Abelian Gauge Theory

An important aspect of the SM is the notion of gauge symmetry. Classically, the Dirac eq. has the gauge symmetry

$$\psi(x) \to e^{i\alpha}\psi(x)$$

for any constant α. This transformation leaves all observable physics unchanged, and $e^{i\alpha} \in U(1)$ is an Abelian group. This transformation is called a gauge symmetry. Specifically, the Lagrange density is invariant under a global U(1) symmetry,

$$e^{i\alpha} \in U(1).$$

Let us check that $\mathcal{L} = \frac{i}{2}\bar{\psi}\overleftrightarrow{\partial}\psi - m\bar{\psi}\psi$ is invariant.

$$\psi \to e^{i\alpha}\psi$$

$$\bar{\psi} = \psi^\dagger\gamma^0 \to (e^{i\alpha}\psi)^\dagger\gamma^0 = e^{-i\alpha}\bar{\psi}$$

So,

$$\bar{\psi}\psi \to (e^{-i\alpha}\bar{\psi})(e^{i\alpha}\psi) = \bar{\psi}\psi$$

$$\bar{\psi}\overleftrightarrow{\partial}\psi \to (e^{-i\alpha}\bar{\psi})\overleftrightarrow{\partial}(e^{i\alpha}\psi)$$

$$= e^{-i\alpha}\bar{\psi}e^{i\alpha}\overleftrightarrow{\partial}\psi = \bar{\psi}\overleftrightarrow{\partial}\psi$$

α = constant

$$\Rightarrow \boxed{\mathcal{L} \text{ is invariant}}$$

Recall: <u>Noether's theorem</u>

If $\mathcal{L}$ is invariant under a continuous symmetry transformation, then $\exists$ a conserved charge and current, and vice versa.

For a spinor field, the conserved current is

$$J^\mu = -\frac{\delta \mathcal{L}}{\delta(\partial_\mu \psi)} \frac{\delta \psi}{\delta \alpha} - \frac{\delta \bar{\psi}}{\delta \alpha} \frac{\delta \mathcal{L}}{\delta(\partial_\mu \bar{\psi})}$$

with a charge $Q = \int d^3\vec{x}\, J^0$.

So, for $U(1)$ symmetry,

$$\psi \to \psi' \equiv e^{i\alpha}\psi$$
$$\simeq (1 + i\alpha)\psi + \mathcal{O}(\alpha^2)$$
$$\equiv \psi + \alpha \frac{\delta \psi}{\delta \alpha}$$

$$\bar{\psi} \to \bar{\psi}' \equiv e^{-i\alpha}\bar{\psi}$$
$$\simeq (1 - i\alpha)\bar{\psi} + \mathcal{O}(\alpha^2)$$
$$\equiv \bar{\psi} + \alpha \frac{\delta \bar{\psi}}{\delta \alpha}$$

With
$$\mathcal{L} = \frac{i}{2}\bar{\psi}\gamma^\mu\partial_\mu\psi - \frac{i}{2}\partial_\mu\bar{\psi}\gamma^\mu\psi - m\bar{\psi}\psi ,$$

we have
$$J^\mu = -\frac{i}{2}\bar{\psi}\gamma^\mu(i\psi) + \frac{i}{2}(-i\bar{\psi})\gamma^\mu\psi$$
$$= \bar{\psi}\gamma^\mu\psi \quad \Rightarrow \quad \boxed{J^\mu = \bar{\psi}\gamma^\mu\psi}$$

$\llcorner$ conserved $U(1)$ current

Current is conserved, $\partial_\mu J^\mu = 0$

Check

$$\partial_\mu J^\mu = \partial_\mu\bar{\psi}\gamma^\mu\psi + \bar{\psi}\gamma^\mu\partial_\mu\psi$$
$$= \bar{\psi}\overleftarrow{\not{\partial}}\psi + \bar{\psi}\overrightarrow{\not{\partial}}\psi$$

Dirac eq. $(i\not{\partial} - m)\psi = 0$ and $\bar{\psi}(i\overleftarrow{\not{\partial}} + m) = 0$

$$\Rightarrow \partial_\mu J^\mu = im\bar{\psi}\psi + (-im)\bar{\psi}\psi$$
$$= 0 \qquad \blacksquare$$

The corresponding charge $Q = \int d^3\vec{x}\, \bar{\psi}\gamma^0\psi$

is conserved, $\frac{dQ}{dt} = 0$.

