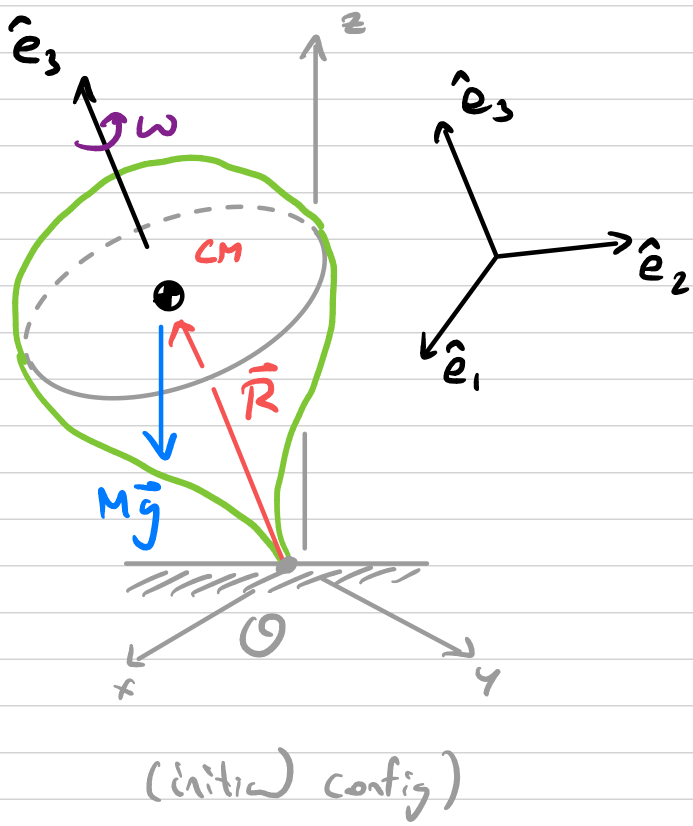
Precession of Symmetric Top due to Weak Torque

Having all the tools in our arsenal, let's apply them for a "simple" physics problem.

Consider a symmetric top with mass M and inertia tensor $\mathbb{I} = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$ in the basis of its principal axes, $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$.

It rotates freely with its tip pivoted at a fixed point O in a lab frame (inertial)

Its CM is at $\vec{R}$.



We assume its angular velocity is initially along its symmetry axis,

$$\vec{\omega} = \omega \hat{e}_3$$

The angular momentum is initially

$$\vec{L} = \mathbb{I}\vec{\omega} = \lambda_3 \omega \hat{e}_3$$

If we release the top, gravity will exert a torque, and cause a change of angular momentum.

The torque due to gravity is $\vec{\Gamma} = \vec{R} \times M\vec{g}$, at the CM wrt fixed point O.

- If $\vec{\Gamma} = \vec{0}$ (e.g., $\vec{R} \times \vec{g} = \vec{0}$)

$$\Rightarrow \vec{L} = \text{const.}$$

To see, take

$$\left(\frac{d\vec{L}}{dt}\right)_{lab} = \lambda_3 \frac{d}{dt}\left(\omega \hat{e}_3\right)_{lab}$$

$$= \lambda_3\left(\dot{\omega}\hat{e}_3 + \omega\left(\frac{d\hat{e}_3}{dt}\right)_{lab}\right) = 0$$

Since $\vec{\omega} \parallel \hat{e}_3 \Rightarrow \left(\frac{d\hat{e}_3}{dt}\right)_{lab} = \vec{\omega} \times \hat{e}_3 = \vec{0}$

That is, $\hat{e}_3$ is fixed in the lab frame too!

$$\Rightarrow \dot{\omega} = 0$$

- If $\vec{\Gamma} \neq \vec{0}$, but ω_1, ω_2 are small, so $\vec{\omega} \simeq \omega_3 \hat{e}_3$

$$\Rightarrow \vec{\Gamma} \perp \vec{\omega} \quad \text{and} \quad \dot{\omega} \text{ remains small}$$

So, EOM gives $\left(\frac{d\vec{L}}{dt}\right)_{lab} = \vec{\Gamma}$

$$\Rightarrow \lambda\left(\cancelto{0}{\dot{\omega}}\hat{e}_3 + \omega\left(\frac{d\hat{e}_3}{dt}\right)_{lab}\right) = \vec{R} \times M\vec{g}$$

Since $\dot{\omega} \simeq 0$, $\left(\frac{d\hat{e}_3}{dt}\right)_{lab} = \frac{1}{\lambda\omega}\vec{R} \times M\vec{g}$

