Generating vertex Feynman rules

Given an interaction Lagrange density $\mathcal{L}_{\text{int.}}$ which is a functional of some generic field $\varphi^a_{i_a}$ where $a = 1, \dots, n$ denotes the type of field and i_a is its a generic index for the representation of the field under Poincaré and internal symmetry transformations. The momentum space interaction vertex $i\Gamma(p_1, \dots, p_n)$ is then given by

$$(2\pi)^4\,\delta^{(4)}(p_1 + p_2 + \cdots + p_n)\,i\Gamma_{i_1\cdots i_n}(p_1, \dots, p_n) = \prod_{a=1}^{n} \frac{\delta}{\delta\tilde{\varphi}^a_{i_a}(p_a)} \left(i \int \mathrm{d}^4x\, \mathcal{L}_{\text{int.}} \right), \tag{1}$$

where $\tilde{\varphi}^a_{i_a}(p)$ is the Fourier transform of the field,

$$\varphi^a_{i_a}(x) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\, e^{-ik\cdot x}\, \tilde{\varphi}^a_{i_a}(p)\,. \tag{2}$$

We assume that all momenta are flowing into the vertex. We will give two examples of how to generate the vertex function $i\Gamma$.

Self-Interacting Scalar Field Theory

Fir consider φ^4 theory with interacting Lagrange density

$$\mathcal{L}_{\text{int.}} = -\frac{\lambda}{4!}\varphi(x)^4\,. \tag{3}$$

The vertex function is defined by

$$(2\pi)^4\,\delta^{(4)}(p_1 + p_2 + p_3 + p_4)\,i\Gamma = \frac{\delta}{\delta\tilde{\varphi}(p_1)}\frac{\delta}{\delta\tilde{\varphi}(p_2)}\frac{\delta}{\delta\tilde{\varphi}(p_3)}\frac{\delta}{\delta\tilde{\varphi}(p_4)} \left(-i\frac{\lambda}{4!} \int \mathrm{d}^4x\, \varphi(x)^4 \right), \tag{4}$$

where the Fourier transform of the field is given by

$$\varphi(x) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\, e^{-ik\cdot x}\, \tilde{\varphi}(p)\,. \tag{5}$$

Inserting this into the vertex function, we find

$$\begin{aligned}
&(2\pi)^4\,\delta^{(4)}(p_1 + p_2 + p_3 + p_4)\,i\Gamma \\
&\quad = -i\frac{\lambda}{4!}\frac{\delta}{\delta\tilde{\varphi}(p_1)}\frac{\delta}{\delta\tilde{\varphi}(p_2)}\frac{\delta}{\delta\tilde{\varphi}(p_3)}\frac{\delta}{\delta\tilde{\varphi}(p_4)} \int\frac{\mathrm{d}^4k_1}{(2\pi)^4}\int\frac{\mathrm{d}^4k_2}{(2\pi)^4}\int\frac{\mathrm{d}^4k_3}{(2\pi)^4}\int\frac{\mathrm{d}^4k_4}{(2\pi)^4} \\
&\qquad \times \int \mathrm{d}^4x\, e^{-i(k_1+k_2+k_3+k_4)\cdot x}\, \tilde{\varphi}(k_1)\tilde{\varphi}(k_2)\tilde{\varphi}(k_3)\tilde{\varphi}(k_4)\,.
\end{aligned} \tag{6}$$

Note that the integration over x can be performed to give a Dirac delta function,

$$(2\pi)^4\,\delta^{(4)}(k_1 + k_2 + k_3 + k_4) = \int \mathrm{d}^4x\, e^{-i(k_1+k_2+k_3+k_4)\cdot x}\,. \tag{7}$$

One of the momentum integrals can then be performed (we choose k_4), so that $k_4 = -k_1 - k_2 - k_3$,

$$\begin{aligned}
(2\pi)^4\,\delta^{(4)}(p_1 + p_2 + p_3 + p_4)\,i\Gamma &= -i\frac{\lambda}{4!}\frac{\delta}{\delta\tilde{\varphi}(p_1)}\frac{\delta}{\delta\tilde{\varphi}(p_2)}\frac{\delta}{\delta\tilde{\varphi}(p_3)}\frac{\delta}{\delta\tilde{\varphi}(p_4)} \int\frac{\mathrm{d}^4k_1}{(2\pi)^4}\int\frac{\mathrm{d}^4k_2}{(2\pi)^4}\int\frac{\mathrm{d}^4k_3}{(2\pi)^4} \\
&\qquad \times \tilde{\varphi}(k_1)\tilde{\varphi}(k_2)\tilde{\varphi}(k_3)\tilde{\varphi}(-k_1 - k_2 - k_3)\,,
\end{aligned} \tag{8}$$