Notice, when ψ (at particular x) $\to e^{i\alpha}\psi$,
all ψ (at other x) also rotate the same way by the same amount.

This seems to contradict the "spirit of relativity", i.e., would expect to do this transformation only locally.
So, might expect $\alpha = \alpha(x)$, with

$$\psi(x) \to e^{i\alpha(x)}\psi(x)$$

which is a <u>local gauge transformation</u>.
<u>Notice</u>: Promotes α to a scalar field.

There is a consequence though.
Consider

$$\partial_\mu \psi \longrightarrow \partial_\mu\left(e^{i\alpha(x)}\psi\right)$$
$$= \underbrace{i(\partial_\mu \alpha)e^{i\alpha}\psi}_{\text{"extra piece"}} + e^{i\alpha}\partial_\mu \psi$$

as before

$$\partial_\mu \psi \nrightarrow e^{i\alpha}\partial_\mu \psi$$

Therefore, $\mathcal{L}$ is <u>No Longer invariant</u>

check, $\bar{\psi}\psi \to e^{-i\alpha}\bar{\psi} e^{i\alpha}\psi = \bar{\psi}\psi$ ✓ as before

and

$$\bar{\psi}\overleftrightarrow{\partial\!\!\!/}\psi = \bar{\psi}\partial\!\!\!/\psi - (\partial_\mu\bar{\psi})\gamma^\mu\psi$$

$$\longrightarrow e^{-i\alpha}\bar{\psi}[i(\partial\!\!\!/\alpha)e^{i\alpha}\psi + e^{i\alpha}\partial\!\!\!/\psi]$$

$$-[-i(\partial_\mu\alpha)e^{-i\alpha}\bar{\psi} + e^{-i\alpha}\partial_\mu\bar{\psi}]\gamma^\mu e^{i\alpha}\psi$$

$$= \bar{\psi}\overleftrightarrow{\partial\!\!\!/}\psi + 2i\bar{\psi}\gamma^\mu\psi\partial_\mu\alpha \quad \times$$

So,

$$\mathcal{L} = \frac{i}{2}\bar{\psi}\overleftrightarrow{\partial\!\!\!/}\psi - m\bar{\psi}\psi$$

$$\longrightarrow \frac{i}{2}\bar{\psi}\overleftrightarrow{\partial\!\!\!/}\psi - m\bar{\psi}\psi - \bar{\psi}\gamma^\mu\psi\partial_\mu\alpha$$

$$= \mathcal{L} - \bar{\psi}\gamma^\mu\psi\partial_\mu\alpha$$

$\Rightarrow \mathcal{L}$ is NOT Invariant!

Promoting $\alpha \Rightarrow \alpha(x)$ destroys $U(1)$ invariance.
Can we fix this conflict?

Yes. Define a modified derivative, the covariant derivative D_μ, that does transform covariantly.

Define D_μ such that

$$D_\mu \psi \xrightarrow[\text{U(1)}]{\text{local}} e^{i\alpha(x)} D_\mu \psi$$

$$\Rightarrow \quad \bar{\psi} \not{D} \psi \longrightarrow \bar{\psi} \not{D} \psi$$

"correction factor"