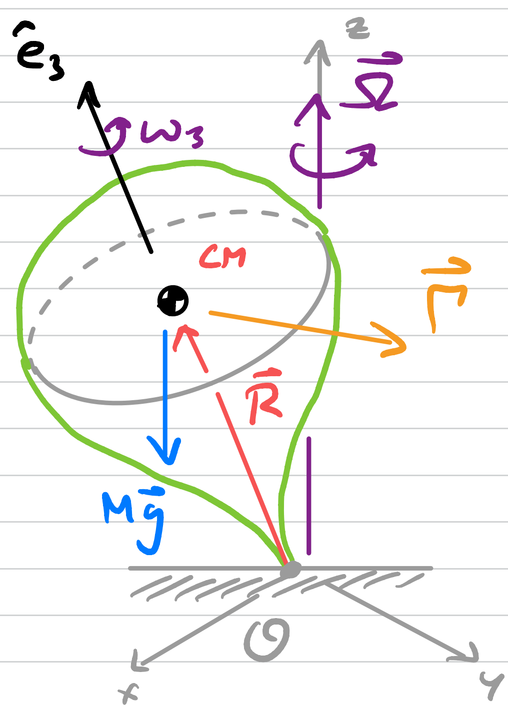
Now, $\vec{R} = R\hat{e}_3$, $\vec{g} = -g\hat{z}$ $\Rightarrow$ $\vec{R} \times \vec{g} = Rg\,\hat{z} \times \hat{e}_3$

$$\Rightarrow \boxed{\left(\frac{d\hat{e}_3}{dt}\right)_{lab} = \left(\frac{MRg}{\lambda\omega}\right)\hat{z} \times \hat{e}_3}$$

Define $\vec{\Omega} = \frac{MRg}{\lambda\omega}\hat{z}$

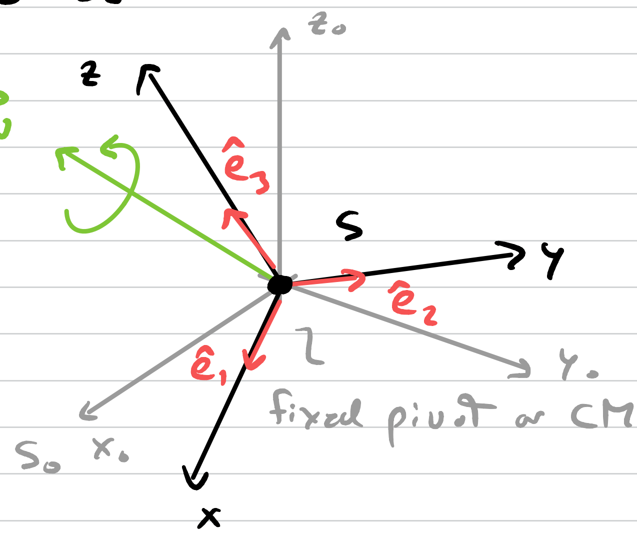
$\Rightarrow$ the symmetry axis of the top rotates w/ $\Omega = \frac{MRg}{\lambda\omega}$ about $\hat{z}$ axis

$\Rightarrow$ precession



Eulers Equations

Newton's Law's of motion for a body rotating in some inertial (space-fixed) frame S_0



$$\left(\frac{d\vec{L}}{dt}\right)_{S_0} = \vec{\Gamma}$$

Recall that

$$\left(\frac{d\vec{L}}{dt}\right)_{S_0} = \left(\frac{d\vec{L}}{dt}\right)_{S} + \vec{\omega}\times\vec{L}$$

Where S is a frame fixed to the body.

Therefore, we find

$$\boxed{\dot{\vec{L}} + \vec{\omega}\times\vec{L} = \vec{\Gamma}}$$ in body-fixed frame S

Notice: If $\vec{\Gamma}=\vec{0} \Rightarrow \vec{L}$ is conserved in S_0

BUT NOT in S!

The use of the body-fixed frame makes it possible to choose the principle axes of the moment of inertia, $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$, as the coordinate axes.

Therefore,

$$\vec{L} = \mathbb{I}\,\vec{\omega} = \lambda_1 \omega_1 \hat{e}_1 + \lambda_2 \omega_2 \hat{e}_2 + \lambda_3 \omega_3 \hat{e}_3$$

