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PHYS 303 – Classical Mechanics of Particles and Waves II

Problem Set 1

Due: Thursday, September 05 at 5:00pm

Term: Fall 2024

Instructor: Andrew W. Jackura

Readings

Review chapters 1–7 of *Classical Mechanics*, by Taylor. Read chapters 8.1–8.5 of Taylor.

Problems

Problem 1. [15 pts.] – Approximate Harmonic Oscillator

The potential energy of a one-dimensional mass m at a distance r from the origin is

$$U(r) = U_0 \left(\frac{r}{R} + \lambda^2 \frac{R}{r} \right) ,$$

for $0 < r < \infty$, with U_0, R, and λ all positive constants.

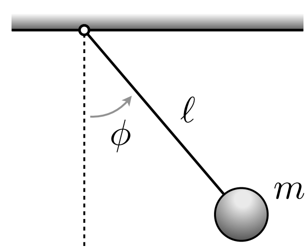
(**a**) **[3 pts.]** What are the physical dimensions of U_0, R, and λ?

(**b**) **[7 pts.]** Find the equilibrium position r_0. Let x be the distance from equilibrium and show that, for small x, the potential energy has the form $U = \text{const} + \frac{1}{2}kx^2$.

(**c**) **[5 pts.]** Determine the angular frequency of small oscillations.

Problem 2. [35 pts.] – Oscillations of a Pendulum

Consider a simple pendulum, consisting of a point mass m, fixed to the end of a massless rod of length ℓ, whose other end is pivoted from the ceiling to let it swing freely in a vertical plane. The pendulum's position is specified by its angle ϕ from its equilibrium position.



(**a**) **[5 pts.]** Show that the pendulum's potential energy, measured from the equilibrium point, is

$$U(\phi) = mg\ell(1 - \cos\phi) .$$

Write down the total energy E as a function of ϕ and $\dot{\phi}$.

(**b**) **[10 pts.]** Differentiate E with respect to t to derive an equation of motion for ϕ. Assuming that the angle ϕ remains small throughout its motion, solve for $\phi(t)$ and show that the motion is periodic with period

$$\tau_0 = 2\pi\sqrt{\ell/g} .$$

Hint: Recall the Taylor expansion $\sin x = x - x^3/3! + \mathcal{O}(x^5)$.

(**c**) **[10 pts.]** Now consider the pendulum is released from rest at some *not necessarily small* angle Φ at $t = 0$. Use conservation of energy to write a relation for $\dot{\phi}$ in terms of ϕ. Show that the period of oscillation is given by

$$\tau = 4\int_{\Phi}^{0} \frac{\mathrm{d}\phi}{\dot{\phi}} = \frac{\tau_0}{\pi}\int_0^{\Phi} \mathrm{d}\phi \, \frac{1}{\sqrt{\sin^2(\Phi/2) - \sin^2(\phi/2)}} \, .$$

Defining $A = \sin(\Phi/2)$, manipulate the integral to show that the period is

$$\tau = \frac{2\tau_0}{\pi}\int_0^1 \mathrm{d}u \, \frac{1}{\sqrt{1-u^2}\sqrt{1-A^2u^2}} .$$

Hint: Recall the half-angle identities for the trigonometric functions.

(**d**) **[10 pts.]** The integral is known as the *complete elliptic integral of the first kind*, and cannot be evaluated analytically. Assume that the release angle Φ is small. Show that

$$A = \frac{1}{2}\Phi - \frac{1}{48}\Phi^3 + \mathcal{O}(\Phi^5) \, ,$$

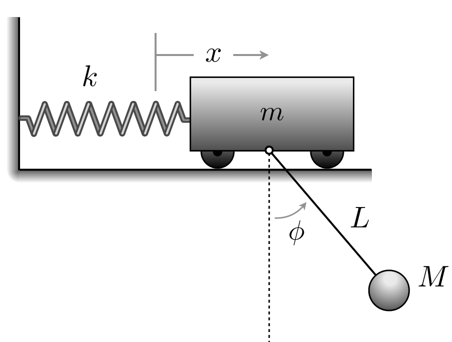
and hence A is also small. Use the binomial expansion $1/\sqrt{1-A^2u^2} = 1 + A^2u^2/2 + \mathcal{O}(A^4)$ to show that the leading order approximation gives the usual period for a simple harmonic oscillator $\tau = \tau_0 + \mathcal{O}(\Phi^2)$. Determine the $\mathcal{O}(\Phi^2)$ correction to the period $\Delta\tau \equiv \tau - \tau_0$.