We now perform the functional derivatives. The first derivative gives 4 terms,

$$\begin{aligned}
&\frac{\delta}{\delta\tilde{\varphi}(p_1)}\frac{\delta}{\delta\tilde{\varphi}(p_2)}\frac{\delta}{\delta\tilde{\varphi}(p_3)}\frac{\delta}{\delta\tilde{\varphi}(p_4)}\,\tilde{\varphi}(k_1)\tilde{\varphi}(k_2)\tilde{\varphi}(k_3)\tilde{\varphi}(-k_1-k_2-k_3)\,,\\
&\quad=(2\pi)^4\,\frac{\delta}{\delta\tilde{\varphi}(p_1)}\frac{\delta}{\delta\tilde{\varphi}(p_2)}\frac{\delta}{\delta\tilde{\varphi}(p_3)}\Big(\delta^{(4)}(k_1-p_4)\tilde{\varphi}(k_2)\tilde{\varphi}(k_3)\tilde{\varphi}(-k_1-k_2-k_3)\\
&\qquad+\tilde{\varphi}(k_1)\delta^{(4)}(k_2-p_4)\tilde{\varphi}(k_3)\tilde{\varphi}(-k_1-k_2-k_3)\\
&\qquad+\tilde{\varphi}(k_1)\tilde{\varphi}(k_2)\delta^{(4)}(k_3-p_4)\tilde{\varphi}(-k_1-k_2-k_3)\\
&\qquad+\tilde{\varphi}(k_1)\tilde{\varphi}(k_2)\tilde{\varphi}(k_3)\delta^{(4)}(p_4+k_1+k_2+k_3)\Big)\,.
\end{aligned}\tag{9}$$

We can repeat taking derivatives. It is easy to see that taking the second derivative will give 3 terms for each of the 4 terms. Therefore, the total number of terms will be 4×3, leaving only two fields left on each term. The third derivative then has two options for the 4×3 terms, which gives $4\times 3\times 2$ terms with one remaining field. The final derivative will eliminate the last field. Each of the $4\times 3\times 2\times 1 = 4!$ terms contain 4 delta functions. The three momentum integrals will yield a single momentum conserving delta function, $\delta(p_1+p_2+p_3+p_4)$. Therefore, since there is no momentum dependence in the interaction, all 4! terms are identical, and cancel the 4! in the denominator. This finally gives

$$i\Gamma = -i\lambda\,. \tag{10}$$

Quantum Electrodynamics

The QED interaction Lagrange density is

$$\mathcal{L}_{\text{int.}} = -Q\bar{\psi}(x)\not{A}(x)\psi(x),, \tag{11}$$

where Q is the coupling. The vertex is then

$$(2\pi)^4\,\delta^{(4)}(q+p_1+p_2)\,i\Gamma^{\mu}_{\alpha\beta} = \frac{\delta}{\delta\tilde{A}_\mu(q)}\frac{\delta}{\delta\tilde{\psi}_\alpha(p_1)}\frac{\delta}{\delta\tilde{\bar{\psi}}_\beta(p_2)}\left(-iQ\int \mathrm{d}^4x\,\bar{\psi}_\epsilon(x)\gamma^{\nu}_{\epsilon\delta}\psi_\delta(x)\,A_\nu(x)\right)\,, \tag{12}$$

where the Fourier transforms of the fields are

$$\begin{aligned}
A_\nu(x) &= \int\frac{\mathrm{d}^4\bar{q}}{(2\pi)^4}\,e^{-i\bar{q}\cdot x}\,\tilde{A}_\nu(\bar{q})\,,\\
\psi_\delta(x) &= \int\frac{\mathrm{d}^4k_1}{(2\pi)^4}\,e^{-ik_1\cdot x}\,\tilde{\psi}_\delta(k_1)\,,\\
\bar{\psi}_\epsilon(x) &= \int\frac{\mathrm{d}^4k_2}{(2\pi)^4}\,e^{-ik_2\cdot x}\,\tilde{\bar{\psi}}_\epsilon(k_2)\,.
\end{aligned}\tag{13}$$

