William & Mary
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PHYS 303 – Classical Mechanics of Particles and Waves II

Example Problem Set

Due: Not for Credit

Term: Fall 2024

Instructor: Andrew W. Jackura

Readings

Read chapter 2 of Taylor.

Problems

Problem 1. [30 pts.] – Projectile Motion in Linear Resistive Medium

A projectile of mass m is launched from the surface of the Earth with an initial velocity $\mathbf{v}_0$ at time $t = 0$. The projectiles trajectory is such that the gravity field $\mathbf{g}$ is assumed constant. The projectile moves in a medium which retards its motion with a magnitude proportional to its velocity, $\mathbf{F}_{\text{drag}} = -b\mathbf{v}$, with b being the positive drag coefficient.

(**a**) **[5 pts.]** Determine the projectiles equations of motion. Show that the velocity $\mathbf{v}$ as a function of time t is given by

$$\mathbf{v}(t) = \mathbf{g}\tau + (\mathbf{v}_0 - \mathbf{g}\tau)\, e^{-t/\tau}\,,$$

where $\tau = m/b$. What are the physical dimensions of b and τ?

(**b**) **[5 pts.]** If the projectile is launched from the origin, show that its position as a function of time is

$$\mathbf{r}(t) = \mathbf{g}\tau t + \tau\,(\mathbf{v}_0 - \mathbf{g}\tau)\,\left(1 - e^{-t/\tau}\right)\,.$$

Show that if the effect of drag is negligible, that is $b \sim \tau^{-1} \ll 1$, that $\mathbf{r}(t)$ reduces to the trajectory for a projectile in a uniform gravity field with no drag. *Hint:* Recall the Taylor expansion $e^x = 1 + x + \mathcal{O}(x^2)$.

(**c**) **[10 pts.]** Consider now a coordinate system such that $\mathbf{g} = (0, 0, -g)$ and $\mathbf{v}_0 = v_0(\cos\theta_0, 0, \sin\theta_0)$ where θ_0 is the launch angle with respect to the Earth's surface. Show that the equation for the range R of the projectile is of the form

$$\tan\theta_0 R + \frac{g\tau}{v_0\cos\theta_0}R + g\tau^2\ln\left(1 - \frac{R}{v_0\tau\cos\theta_0}\right) = 0\,.$$

This equation is transcendental in R, and thus cannot be solved analytically. An approximate solution can be made for systems where the effect of drag is small, that is $b \sim \tau^{-1} \ll 1$. Use the Taylor expansion $\ln(1 + x) = x - x^2/2 + x^3/3 + \mathcal{O}(x^4)$ to generate the approximate algebraic equation

$$\tan\theta_0 - \frac{g}{2}\frac{R}{v_0^2\cos\theta_0} - \frac{g}{3}\frac{R^2}{\tau v_0^3\cos^3\theta_0} + \mathcal{O}\big(\tau^{-2}\big) = 0\,.$$

(**d**) **[10 pts.]** To solve the approximate equation for R in part (**c**), we construct a perturbative expansion for R of the form

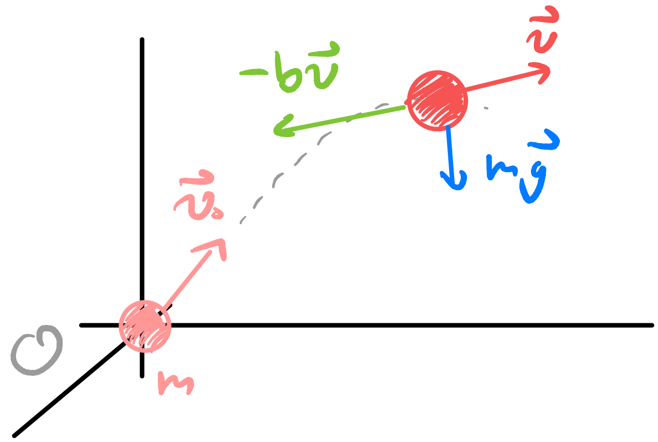
$$R = R_0 - \alpha\tau^{-1} + \mathcal{O}(\tau^{-2}),$$

where R_0 is the range of the projectile when drag is ignored and α is a positive constant to be determined. Determine α, and the resulting shift in the range $\Delta R = R_0 - R$ in terms of v_0, θ_0, g, and τ. Verify that the shift decreases the range with respect to the vacuum case.