So, $\dot{\vec{L}} + \vec{\omega} \times \vec{L} = \vec{\Gamma}$

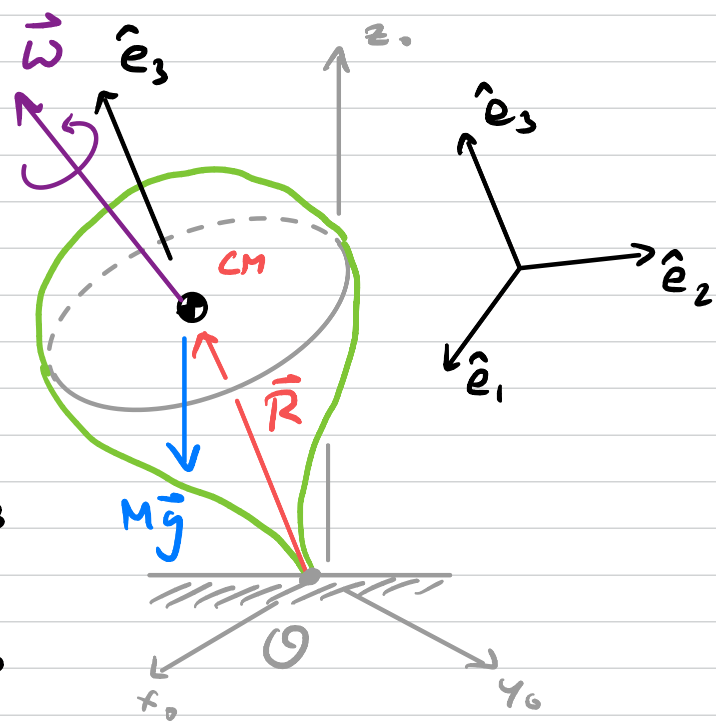
$$\boxed{\begin{aligned} \lambda_1 \dot{\omega}_1 - (\lambda_2 - \lambda_3)\,\omega_2 \omega_3 &= \Gamma_1 \\ \lambda_2 \dot{\omega}_2 - (\lambda_3 - \lambda_1)\,\omega_3 \omega_1 &= \Gamma_2 \\ \lambda_3 \dot{\omega}_3 - (\lambda_1 - \lambda_2)\,\omega_1 \omega_2 &= \Gamma_3 \end{aligned}}$$

These are called the Euler Equations of rotational motion

An example is the symmetric spinning top we considered before

$\lambda_1 = \lambda_2$

& $\Gamma_3 = 0 \Rightarrow \lambda_3 \dot{\omega}_3 = 0$



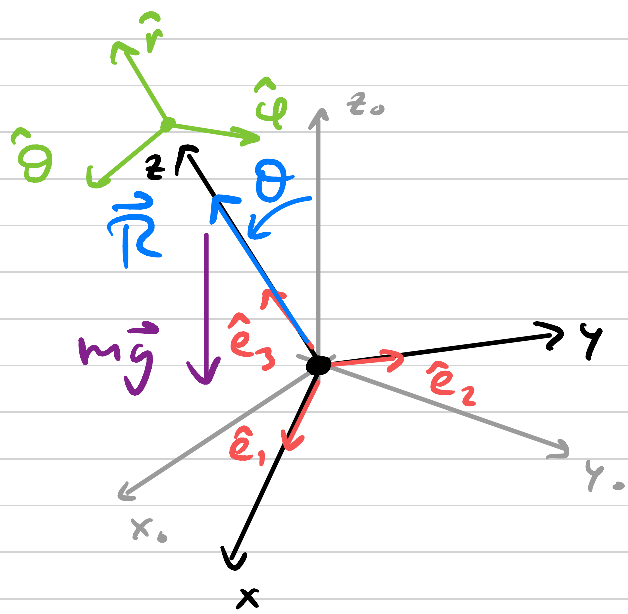
The other two Euler eqns. are

$$\begin{cases} \dot{\omega}_1 = \dfrac{1}{\lambda_1}\Gamma_1 - \left(\dfrac{\lambda_3}{\lambda_1} - 1\right)\omega_2 \omega_3 \\ \dot{\omega}_2 = \dfrac{1}{\lambda_2}\Gamma_2 + \left(\dfrac{\lambda_3}{\lambda_1} - 1\right)\omega_1 \omega_3 \end{cases}$$

Take the axis ($\hat{e}_3$) to be at spherical angles (θ, φ) in the space frame, we find $\vec{\Gamma} \parallel \hat{\varphi}$.

Let's choose the axes at time $t=0$ such that

$\hat{e}_2$? $\hat{e}_1 \parallel \vec{\Gamma} \parallel \hat{y}_0$ and $\hat{e}_3$ is in the xz plane.



The CM is at $\vec{R} = R\hat{r}$

where $m\vec{g}$ is given by

$$m\vec{g} = mg(-\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta})$$

At time t, $\hat{e}_1$ will have rotated by an angle $\omega_3 t$ around $\hat{e}_3$ (since $\dot{\omega}_3 = 0 \Rightarrow \omega_3 = \text{const.}$)