Therefore, the vertex function is given by

$$
\begin{aligned}
(2\pi)^4\,\delta^{(4)}(q+p_1+p_2)\,i\Gamma^{\mu}_{\alpha\beta} &= -iQ\,\frac{\delta}{\delta\tilde{A}_\mu(q)}\frac{\delta}{\delta\tilde{\psi}_\alpha(p_1)}\frac{\delta}{\delta\tilde{\bar{\psi}}_\beta(p_2)}\int \mathrm{d}^4x\,\bar{\psi}_\epsilon(x)\gamma^{\nu}_{\epsilon\delta}\psi_\delta(x)\,A_\nu(x)\,,\\
&= -iQ\,\frac{\delta}{\delta\tilde{A}_\mu(q)}\frac{\delta}{\delta\tilde{\psi}_\alpha(p_1)}\frac{\delta}{\delta\tilde{\bar{\psi}}_\beta(p_2)}\int\frac{\mathrm{d}^4\bar{q}}{(2\pi)^4}\int\frac{\mathrm{d}^4k_1}{(2\pi)^4}\int\frac{\mathrm{d}^4k_2}{(2\pi)^4}\\
&\qquad\times\int\mathrm{d}^4x\,e^{-i(\bar{q}+k_1+k_2)\cdot x}\,\tilde{\bar{\psi}}_\epsilon(k_2)\gamma^{\nu}_{\epsilon\delta}\tilde{\psi}_\delta(k_1)\,\tilde{A}_\nu(\bar{q})\,,\\
&= -iQ\,\frac{\delta}{\delta\tilde{A}_\mu(q)}\frac{\delta}{\delta\tilde{\psi}_\alpha(p_1)}\frac{\delta}{\delta\tilde{\bar{\psi}}_\beta(p_2)}\int\frac{\mathrm{d}^4\bar{q}}{(2\pi)^4}\int\frac{\mathrm{d}^4k_1}{(2\pi)^4}\int\frac{\mathrm{d}^4k_2}{(2\pi)^4}\\
&\qquad\times(2\pi)^4\,\delta^{(4)}(\bar{q}+k_1+k_2)\,\tilde{\bar{\psi}}_\epsilon(k_2)\gamma^{\nu}_{\epsilon\delta}\tilde{\psi}_\delta(k_1)\,\tilde{A}_\nu(\bar{q})\,,\\
&= -iQ\,\frac{\delta}{\delta\tilde{A}_\mu(q)}\frac{\delta}{\delta\tilde{\psi}_\alpha(p_1)}\frac{\delta}{\delta\tilde{\bar{\psi}}_\beta(p_2)}\int\frac{\mathrm{d}^4k_1}{(2\pi)^4}\int\frac{\mathrm{d}^4k_2}{(2\pi)^4}\,\tilde{\bar{\psi}}_\epsilon(k_2)\gamma^{\nu}_{\epsilon\delta}\tilde{\psi}_\delta(k_1)\,\tilde{A}_\nu(-k_1-k_2)\,.
\end{aligned}
$$

Taking now the functional derivatives,

$$
\frac{\delta}{\delta f(p)}f(k) = (2\pi)^4\,\delta^{(4)}(p-k)
$$

wer find

$$
\begin{aligned}
(2\pi)^4\,\delta^{(4)}(q+p_1+p_2)\,i\Gamma^{\mu}_{\alpha\beta} &= -iQ\,\frac{\delta}{\delta\tilde{A}_\mu(q)}\frac{\delta}{\delta\tilde{\psi}_\alpha(p_1)}\frac{\delta}{\delta\tilde{\bar{\psi}}_\beta(p_2)}\int\frac{\mathrm{d}^4k_1}{(2\pi)^4}\int\frac{\mathrm{d}^4k_2}{(2\pi)^4}\,\tilde{\bar{\psi}}_\epsilon(k_2)\gamma^{\nu}_{\epsilon\delta}\tilde{\psi}_\delta(k_1)\,\tilde{A}_\nu(-k_1-k_2)\,,\\
&= -iQ\,\frac{\delta}{\delta\tilde{A}_\mu(q)}\,\gamma^{\nu}_{\beta\alpha}\,\tilde{A}_\nu(-p_1-p_2)\,,\\
&= (2\pi)^4\,\delta^{(4)}(q+p_1+p_2)\,\left(-iQ\gamma^{\mu}_{\beta\alpha}\right) \qquad (14)
\end{aligned}
$$

Thus, we conclude that the vertex function for QED is

$$
\Gamma_{\beta\alpha} = -iQ\gamma^{\mu}_{\beta\alpha}\,. \qquad (15)
$$