Simplest choice $\boxed{D_\mu = \partial_\mu + igA_\mu(x)}$

gauge field (real)

convenient choice of phase

AND require that

$$\boxed{A_\mu \longrightarrow A_\mu - \frac{1}{g}\partial_\mu \alpha}$$

Note: sign in D_μ chosen such that

$$D_\mu \psi = \partial_\mu \psi \boxed{+} igA_\mu \psi$$

For $\bar{\psi}$, we have

$$D_\mu \bar{\psi} = \partial_\mu \bar{\psi} \boxed{-} igA_\mu \bar{\psi}$$

Claim

If $D_\mu = \partial_\mu + igA_\mu$ and $A_\mu \to A_\mu - \frac{1}{g}\partial_\mu \alpha$

under $\psi \to e^{i\alpha}\psi$, then

$$D_\mu \psi \to e^{i\alpha} D_\mu \psi.$$

Proof

$$
\begin{aligned}
D_\mu \psi &= (\partial_\mu + igA_\mu)\psi \\
&\longrightarrow \left(\partial_\mu + ig\left(A_\mu - \tfrac{1}{g}\partial_\mu \alpha\right)\right) e^{i\alpha}\psi \\
&= \cancel{i(\partial_\mu \alpha) e^{i\alpha}\psi} + e^{i\alpha}\partial_\mu \psi \\
&\quad + igA_\mu e^{i\alpha}\psi - \cancel{i(\partial_\mu \alpha) e^{i\alpha}\psi} \\
&= e^{i\alpha}[\partial_\mu + igA_\mu]\psi \\
&= e^{i\alpha} D_\mu \psi \qquad \blacksquare
\end{aligned}
$$

So, we consider a new theory which is invariant under a local $U(1)$ symmetry

$$\mathcal{L} = \frac{i}{2}\bar{\psi}\not{D}\psi + h.c. - m\bar{\psi}\psi$$

where $\not{D} = \gamma^\mu D_\mu$

This $\mathcal{L}$ is invariant under local $U(1)$ symmetry.
But, this is not the same theory we started with.
To see, write out D_μ

$$\frac{i}{2}\bar{\psi}\not{D}\psi = \frac{i}{2}\bar{\psi}\not{\partial}\psi - \frac{1}{2}g\bar{\psi}\not{A}\psi$$

$$\left(\frac{i}{2}\bar{\psi}\not{D}\psi\right)^{\dagger} = \left(\frac{i}{2}\bar{\psi}\not{\partial}\psi\right)^{\dagger} - \frac{1}{2}g\bar{\psi}\not{A}\psi$$

$$\Rightarrow \mathcal{L} = \frac{i}{2}\bar{\psi}\not{\partial}\psi + \text{h.c.} - m\bar{\psi}\psi - g\bar{\psi}\not{A}\psi$$

Recall the Noether current, $J^{\mu} = \bar{\psi}\gamma^{\mu}\psi$.

So,

$$\mathcal{L} = \underbrace{\frac{i}{2}\bar{\psi}\not{\partial}\psi + \text{h.c.} - m\bar{\psi}\psi}_{\text{original theory}} \underbrace{- g A_{\mu} J^{\mu}}_{\text{New term}}$$

The new term is an interaction of the fermion field with the gauge field.

Therefore, promoting global $\Rightarrow$ local gauge symmetry introduces interactions with gauge field $A_{\mu}(x)$. The fields are coupled through coupling (or coupling constant*) g.

* These are not constant in QFT...

We have now introduced the gauge field $A_\mu(x)$. If we want to think of A_μ as some physical field, then we should think about adding some more dynamics of the field, i.e., a kinetic term. Since A_μ must transform as a Lorentz vector, we must build ~~in~~ not only a term invariant under gauge transformations, but also under Poincaré transformations. Let's review a famous vector field, the electromagnetic field.

Electromagnetic Field

Consider the four-potential $A_\mu(x)$.
The field-strength tensor is

$$\boxed{\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \\ &= -F_{\nu\mu} \end{aligned}}$$

In natural units, $A^\mu = (\varphi, \vec{A})$ (cf. Jackson)

and, $\vec{E} = -\vec{\nabla}\varphi - \dfrac{\partial \vec{A}}{\partial t}$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

so,

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ & 0 & -B^3 & -B^2 \\ & & 0 & -B^1 \\ & & & 0 \end{pmatrix}$$

For a given $F_{\mu\nu}$, the four-potential A_μ is NOT unique.