Therefore, $\vec{\Gamma}_1 = \Gamma \sin\omega_3 t$, $\vec{\Gamma}_2 = \Gamma \cos\omega_3 t$

where $$\Gamma = |\vec{\Gamma}| = MgR\sin\theta$$

To solve the remaining Euler eqns., we introduce a complex variable

$$\eta = \omega_1 + i\omega_2$$

Therefore,

$$\dot{\eta} = \dot{\omega}_1 + i\dot{\omega}_2$$
$$= \frac{1}{\lambda_1}(\Gamma_1 + i\Gamma_2) + \left(\frac{\lambda_3}{\lambda_1} - 1\right)\omega_3(-\omega_2 + i\omega_1)$$

$$\Rightarrow \dot{\eta} = \frac{i\Gamma}{\lambda_1} e^{-i\omega_3 t} + i\left(\frac{\lambda_3}{\lambda_1} - 1\right)\omega_3 \eta .$$

We can write the solution as $\eta = \eta_h + \eta_p$

homogeneous general solution

particular solution

The homogeneous eqn is

$$\dot{\eta} = i\left(\frac{\lambda_3}{\lambda_1} - 1\right)\omega_3 \eta ,$$

which by direct integration gives homogeneous solution

$$\eta_h = A \exp\left[i\left(\frac{\lambda_3}{\lambda_1} - 1\right)\omega_3 t\right]$$

TBD from ICs

The particular solution is $\eta_p = \eta_0 e^{-i\omega_3 t}$

Fix η_0 by substituting into EOM,

$$-i\omega_3\eta_0 = \frac{i\Gamma}{\lambda_1} + i\left(\frac{\lambda_3}{\lambda_1} - 1\right)\omega_3\eta_0$$

$$\Rightarrow \eta_0 = -\frac{\Gamma}{\lambda_3\omega_3}$$

To fix A, the ICs are $\omega_1 = \omega_2 = 0$ @ $t=0$

$$\Rightarrow \quad \eta(0) = 0 = A + \eta_0 \quad \Rightarrow \quad A = -\eta_0$$

So, altogether

$$\eta(t) = \frac{\Gamma}{\lambda_3 \omega_3} \left[e^{i\left(\frac{\lambda_3}{\lambda_1} - 1\right)\omega_3 t} - e^{-i\omega_3 t} \right]$$

Writing $\vec{\omega} = \vec{\omega}_\perp + \omega_3 \hat{e}_3$, the small torque condition is

$$|\eta| = |\vec{\omega}_\perp| \ll \omega_3 \qquad \& \quad \theta \ll 1$$

$$\Rightarrow \quad \frac{\Gamma}{\lambda_3 \omega_3} \ll \omega_3 \quad \Rightarrow \quad \underbrace{\frac{MgR}{\lambda_3 \omega_3}}_{\equiv \Omega = \text{ the precession frequency}} \ll \omega_3$$

Zero Torque

The zero-torque Euler eqns. are

$$\boxed{\begin{aligned} \lambda_1 \dot{\omega}_1 &= (\lambda_2 - \lambda_3)\omega_2 \omega_3 \\ \lambda_2 \dot{\omega}_2 &= (\lambda_3 - \lambda_1)\omega_3 \omega_1 \\ \lambda_3 \dot{\omega}_3 &= (\lambda_1 - \lambda_2)\omega_1 \omega_2 \end{aligned}}$$

Let's consider again the "free" motion of a symmetric top, $\lambda_1 = \lambda_2$ w/ $\hat{e}_3$ being the symmetry axis.

Then $\lambda_3 \dot{\omega}_3 = 0$

& $\dot{\omega}_1 = -\left(\frac{\lambda_3}{\lambda_1} - 1\right)\omega_2\omega_3$, $\dot{\omega}_2 = +\left(\frac{\lambda_3}{\lambda_1} - 1\right)\omega_1\omega_3$

As before, introduce $\eta = \omega_1 + i\omega_2$

$\Rightarrow$ $\dot{\eta} = i\Omega_b \eta$ where $\Omega_b = \left(\frac{\lambda_3}{\lambda_1} - 1\right)\omega_3$

The general solution is $\eta = A e^{i\Omega_b t}$

So, free precession

$$\boxed{\vec{\omega} = A\cos\Omega_b t\,\hat{e}_1 + A\sin\Omega_b t\,\hat{e}_2 + \omega_3\hat{e}_3}$$

Notice that since $\vec{L} = \lambda_1\omega_1\hat{e}_1 + \lambda_2\omega_2\hat{e}_2 + \lambda_3\omega_3\hat{e}_3$

$= \lambda_1 A\cos\Omega_b t\,\hat{e}_1 + \lambda_2 A\sin\Omega_b t\,\hat{e}_2 + \lambda_3\omega_3\hat{e}_3$