Problem 3. [20 pts.] – Coupled Oscillators

A simple pendulum (mass M and length L) is suspended from a cart (mass m) that can oscillate on the end of a spring of force constant k.

(**a**) **[10 pts.]** Write down the Lagrangian in terms of the two generalized coordinates x and ϕ, where x is the extension of the spring from its equilibrium length and ϕ is the angle of the pendulum measured from its equilibrium position.

(**b**) **[10 pts.]** Find the two Lagrange equations of motion. Simplify the two equations in the case where *both* x and ϕ are small.



Problem 4. [30 pts.] – Rocket Motion under Gravity

Consider a toy rocket with mass m and velocity v traveling in a straight line. The rocket is subject to an external force F acting along the same line. The rocket expels fuel at a constant velocity $-v_{\text{ex}}$ relative to itself.

(**a**) **[5 pts.]** Show that the rocket's equation of motion is

$$m\frac{\mathrm{d}v}{\mathrm{d}t} + v_{\text{ex}}\frac{\mathrm{d}m}{\mathrm{d}t} = F\,.$$

(**b**) **[5 pts.]** Suppose the rocket takes off vertically from rest in a constant gravity field g. Assume that the rocket ejects a mass at constant rate $k = -\dot{m}$ (where $k > 0$) with m_0 being the initial mass of the rocket measured at $t = 0$. Solve the equation of motion for $v(t)$.

(**c**) **[5 pts.]** Integrate $v(t)$ and show that the rockets height y as a function of t is

$$y(t) = v_{\text{ex}}t - \frac{1}{2}gt^2 - \frac{mv_{\text{ex}}}{k}\ln\left(\frac{m_0}{m}\right)\,,$$

where we have set the initial position at the origin.

(**d**) **[5 pts.]** Let m_e be the mass of the empty rocket, which is 10% of the initial mass of the rocket. Determine the height of the rocket at burnout, i.e., the point the fuel is depleted, in terms of the given variables m_0, k, g, and v_{ex}.

(**e**) **[10 pts.]** After burnout, the empty rocket continues on its trajectory under the influence of gravity. Find the maximum height of the rocket with respect to the ground. What is the total flight time of the rocket in terms of the given variables?

Extra Credit [15 pts.] – Rocket Motion with Drag

Repeat **Problem 4 (b)** where now the rocket is also subjected to a linear resistive force $\mathbf{F}_{\text{drag}} = -b\mathbf{v}$, $b > 0$. Derive a result for the rockets velocity $v(t)$ in terms of g, b, k, m_0, and m. Show that if we ignored the effect of gravity, the rockets speed reduces to

$$v(t) = \frac{k}{b}\,v_{\text{ex}}\left[1 - \left(\frac{m}{m_0}\right)^{b/k}\right]\,.$$

Further, show that if we also ignored the effect of drag, that the speed simplifies to $v(t) = v_{\text{ex}}\ln(m_0/m)$. *Hint:* Recall the change-of-base formula $a^x = e^{x\ln(a)}$, and the Taylor expansion $e^x = 1 + x + \mathcal{O}(x^2)$. A general first order linear ordinary differential equation (ODE) is of the form

$$\dot{v}(t) + P(t)v(t) = Q(t)\,,$$

where $P(t)$ and $Q(t)$ are general functions. This class of ODEs can be solved by introducing an integrating factor $\mu(t)$,

$$\mu(t) = \exp\left(\int_{t_0}^{t}\mathrm{d}t'\,P(t')\right)\,,$$

such that

$$v(t) = \frac{1}{\mu(t)}\int_{t_0}^{t}\mathrm{d}t'\,\mu(t')Q(t')\,.$$