- The field strength is invariant under gauge transformation

$$A_\mu \Rightarrow A_\mu - \partial_\mu \Lambda$$

Check

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \\ &\longrightarrow \partial_\mu A_\nu - \cancel{\partial_\mu \partial_\nu \Lambda} \\ &\qquad - \partial_\nu A_\mu + \cancel{\underbrace{\partial_\nu \partial_\mu \Lambda}_{= \partial_\mu \partial_\nu \Lambda}} = F_{\mu\nu} \end{aligned}$$

The Lagrange density is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - A_\mu J^\mu$$

Source current $J^\mu = (\rho, \vec{J})$

Can use Euler-Lagrange eqns. to get equations of motion

$$\partial_\mu \left(\frac{\delta \mathcal{L}}{\delta(\partial_\mu A_\nu)} \right) - \frac{\delta \mathcal{L}}{\delta A_\nu} = 0$$

Now,

$$\frac{\delta(F_{\alpha\beta}F^{\alpha\beta})}{\delta(\partial_\mu A_\nu)} = 2F^{\alpha\beta}\frac{\delta F_{\alpha\beta}}{\delta(\partial_\mu A_\nu)}$$

$$= 2F^{\alpha\beta}\left(\frac{\delta(\partial_\alpha A_\beta)}{\delta(\partial_\mu A_\nu)} - \frac{\delta(\partial_\beta A_\alpha)}{\delta(\partial_\mu A_\nu)}\right)$$

$$= 2F^{\alpha\beta}\left(\delta^\mu_\alpha \delta^\nu_\beta - \delta^\mu_\beta \delta^\nu_\alpha\right)$$

$$= 2F^{\mu\nu} - 2F^{\nu\mu}$$

$$\llcorner\ F^{\nu\mu} = -F^{\mu\nu}$$

$$= 4F^{\mu\nu}$$

So,
$$\partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu A_\nu)}\right) = -\partial_\mu F^{\mu\nu}$$

and
$$\frac{\delta\mathcal{L}}{\delta A_\nu} = -J^\nu$$

Therefore,
$$\partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu A_\nu)}\right) - \frac{\delta\mathcal{L}}{\delta A_\nu} = 0$$

$$\Rightarrow \boxed{\partial_\mu F^{\mu\nu} = J^\nu}$$

The eqns. of motion contains two Maxwell eqns.

$$\partial_\mu F^{\mu\nu} = J^\nu \Rightarrow \begin{cases} \vec{\nabla}\cdot\vec{E} = \rho \\ \vec{\nabla}\times\vec{B} - \dfrac{\partial\vec{E}}{\partial t} = \vec{J} \end{cases}$$

Note:

$$\underbrace{\partial_\mu \partial_\nu}_{\text{symmetric}} \underbrace{F^{\mu\nu}}_{\text{antisymmetric}} = 0 \quad \Rightarrow \quad \partial_\nu J^\nu = 0$$

So, current must be conserved for consistency

$$\partial_\nu J^\nu = 0 \quad \Rightarrow \quad \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$$

Definition: Dual field strength

$$\tilde{F}^{\mu\nu} \equiv \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} = -\tilde{F}^{\nu\mu}$$

$$= \begin{pmatrix} 0 & -B^1 & -B^2 & -B^3 \\ & 0 & +E^3 & -E^2 \\ & & 0 & +E^1 \\ & & & 0 \end{pmatrix}$$

Can show it obeys identity $\partial_\mu \tilde{F}^{\mu\nu} = 0$, get other two Maxwell eqns.

$$\partial_\mu \tilde{F}^{\mu\nu} = 0 \quad \Rightarrow \quad \begin{cases} \vec{\nabla} \cdot \vec{B} = 0 \\ \vec{\nabla} \times \vec{E} + \dfrac{\partial \vec{B}}{\partial t} = \vec{0} \end{cases}$$

Interacting Spinor Field theory

Going back to our spinor field theory which is invariant under local $U(1)$ gauge transformations

$$\mathcal{L} = \frac{i}{2}\bar{\psi}\not{D}\psi + h.c. - m\bar{\psi}\psi$$

$$= \frac{i}{2}\bar{\psi}\not{\partial}\psi + h.c. - m\bar{\psi}\psi - gA_\mu J^\mu$$

with conserved current $J^\mu = \bar{\psi}\gamma^\mu\psi$.