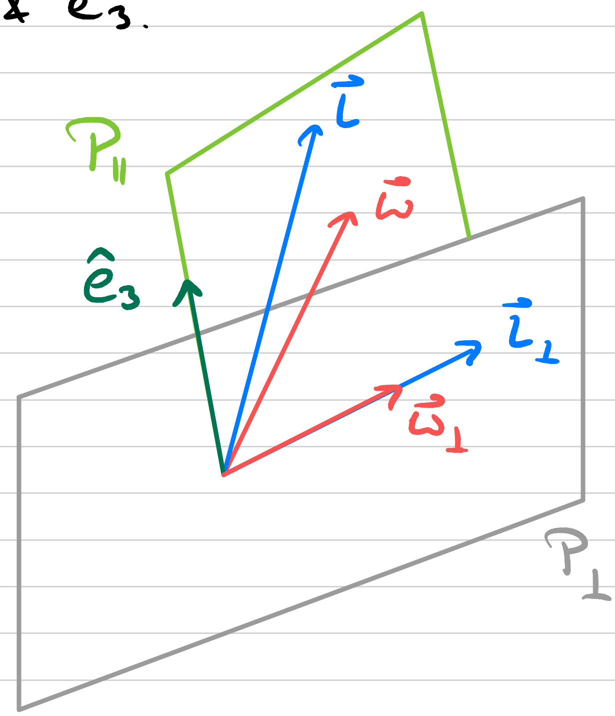
But, $\lambda_1 = \lambda_2$ $\Rightarrow$ $\vec{L} = \lambda_1 A\left(\cos\Omega_b t\,\hat{e}_1 + \sin\Omega_b t\,\hat{e}_2\right) + \lambda_3\omega_3\hat{e}_3$

Define $\vec{\omega}_\perp$ via $\vec{\omega} = \vec{\omega}_\perp + \omega_3\hat{e}_3$

where $\vec{\omega}_\perp \cdot \hat{e}_3 = 0$

Then,
$$\vec{L} = \mathbb{I}\cdot\vec{\omega} = \lambda_1\vec{\omega}_\perp + \lambda_3\omega_3\hat{e}_3$$

Geometrically, since the perpendicular component of $\vec{L}$,
$\vec{L}_\perp = \lambda_1 \vec{\omega}_\perp$, is parallel to $\vec{\omega}_\perp$, $\vec{L}$ should be in the same plane as $\vec{\omega}$ & $\hat{e}_3$.



Analytically, we can show three vectors, $\vec{A}, \vec{B}, \vec{C}$ are in the same plane iff
$\vec{A}\cdot(\vec{B}\times\vec{C}) = 0$

$$\vec{L}\cdot(\hat{e}_3\times\vec{\omega})$$
$$= \lambda_1 \vec{\omega}_\perp \cdot (\hat{e}_3 \times \vec{\omega}_\perp) + \lambda_3 \omega_3 \hat{e}_3 \cdot (\hat{e}_3 \times \vec{\omega}_\perp)$$
$$= 0$$

Thus, we see that $\vec{\omega}$, $\vec{L}$, & $\hat{e}_3$ all lie in the same plane.

What if $\lambda_1 \neq \lambda_2$?

For a body whose principal moments of inertia are all different, we may assume $\lambda_1 < \lambda_3 < \lambda_2$ to find

$$\begin{cases} \dot{\omega}_1 = \left(\dfrac{\lambda_2 - \lambda_3}{\lambda_1}\omega_3\right)\omega_2 \\ \dot{\omega}_2 = \left(\dfrac{\lambda_3 - \lambda_1}{\lambda_2}\omega_3\right)\omega_1 \end{cases}$$

As long as ω_1 & ω_2 are small, $\dot{\omega}_3$ remains small & we can take ω_3 to be constant. Then, the two coupled equations for ω_1 & ω_2 can be solved easily

$$\ddot{\omega}_1 = \left(\frac{\lambda_2-\lambda_3}{\lambda_1}\,\omega_3\right)\dot{\omega}_2 = \left[\frac{(\lambda_2-\lambda_3)(\lambda_3-\lambda_1)}{\lambda_1\lambda_2}\,\omega_3^2\right]\omega_1$$

Since $(\lambda_2-\lambda_3)(\lambda_3-\lambda_1)>0$, this solution will have a real exponential solution — which is unstable & grows rapidly.

In fact, a very strong statement about this instability can be shown to be true:

There is no purely decaying solution $\sim e^{-\alpha t}$, $\alpha>0$ for any initial conditions.

To see this, note that $\mathrm{sgn}(\dot{\omega}_1) = \mathrm{sgn}(\omega_3)\cdot\mathrm{sgn}(\omega_2)$

For an initial value $\omega_1(t=0) = \omega_{1,0}$, we have

$\mathrm{sgn}(\dot{\omega}_2(t=0)) = \mathrm{sgn}(\omega_3)\cdot\mathrm{sgn}(\omega_{1,0})$.

