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PHYS 303 – Classical Mechanics of Particles and Waves II

Problem Set 3

Due: Thursday, September 19 at 5:00pm

Term: Fall 2024

Instructor: Andrew W. Jackura

Readings

Read sections 14.1–14.8 of Taylor.

Problems

Problem 1. [10 pts.] – Free Particle Motion in Polar Coordinates

We saw that the trajectory of a free particle in polar coordinates is

$$r(\phi) = \frac{r_0}{\cos(\phi - \phi_0)}\,,$$

where $r_0 = r(\phi_0)$. Show that this is an equation of a straight line by writing it in Cartesian coordinates (x, y) and casting it in standard form $ax + by = c$, where a, b, and c are constants to be determined.

Problem 2. [10 pts.] – Kepler's First Law

We showed in lecture that any Kepler orbit of an inverse-square force law can be written in the form

$$r(\phi) = \frac{c}{1 + \epsilon \cos\phi}\,,$$

where $c > 0$ and $\epsilon > 0$. For the case $0 \le \epsilon < 1$, rewrite this equation in Cartesian coordinates (x, y) and prove that this equation can be cast in the form

$$\frac{(x+d)^2}{a^2} + \frac{y^2}{b^2} = 1\,,$$

where a, b, and d are constants to be determined in terms of c and ϵ.

Problem 3. [15 pts.] – Halley's Comet

Halley's comet, which passed around the sun early in 1986, moves in a highly elliptical orbit of eccentricity 0.967 and a period of 76 years. Calculate its minimum and maximum distances from the sun.

Problem 4. [25 pts.] – **Logarithmic Spiral**

The relative motion of a two-body system with reduced mass μ is that of a logarithmic spiral given by $r = ke^{\alpha\varphi}$, where k and α are constants.

(**a**) **[5 pts.]** Find the central force law responsible for this relative motion in terms of k, α, μ, and the angular momentum ℓ. *Hint:* Recall that in cylindrical polar coordinates (r, φ) the equations of motion are

$$\mu r^2 \dot{\varphi} = \ell\,, \qquad \mu \ddot{r} = \mu r \dot{\varphi}^2 - \frac{\partial U(r)}{\partial r}\,,$$

where $U(r)$ is the potential energy of the central force.

(**b**) **[10 pts.]** Unlike inverse-square force laws, we can solve for the time dependence of both r and φ. If given that $\varphi = 0$ at $t = 0$, determine $\varphi(t)$ and $r(t)$.

(**c**) **[10 pts.]** Determine the total energy E of the orbit, assuming we define $U(r \to \infty) = 0$.

Problem 5. [20 pts.] – **Precession**

A particle of mass m moves with angular momentum ℓ in the field of a fixed force center with

$$F(r) = -\frac{k}{r^2} + \frac{\lambda}{r^3}\,,$$

where k and λ are positive constants. Recall that the equations of motion for this system are

$$mr^2\dot{\varphi} = \ell\,, \qquad m\ddot{r} = F(r) + \frac{\ell^2}{mr^3}\,,$$

(**a**) **[10 pts.]** Show that under a transformation $u = 1/r$ that the radial equation becomes

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\varphi^2} = -\alpha^2 u + \beta\,,$$

where α, and β are positive constants to be determined in terms of m, ℓ, k and λ.

(**b**) **[10 pts.]** Assuming that $\varphi = 0$ is defined to align with the periapsis, show that the solution to the radial equation gives an orbit

$$r(\varphi) = \frac{c}{1 + \epsilon\cos(\alpha\varphi)}\,,$$

where c and ϵ are positive constants. Find c in terms of the given parameters. Describe the orbit for the case $0 < \epsilon < 1$.

(**c**) **[5 pts.]** Are there any values of α for which the orbit is closed?