If A_μ is dynamical, we should add gauge invariant kinetic term.

Consider $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, which is gauge invariant, with kinetic Lagrange density

$$\mathcal{L}_G = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

This is locally $U(1)$ invariant, Poincaré invariant, and is an appropriate kinetic term for A_μ

$$\mathcal{L}_G = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)$$

$$= \underbrace{-\frac{1}{2}\partial_\mu A_\nu \partial^\mu A_\nu}_{\text{kinetic term}} + \underbrace{\frac{1}{2}(\partial_\mu A^\nu)^2}_{\text{other term?}}$$

This Lagrange density is of a free electromagnetic field!

$\Rightarrow$ The EM field arises naturally by requiring global $U(1) \to$ local $U(1)$.

We will see that other forces of the Standard Model arise in a similar way.

This theory is spinor electrodynamics. The corresponding quantum theory is Quantum Electrodynamics.

$$\mathcal{L} = \frac{i}{2}\bar{\psi}\not{\partial}\psi + h.c. - m\bar{\psi}\psi - gA_\mu J^\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

This is then the electromagnetic coupling of matter to the gauge field.
$\Rightarrow$ electromagnetic charge of the fermion field!

Heuristic argument that $g = q$, charge of particle.

Classically, to get Hamiltonian for relativistic particle coupled to EM field A_μ, replace kinetic momentum

$$p_\mu \to p_\mu - q A_\mu$$

Now, classical $\to$ quantum transition

$$p_\mu \to i\partial_\mu$$

So, might expect

$$i\partial_\mu \to i\partial_\mu - q A_\mu$$
$$= i(\partial_\mu + iq A_\mu) = i D_\mu$$

with $g = q$.

So, $\boxed{g \to q = eQ}$

charge

fundamental electron charge

$e = \sqrt{4\pi\alpha} \simeq 0.303$

$Q = -1$ for electron field

$= +1$ for positron field

$= +2/3$ for up quark field

Notice that A_μ must be a massless field, because a mass term is not $U(1)$ invariant.

$$m_A^2 A_\mu A^\mu \longrightarrow m_A^2 \left(A_\mu - \frac{1}{q}\partial_\mu \alpha\right)\left(A^\mu - \frac{1}{q}\partial^\mu \alpha\right)$$
$$\neq m_A^2 A_\mu A^\mu$$

Therefore, the excitation of the quantum theory, i.e., the photon, must be massless because of exact local $U(1)$ symmetry.

All together, our local $U(1)$ invariant theory is

Spinor Electrodynamics

$$\boxed{\begin{aligned} &\mathcal{L} = \frac{1}{2} i \bar{\psi} \not{D} \psi + \text{h.c.} - m\bar{\psi}\psi - \frac{1}{4} F_{\mu\nu}F^{\mu\nu} \\ &\text{with} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \\ &\text{and} \quad D_\mu \psi = \partial_\mu \psi + i \underbrace{q}_{\text{charge of fermion field}} A_\mu \psi \end{aligned}}$$

This is our first example of a gauge field theory. This theory describes charged fermions interacting with the EM field.

Functional Quantization of Abelian Gauge theories

We proceed with constructing the quantum theory of spinor electrodynamics. Let us consider the path integral for pure electrodynamics

$$\int \mathcal{D}A \, e^{iS[A]}$$

where $\mathcal{D}A = \prod_{\mu=0}^{3} \mathcal{D}A_\mu = \mathcal{D}A_0 \mathcal{D}A_1 \mathcal{D}A_2 \mathcal{D}A_3$,

and the action is

$$S[A] = \int d^4x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right]$$

$$= \int d^4x \; \frac{1}{2} A_\mu \left(\partial^2 g^{\mu\nu} - \partial^\mu \partial^\nu \right) A_\nu$$

$$= \int \frac{d^4k}{(2\pi)^4} \; \frac{1}{2} A_\mu(k) \underbrace{\left(-k^2 g^{\mu\nu} + k^\mu k^\nu \right)}_{= i D^{-1}_{\mu\nu}(k)} A_\nu(k)$$

Can we interpret this as a transition amplitude?