Shortly after $t=0$, $\mathrm{sgn}(\omega_2) = \mathrm{sgn}(\dot{\omega}_2(t=0)) = \mathrm{sgn}(\omega_3)\cdot\mathrm{sgn}(\omega_{1,0})$

So,

$$\mathrm{sgn}(\dot{\omega}_1(t=0)) = \mathrm{sgn}(\omega_3)\cdot\mathrm{sgn}(\omega_2) = \mathrm{sgn}(\omega_{1,0}).$$

This is a constraint that must be satisfied by all initial conditions & solutions.

$\Rightarrow$ It is NOT satisfied by a purely decaying solution:

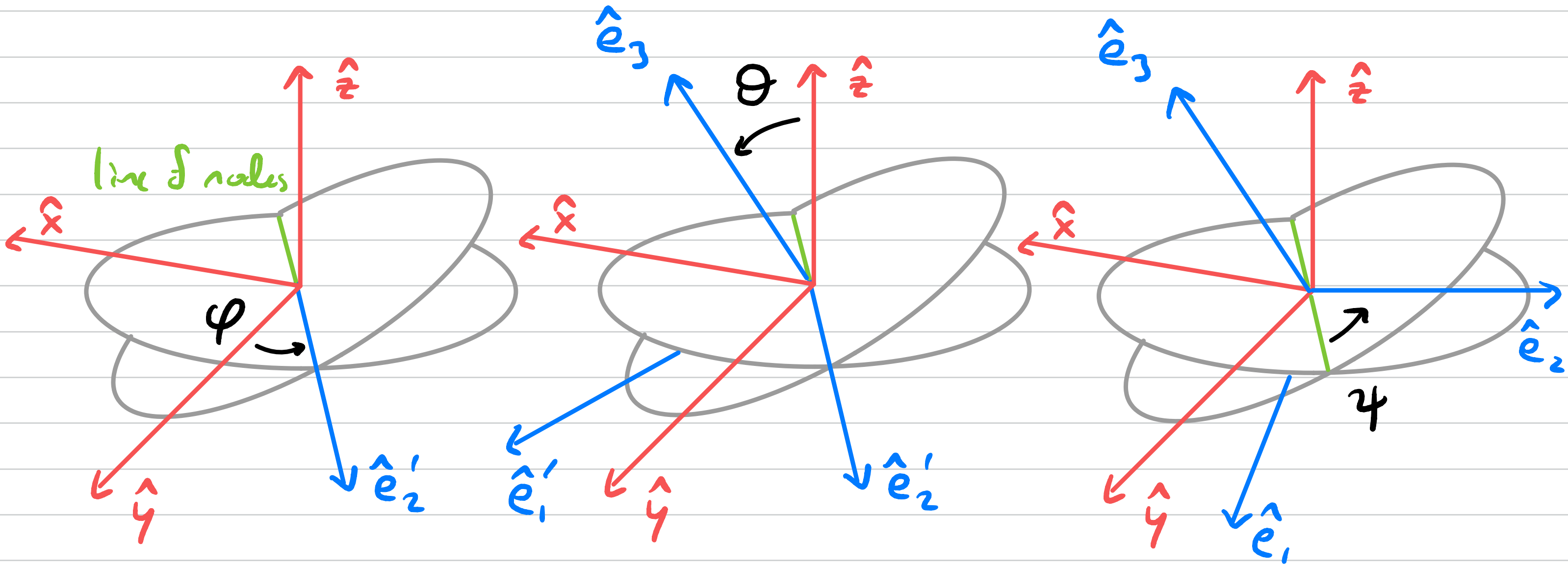
$$\omega_1 \sim e^{-\alpha t} \Rightarrow \dot{\omega}_{1,0} = -\alpha\,\omega_{1,0}$$

$$\Rightarrow \mathrm{sgn}(\dot{\omega}_{1,0}) = -\mathrm{sgn}(\omega_{1,0})$$

which is a contradiction.

Euler Angles

The Euler angles are a special set of angles which are frequently used to describe rotational motion



From some fixed coordinate system (x, y, z) [zyz convention]

1. Rotate about $\hat{z}$ by φ
2. Rotate about $\hat{e}_2'$ by θ
3. Rotate about $\hat{e}_3$ by ψ

By definition, $\vec{\omega} = \dot{\varphi}\,\hat{z} + \dot{\theta}\,\hat{e}_2' + \dot{\psi}\,\hat{e}_3$

Can decompose $\hat{z}$, $\hat{e}_1'$, & $\hat{e}_2'$ into $\hat{e}_1, \hat{e}_2, \hat{e}_3$

$$\begin{cases} \hat{z} = \cos\theta\,\hat{e}_3 - \sin\theta\,\hat{e}_1' \\ \hat{e}_1' = \cos\psi\,\hat{e}_1 - \sin\psi\,\hat{e}_2 \\ \hat{e}_2' = \sin\psi\,\hat{e}_1 + \cos\psi\,\hat{e}_2 \end{cases}$$

