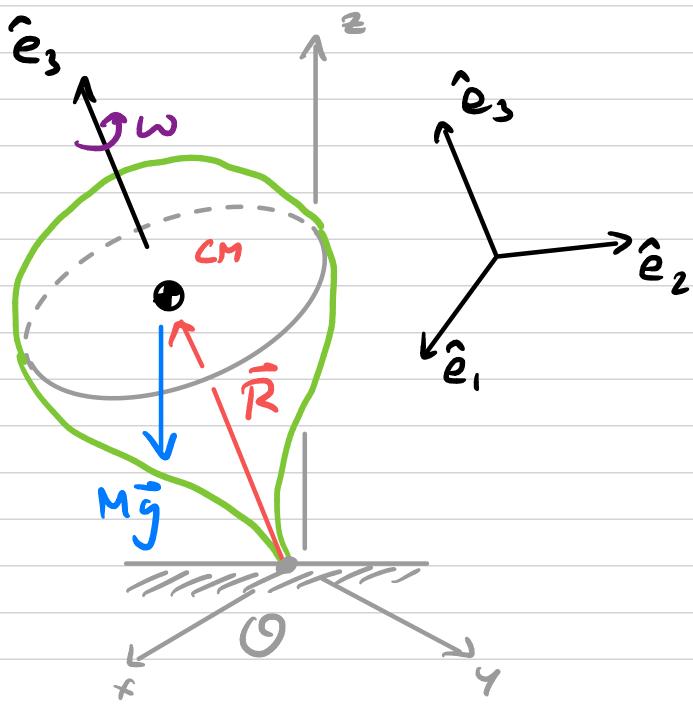
Precession of Symmetric Top due to Weak Torque

Having all the tools in our arsenal, let's apply them for a "simple" physics problem.

Consider a symmetric top with mass M and inertia tensor $\mathbb{I} = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$ in the basis of its principal axes, $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$.

It rotates freely with its tip pivoted at a fixed point O in a lab frame (inertial)

Its CM is at $\vec{R}$.



(initial config)

We assume its angular velocity is initially along its symmetry axis,

$$\vec{\omega} = \omega \hat{e}_3$$

The angular momentum is initially

$$\vec{L} = \mathbb{I}\vec{\omega} = \lambda_3 \omega \hat{e}_3$$

If we release the top, gravity will exert a torque, and cause a change of angular momentum.

The torque due to gravity is $\vec{\Gamma} = \vec{R} \times M\vec{g}$, at the CM wrt fixed point O.

- If $\vec{\Gamma} = \vec{0}$ (e.g., $\vec{R} \times \vec{g} = \vec{0}$)

  $\Rightarrow \vec{L} = \text{const.}$

To see, take

$$\left(\frac{d\vec{L}}{dt}\right)_{lab} = \lambda \frac{d}{dt}\left(\omega \hat{e}_3\right)_{lab}$$

$$= \lambda\left(\dot{\omega}\hat{e}_3 + \omega\left(\frac{d\hat{e}_3}{dt}\right)_{lab}\right) = 0$$

Since $\vec{\omega} \parallel \hat{e}_3 \Rightarrow \left(\frac{d\hat{e}_3}{dt}\right)_{lab} = \vec{\omega} \times \hat{e}_3 = \vec{0}$

That is, $\hat{e}_3$ is fixed in the lab frame too!

$\Rightarrow \dot{\omega} = 0$

- If $\vec{\Gamma} \neq \vec{0}$, but ω_1, ω_2 are small, so $\vec{\omega} \simeq \omega_3 \hat{e}_3$

$$\Rightarrow \vec{\Gamma} \perp \vec{\omega} \text{ and } \dot{\omega} \text{ remains small}$$

So, EOM gives $\left(\dfrac{d\vec{L}}{dt}\right)_{lab} = \vec{\Gamma}$

$$\Rightarrow \lambda\left(\cancelto{0}{\dot{\omega}}\,\hat{e}_3 + \omega\left(\frac{d\hat{e}_3}{dt}\right)_{lab}\right) = \vec{R} \times M\vec{g}$$

Since $\dot{\omega} \simeq 0$, $\left(\dfrac{d\hat{e}_3}{dt}\right)_{lab} = \dfrac{1}{\lambda\omega}\vec{R} \times M\vec{g}$

Now, $\vec{R} = R\,\hat{e}_3$, $\vec{g} = -g\,\hat{z}$ $\Rightarrow$ $\vec{R} \times \vec{g} = Rg\,\hat{z} \times \hat{e}_3$

$$\Rightarrow \boxed{\left(\frac{d\hat{e}_3}{dt}\right)_{lab} = \left(\frac{MRg}{\lambda\omega}\right)\hat{z} \times \hat{e}_3}$$

Define $\vec{\Omega} = \dfrac{MRg}{\lambda\omega}\,\hat{z}$

$\Rightarrow$ the symmetry axis of the top rotates w/ $\Omega = \dfrac{MRg}{\lambda\omega}$ about $\hat{z}$ axis

$\Rightarrow$ <u>precession</u>