The action vanishes when $A_\mu(k) = k_\mu \alpha$, i.e., when $A_\mu(x)$ is a pure gauge $\partial_\mu \alpha$.

Because this set is infinite (arbitrary $\alpha(x)$), the PI is badly divergent, $\int \mathcal{D}A\, e^{iS} \propto \int \mathcal{D}\alpha \ldots \to \infty \times \ldots$

Similarly, each field $A_\mu \neq \partial_\mu \alpha$ can be associated with an infinite set of equivalent configurations

$$A_\mu^\alpha = A_\mu - \frac{1}{q}\partial_\mu \alpha$$

The PI is thus badly divergent and not normalizable. This obviously results from it containing integrals over ∞ many physically equivalent configurations. Indeed, if we tried to include a source term and perform the Gaussian integral of the PI, we would fail because the function

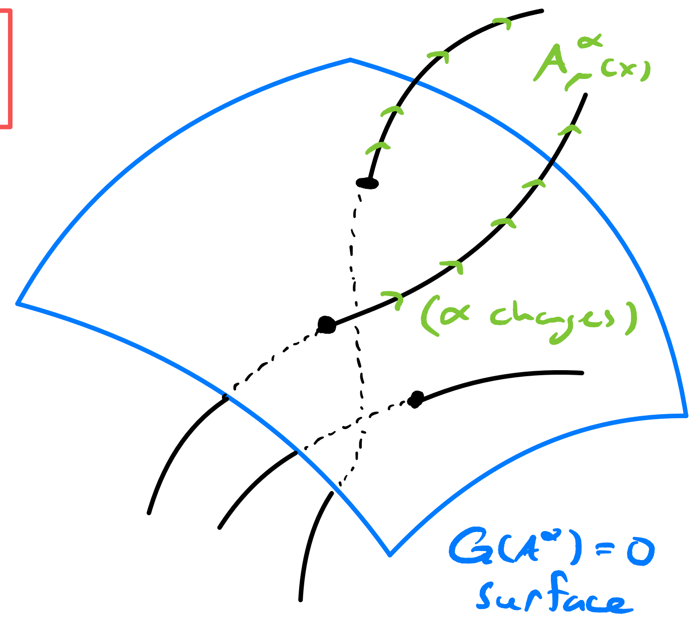
$$-i D_{\mu\nu}^{-1}(k) = -k^2 g_{\mu\nu} + k_\mu k_\nu$$

is singular. That is, it does not have an inverse, i.e., no propagator!

In the PI formalism, this problem is formally trivial to solve – we only need to constrain the PI with a suitable gauge constraint

$$\boxed{G(A^\alpha) = 0}$$

such that out of each α-path, only one member contributes. So, we "only" need to set a functional δ-function $\delta(G(A^\alpha))$ into the PI to constrain it surface $G(A^\alpha)=0$.



This has to be done without changing the PI measure.

To define a consistent procedure, we introduce the following identity

$$\boxed{1 \equiv \Delta_{FP}(A_\mu)\int \mathcal{D}\alpha\ \delta(G(A_\mu^\alpha))}$$

where the Faddeev-Popov determinant Δ_{FP} guarantees that the measure of the PI is preserved.

The Faddeev-Popov determinant is gauge invariant

$$\Delta_{FP}^{-1}(A_\mu^{\alpha'}) = \int \mathcal{D}\alpha\, \delta(G(A^{\alpha'\alpha}))$$

$$= \int \mathcal{D}(\alpha'\alpha)\, \delta(G(A^{\alpha'\alpha}))$$

$$= \int \mathcal{D}\alpha''\, \delta(G(A^{\alpha''})) = \Delta_{FP}^{-1}(A_\mu)$$

Formally, Δ_{FP} takes the form

$$\boxed{\Delta_{FP}(A_\mu) = \det\left(\frac{\delta G[A_\mu^\alpha]}{\delta\alpha}\right)}$$

↳ understood as a limiting process from a discretized spacetime.