Therefore

$$\boxed{\begin{aligned}\vec{\omega} = &(-\dot{\varphi}\sin\theta\cos\psi + \dot{\theta}\sin\psi)\hat{e}_1 \\ &+ (\dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi)\hat{e}_2 \\ &+ (\dot{\varphi}\cos\theta + \dot{\psi})\hat{e}_3\end{aligned}}$$

We can also express $\vec{\omega}$ in terms of $\hat{x}, \hat{y}, \hat{z}$

$$\begin{cases} \hat{e}_2' = -\sin\varphi\,\hat{x} + \cos\varphi\,\hat{y} \\ \hat{e}_3 = \sin\theta\cos\varphi\,\hat{x} + \sin\theta\sin\varphi\,\hat{y} + \cos\theta\,\hat{z} \end{cases}$$

Therefore,

$$\boxed{\begin{aligned}\vec{\omega} = &(-\dot{\theta}\sin\varphi + \dot{\psi}\sin\theta\cos\varphi)\hat{x} \\ &+ (\dot{\theta}\cos\varphi + \dot{\psi}\sin\theta\sin\varphi)\hat{y} \\ &+ (\dot{\varphi} + \dot{\psi}\cos\theta)\hat{z}\end{aligned}}$$

Note that for a body w/ 2 axis of symmetry parallel to $\hat{e}_3$, then $\lambda_1 = \lambda_2$ for any chosen axes $\perp$ to $\hat{e}_3$.

For example, $\hat{e}_1'$ & $\hat{e}_2'$ work just as well as $\hat{e}_1$ & $\hat{e}_2$

$$\Rightarrow \quad \vec{\omega} = -\dot{\varphi}\sin\theta\,\hat{e}_1' + \dot{\theta}\,\hat{e}_2' + (\dot{\varphi}\cos\theta + \dot{\psi})\hat{e}_3$$

We find then ($\lambda_1 = \lambda_2$)

$$\vec{L} = \mathbb{I}\,\vec{\omega}$$
$$= (-\lambda_1 \dot{\varphi} \sin\theta)\,\hat{e}_1' + \lambda_1 \dot{\theta}\,\hat{e}_2' + \lambda_3 (\dot{\varphi}\cos\theta + \dot{\psi})\,\hat{e}_3$$

Note that

$$\hat{e}_1' \cdot \hat{z} = -\sin\theta \,, \quad \hat{e}_2' \cdot \hat{z} = 0 \,, \quad \hat{e}_3 \cdot \hat{z} = \cos\theta$$

$$\Rightarrow \begin{cases} L_z = \vec{L}\cdot\hat{z} = \lambda_1 \dot{\varphi} \sin^2\theta + L_3 \cos\theta \\ L_3 = \lambda_3 (\dot{\varphi}\cos\theta + \dot{\psi}) \end{cases}$$

Finally, the kinetic energy can be written

$$\boxed{T = \frac{1}{2}\vec{\omega}\cdot\vec{L} = \frac{1}{2}\lambda_1 (\dot{\varphi}^2 \sin^2\theta + \dot{\theta}^2) + \frac{1}{2}\lambda_3 (\dot{\varphi}\cos\theta + \dot{\psi})^2}$$

Spinning Top

We are now in position to the solve for the full motion of a symmetric spinning top.
The Lagrangian is

$$\mathcal{L} = T - U$$
$$= \frac{1}{2}\lambda_1 (\dot{\varphi}^2 \sin^2\theta + \dot{\theta}^2) + \frac{1}{2}\lambda_3 (\dot{\varphi}\cos\theta + \dot{\psi})^2 - MgR\cos\theta$$

Notice that φ is cyclic

$$p_\varphi = \frac{\partial \mathcal{L}}{\partial \dot\varphi} = \lambda_1 \dot\varphi \sin^2\theta + \lambda_3 \cos\theta (\dot\varphi \cos\theta + \dot\psi)$$

$$= \text{const.} \equiv L_z$$

Notice that also ψ is cyclic

$$p_\psi = \frac{\partial \mathcal{L}}{\partial \dot\psi} = \lambda_3 (\dot\varphi \cos\theta + \dot\psi) = \text{const.} \equiv L_3$$

finally, for θ

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot\theta} \Rightarrow \lambda_1 \ddot\theta = \lambda_1 \dot\varphi^2 \sin\theta\cos\theta - \lambda_3 \dot\varphi \sin\theta (\dot\varphi \cos\theta + \dot\psi) + MgR\sin\theta$$

The total energy is conserved

$$E = T + U$$
$$= \frac{1}{2}\lambda_1 (\dot\varphi^2 \sin^2\theta + \dot\theta^2) + \frac{1}{2}\lambda_3 (\dot\varphi\cos\theta + \dot\psi)^2 + MgR\cos\theta$$