Physics 303 - Example Problem Set

Andrew Jackura

1. projectile
   - mass, m
   - initial velocity, $\vec{v}_0$ @ $t=0$



Forces
- gravity, $m\vec{g}$ = const.
- drag, $-b\vec{v}$

a) Find $\vec{v}(t)$

From NII: $m\ddot{\vec{r}} = m\vec{g} - b\vec{v}$

Now, recall $\dot{\vec{r}} = \vec{v}$, so EOM is

$$\boxed{\dot{\vec{v}} = \vec{g} - \frac{b}{m}\vec{v}}$$

Define for convenience $\tau = m/b$

Note that $[m] = M$, $\boxed{[b] = MT^{-1}}$

$$\Rightarrow \boxed{[\tau] = [m][b]^{-1} = T}$$

So, we solve $\dfrac{d\vec{v}}{dt} + \dfrac{1}{\tau}\vec{v} = \vec{g}$ via integrating factor

$\mu = e^{t/\tau}$ s.t. $\dfrac{1}{\tau} = \dfrac{d}{dt} e^{t/\tau}$ $\Rightarrow \dfrac{d}{dt}(\mu\vec{v}) = \mu\dot{\vec{v}} + \dfrac{1}{\tau}\vec{v}$

So, EOM is $\dfrac{d}{dt}\left(e^{t/\tau}\vec{v}\right) = \vec{g}\, e^{t/\tau}$

We can integrate easily

$$\Rightarrow \int dt \frac{d}{dt}(e^{t/\tau}\vec{v}) = \int dt\, \vec{g}\, e^{t/\tau}$$

$$\Rightarrow \quad e^{t/\tau}\vec{v} = \tau\vec{g}\, e^{t/\tau} + \vec{c}$$

From IC's, $\vec{v} = \vec{v}_0$ at $t=0$

$$\Rightarrow \quad \vec{v}_0 = \tau\vec{g} + \vec{c} \Rightarrow \quad \vec{c} = \vec{v}_0 - \tau\vec{g}$$

So, solution is

$$\boxed{\vec{v}(t) = \vec{g}\tau + (\vec{v}_0 - \vec{g}\tau)e^{-t/\tau}}$$

as required

b) We now want $\vec{r}(t)$ such that $\vec{r}(0) = \vec{0}$.

Recall $\frac{d\vec{r}}{dt} = \vec{v}$

$$\Rightarrow \quad \vec{r}(t) = \int_0^t dt' \, \vec{v}(t')$$

$$= \int_0^t dt' \left[\vec{g}\tau + (\vec{v}_0 - \vec{g}\tau)e^{-t/\tau}\right]$$

$$= \vec{g}\tau t' \Big|_0^t + (\vec{v}_0 - \vec{g}\tau)(-\tau)e^{-t/\tau}\Big|_0^t$$

$$\boxed{\vec{r}(t) = \vec{g}\tau\, t + \tau(\vec{v}_0 - \vec{g}\tau)(1 - e^{-t/\tau})}$$

If drag is negligible, $b \ll 1 \Rightarrow \tau^{-1} = \frac{b}{m} \ll 1$

Taylor expansion $e^{x} = 1 + x + \frac{1}{2}x^2 + \mathcal{O}(x^3)$

$$\Rightarrow e^{-t/\tau} = 1 - \frac{t}{\tau} + \frac{1}{2}\left(\frac{t}{\tau}\right)^2 + \mathcal{O}(\tau^{-3})$$

So,

$$\vec{r}(t) = g\tau t + \tau(\vec{v}_0 - \vec{g}\tau)\left[1 - \left(1 - \frac{t}{\tau} + \frac{1}{2}\frac{t^2}{\tau^2} + \cdots\right)\right]$$

$$= \cancel{\vec{g}\tau t} + (\vec{v}_0 - \cancel{\vec{g}\tau})\left[t - \frac{1}{2}\frac{t^2}{\tau} + \mathcal{O}(\tau^{-2})\right]$$

$$= \vec{v}_0 t + \frac{1}{2}\vec{g}t^2 - \underbrace{\frac{1}{2}\frac{\vec{v}_0}{\tau}t^2}_{\mathcal{O}(\tau^{-1})} + \mathcal{O}(\tau^{-2})$$

$$= \underbrace{\vec{v}_0 t + \frac{1}{2}\vec{g}t^2}_{\text{trajectory for motion w/o drag.}} + \mathcal{O}(\tau^{-1})$$

c) choosing coordinate system $\vec{g} = (0, 0, -g)$
and $\vec{v}_0 = (v_0\cos\theta_0, 0, v_0\sin\theta_0)$
our equations for $\vec{r}(t)$ in this system are

$$\begin{cases} x(t) = \tau v_0 \cos\theta_0 (1 - e^{-t/\tau}) \\ y(t) = 0 \\ z(t) = -g\tau t + \tau(v_0\sin\theta_0 + g\tau)(1 - e^{-t/\tau}) \end{cases}$$