Inserting the identity into the PI, we get

$$\int \mathcal{D}A \left(\Delta_{FP}(A_\mu)\int \mathcal{D}\alpha\, \delta(G(A_\mu^\alpha))\right) e^{iS[A_\mu]}$$

All invariant under gauge transformation

$\Rightarrow A_\mu \to A_\mu^\alpha$ transformation

$$= \underbrace{\left(\int \mathcal{D}\alpha\right)}_{\text{unphysical } \infty \text{ d.o.f. extracted}} \cdot \int \mathcal{D}A\, \Delta_{FP}(A_\mu)\, \underbrace{\delta(G(A_\mu))}_{\text{constraint to surface } G(A_\mu)=0}\, e^{iS'[A_\mu]}$$

ensures measure is preserved

In general Δ_{FP} can depend on A_μ, and thus change the PI on the surface $G(A_\mu)$. We will see this behavior for non-Abelian gauge theories. For Abelian gauge theories, like QED, there is a class of Lorentz-covariant gauges for which Δ_{FP} = const. wrt A_μ.

Let us choose

$$G_\omega(A_\mu) = \partial_\mu A^\mu - \omega(x) = 0$$

Then,
$$\Delta_{FP} = \det\left(\frac{\delta G(A^\alpha)}{\delta\alpha} \right) = \det\left(\frac{1}{g} \partial^2 \right)$$

independent of A_μ!

So, can take Δ_{FP} out of integrals!

Example

Lorentz gauge : $\omega(x) = 0$

$$\Rightarrow G(A) = \partial_\mu A^\mu = 0$$

We have now gauge-fixed the action. Note that Δ_{FP} is also independent of $\omega(x)$. So, we can integrate over ω with any weight function, effectively averaging over this function. Let us choose to average $\omega(x)$ with a Gaussian weight

$$= N(\xi)\int \mathcal{D}\omega \, e^{-i\int d^4x \frac{\omega^2}{2\xi}} \left(\int \mathcal{D}\alpha\right) \det\left(\frac{1}{e}\partial^2\right) \times \int \mathcal{D}A \, \delta(\partial_\mu A^\mu - \omega) \, e^{iS[A]}$$

$$= \underbrace{N(\xi)\left(\int \mathcal{D}\alpha\right)\det\left(\frac{1}{e}\partial^2\right)}_{\equiv \tilde{N} \,:\, A\text{-independent } \infty \text{ constant}} \int \mathcal{D}A \, e^{iS[A] - i\int d^4x \frac{1}{2\xi}(\partial_\mu A^\mu)^2}$$

additional piece in the action

$$= \tilde{N}\int \mathcal{D}A \, e^{i\int d^4x \left(\mathcal{L}[A] - \frac{1}{2\xi}(\partial_\mu A^\mu)^2\right)}$$

gauge ∞ now here

ξ is an arbitrary constant

The net effect of gauge fixing is an additional term in the Lagrange density

$$\mathcal{L}[A] \longrightarrow \mathcal{L}^{\xi}[A] = \mathcal{L}[A] - \frac{1}{2\xi}(\partial_\mu A^\mu)^2$$

The new term is very important. First note that for a pure gauge

$$\frac{1}{2\xi}(\partial_\mu A^\mu)^2 \longrightarrow \frac{k^4}{2\xi}\alpha^2$$

If $\xi \neq 0$, the integral is now strongly damped.

Moreover,

$$iS[A] - \frac{i}{2\xi}\int d^4x\,(\partial_\mu A^\mu)^2$$

$$= \frac{i}{2}\int d^4x\, A_\mu(x)\left(\partial^2 g^{\mu\nu} - \partial^\mu\partial^\nu + \frac{1}{\xi}\partial^\mu\partial^\nu\right)A_\nu(x)$$

$$= \frac{i}{2}\int \frac{d^4k}{(2\pi)^4}\tilde{A}_\mu(k)\underbrace{\left(-k^2 g^{\mu\nu} + \left(1-\frac{1}{\xi}\right)k^\mu k^\nu\right)}_{-iD_{\mu\nu}^{-1}(k)}\tilde{A}_\nu(k)$$