Writing in terms of L_z & L_3

$$E = \frac{1}{2}\lambda_1 \dot\theta^2 + \frac{1}{2}\lambda_1 \left(\frac{L_z - L_3\cos\theta}{\lambda_1 \sin^2\theta}\right)^2 \sin^2\theta + \frac{1}{2}\lambda_3 \left(\frac{L_3}{\lambda_3}\right)^2 + MgR\cos\theta$$

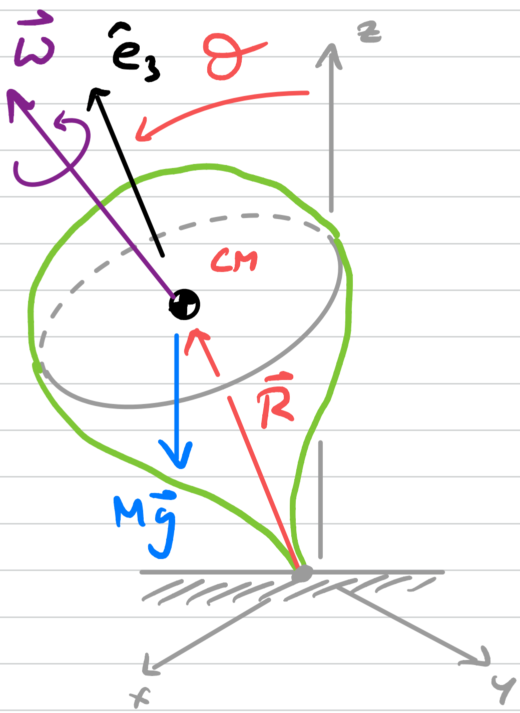
We can write this as

$$E = \frac{1}{2}\lambda_1 \dot{\theta}^2 + U_{eff}(\theta)$$

where the effective potential is

$$\boxed{U_{eff}(\theta) = \frac{(L_z - L_3\cos\theta)^2}{2\lambda_1 \sin^2\theta} + \frac{L_3^2}{2\lambda_3} + MgR\cos\theta}$$

Depending on the energy of the top, the angle θ = angle between the symmetry axis & the z-direction will vary between two limiting values



If the energy is exactly matched to be at the minimum of U_{eff}, θ will be fixed. An especially simple example arises, since

$$\dot{\varphi} = \frac{L_z - L_3\cos\theta}{\lambda_1 \sin^2\theta} = \text{const.}$$

Since $\dot{\varphi} \equiv \Omega$ is constant, the top will precess at a fixed angle θ & constant angular velocity.

From the expression for energy,

$$\frac{dU_{eff}}{d\theta} = 0 \Rightarrow \frac{dU_{eff}}{d\cos\theta} = 0$$

$$\Rightarrow \frac{(L_z - L_3\cos\theta)^2\cos\theta}{\lambda_1 \sin^4\theta} - L_3 \cdot \frac{L_z - L_3\cos\theta}{\lambda_1 \sin^2\theta} + MgR = 0$$

Since $\Omega = \dfrac{L_z - L_3\cos\theta}{\lambda_1 \sin^2\theta}$, we find

$$\underbrace{\lambda_1 \cos\theta}_{a}\, \Omega^2 \underbrace{- L_3}_{b}\, \Omega + \underbrace{MgR}_{c} = 0$$

$$\Rightarrow a\Omega^2 + b\Omega + c = 0$$

There are two solutions: $\Omega_\pm = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

When $b^2 \gg ac$, we expand $\sqrt{b^2 - 4ac} \simeq b\left(1 - \dfrac{2ac}{b^2}\right)$

So,

$$\begin{cases} \Omega_+ \simeq -\dfrac{c}{b} = \dfrac{MgR}{L_3} = \dfrac{MgR}{\lambda_3 \omega_3} & \leftarrow \text{precession} \\ \Omega_- \simeq -\dfrac{b}{a} = \dfrac{L_3}{\lambda_1 \cos\theta} & \leftarrow \text{free precession} \end{cases}$$

For the angular momentum, we have

$$\vec{L} = -\lambda_1 \dot{\varphi} \sin\theta \, \hat{e}_1' + L_3 \hat{e}_3$$

Since $\hat{z}$, $\hat{e}_1'$, & $\hat{e}_3$ are all on the same plane, the horizontal component of $\vec{L}$ is

$$L_h = -\lambda_1 \dot{\varphi} \sin\theta \cos\theta + L_3 \sin\theta$$

For the large solution $\Omega_- \simeq -\frac{b}{a} = + \frac{L_3}{\lambda_1 \cos\theta}$, we find

$$L_h \simeq -\lambda_1 \cdot \frac{L_3}{\lambda_1 \cos\theta} \sin\theta \cos\theta + L_3 \sin\theta = 0$$

That is, $\vec{L}$ is nearly vertical.