Solving for $t = t(x)$,

$$\frac{x}{\tau v_0 \cos\theta_0} = 1 - e^{-t/\tau} \Rightarrow e^{-t/\tau} = 1 - \frac{x}{\tau v_0 \cos\theta_0}$$

or, $$t = -\tau \ln\left(1 - \frac{x}{\tau v_0 \cos\theta_0}\right)$$

so, $z = z(x)$ is

$$z(x) = g\tau^2 \ln\left(1 - \frac{x}{\tau v_0 \cos\theta_0}\right) + \frac{(v_0 \sin\theta_0 + g\tau)\,x}{v_0 \cos\theta_0}$$

When $x = R$, $z(R) = 0$

$$\Rightarrow \boxed{0 = \tan\theta_0 R + \frac{g\tau}{v_0 \cos\theta_0} R + g\tau^2 \ln\left(1 - \frac{R}{\tau v_0 \cos\theta_0}\right)}$$

If drag is small, $b \sim \tau^{-1} \ll 1$,

Taylor expansion $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + O(x^4)$

$$\Rightarrow 0 = \tan\theta_0 R + \cancel{\frac{g\tau}{v_0 \cos\theta_0} R}$$
$$+ g\tau^2\left[\cancel{-\frac{R}{\tau v_0 \cos\theta_0}} - \frac{R^2}{2\tau^2 v_0^2 \cos^2\theta_0} - \frac{R^3}{3\tau^3 v_0^3 \cos^3\theta_0} + \cdots\right]$$

Since $R = 0$ is not physical,

$$\Rightarrow \boxed{0 = \tan\theta_0 - \frac{g}{2 v_0^2 \cos^2\theta_0} R - \frac{g}{3\tau v_0^3 \cos^3\theta_0} R^2 + O(\tau^{-2})}$$

d) Now, we construct expansion

$$R = R_0 - \alpha \tau^{-1} + \mathcal{O}(\tau^{-2})$$

with $\alpha > 0$, & R_0 = range with no drag.

We want to determine $\Delta R = R_0 - R$.

Inserting $R = R_0 - \alpha\tau^{-1}$ into

$$0 = \tan\theta_0 - \frac{g}{2v_0^2\cos^2\theta_0} R - \frac{g}{3\tau v_0^3 \cos^3\theta_0} R^2 + \mathcal{O}(\tau^{-2})$$

$$\Rightarrow 0 = \tan\theta_0 - \frac{g}{2v_0^2\cos^2\theta_0}(R_0 - \alpha\tau^{-1})$$
$$- \frac{g}{3\tau v_0^3\cos^3\theta_0}(R_0^2 \underbrace{- 2\alpha R_0 \tau^{-1} + \alpha^2 \tau^{-2}}_{\text{can ignore} \sim \mathcal{O}(\tau^{-1})}) + \mathcal{O}(\tau^{-2})$$

$$\Rightarrow 0 = \left(\tan\theta_0 - \frac{g}{2v_0^2\cos^2\theta_0} R_0\right)$$
$$+ \left(\frac{g}{2v_0^2\cos^2\theta_0}\alpha - \frac{g}{3v_0^3\cos^3\theta_0} R_0^2\right)\tau^{-1} + \mathcal{O}(\tau^{-2})$$

Each coefficient of τ^{-n} must vanish

$\mathcal{O}(\tau^0)$: $\quad \tan\theta_0 = \frac{g}{2v_0^2\cos^2\theta_0} R_0$

$$\Rightarrow R_0 = \frac{2v_0^2}{g}\sin\theta_0\cos\theta_0$$

$\hookrightarrow$ range with no drag

$O(\tau^{-1})$: $\quad \dfrac{g}{2v_0^2\cos^2\theta_0}\alpha - \dfrac{g}{3v_0^3\cos^3\theta_0}R_0^2 = 0$

$$\Rightarrow \quad \alpha = \frac{2}{3v_0\cos\theta_0}R_0^2$$

$$= \frac{2}{3v_0\cos\theta_0}\left(4\frac{v_0^4}{g^2}\sin^2\theta_0\cos^2\theta_0\right)$$

$$= \frac{8}{3}\frac{v_0^3}{g^2}\sin^2\theta_0\cos\theta_0$$

Therefore $\quad R = R_0 - \alpha\tau^{-1}$

So, $\quad \Delta R = R_0 - R$

$$= \alpha\tau^{-1}$$

$$\Rightarrow \boxed{\begin{aligned}\Delta R &= \frac{8}{3}\frac{v_0^3}{g^2}\sin^2\theta_0\cos\theta_0\\ &= \frac{4}{3}\frac{v_0^3}{g^2}\sin 2\theta_0\sin\theta_0\end{aligned}}$$

$\sin 2\theta = 2\sin\theta\cos\theta$

Notice that $v_0 > 0$, $g > 0$, $\sin^2\theta_0 > 0$ & $\cos\theta_0 > 0$ for $0 \le \theta_0 \le \frac{\pi}{2}$

So, $\Delta R > 0$, thus the shift decreases