The inverse propagator is no longer singular

$$\boxed{\begin{gathered}\langle 0|T\{A_\mu(x)A_\nu(0)\}|0\rangle = \int\frac{d^4k}{(2\pi)^4}\,e^{-ik\cdot x}\,iD^\xi_{\mu\nu}(k)\\ \text{with}\\ iD^\xi_{\mu\nu}(k) = \frac{-i}{k^2+i\epsilon}\left(g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right)\end{gathered}}$$

photon propagator

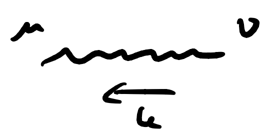
The choice of ξ is a choice of averaging over gauge choices. Physical observables must be gauge-invariant, i.e., independent of ξ.

Some special cases

$\xi = 1$: $\quad iD_{\mu\nu} = -i \dfrac{g_{\mu\nu}}{k^2 + i\epsilon}$ $\quad$ Feynman Gauge

$\xi = 0$: $\quad iD_{\mu\nu} = \dfrac{-i}{k^2 + i\epsilon}\left(g_{\mu\nu} - \dfrac{k_\mu k_\nu}{k^2} \right)$ $\quad$ Landau Gauge

The momentum-space Feynman rule

$iD^{\xi}_{\mu\nu}(k) =$ 

$$= \frac{-i}{k^2 + i\epsilon}\left(g_{\mu\nu} - (1-\xi)\, \frac{k_\mu k_\nu}{k^2} \right)$$

Physical observables must be gauge independent.
For a gauge invariant operator $O[\hat{A}]$, then obviously

$$\boxed{\langle 0|T[O[\hat{A}]]|0\rangle = \frac{\int \mathcal{D}A\, O[A]\, e^{i\int d^4x\left(\mathcal{L} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2\right)}}{\int \mathcal{D}A\, e^{i\int d^4x\left(\mathcal{L} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2\right)}}}$$

We can derive this as we did for the PI.
Eventually, all infinities cancel in the ratio.
Note that gauge invariance of $O[A]$ is an essential requirement for obtaining this.

$\Rightarrow$ From the LSZ theorem, we can get a unitary, gauge-invariant S-matrix.

Quantum Electrodynamics

We are now ready to formulate QED. The gauge-fixed Lagrange density is

$$\mathcal{L}^{\xi}_{QED} = \frac{i}{2}\bar{\psi}\not{D}\psi + \text{h.c} - m\bar{\psi}\psi$$

$$- \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2$$

with

$$D_\mu = \partial_\mu + iqA_\mu ,$$

and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

The generating functional is

$$\boxed{\begin{aligned} Z^{\xi}_{QED}[J^\mu, \eta, \bar{\eta}] &= \frac{1}{N^{\xi}_{vac}} \int \mathcal{D}A\,\mathcal{D}\psi\,\mathcal{D}\bar{\psi}\; e^{i\int_x \mathcal{L}^{\xi}_{QED} + J^\mu A_\mu + \bar{\eta}\psi + \bar{\psi}\eta} \\ &= \frac{1}{N^{\xi}_{vac}}\, e^{-iq\int d^4x \frac{\delta}{\delta\eta}\gamma^\mu \frac{\delta}{\delta\bar{\eta}}\frac{\delta}{\delta J^\mu}}\, Z^{\xi}_{FP}[J^\mu]\, Z_D[\eta,\bar{\eta}] \end{aligned}}$$

interaction

$$\mathcal{L}_{int} = -iq\bar{\psi}\not{A}\psi$$

where the gauge-fixed FP gauge-field generating function is

$$Z^{\xi}_{FP}[J^\mu] = \int \mathcal{D}A \; e^{\frac{i}{2}\int_{x,y} A^\mu(x)\, i D^{-1,\xi}_{\mu\nu}(x,y)\, A^\nu(y) + i\int_x J_\mu(x) A^\mu(x)}$$

$$= e^{\frac{i}{2}\int_{x,y} J^\mu(x)\, i D^{\xi}_{\mu\nu}(x,y)\, J^\nu(y)}$$

and the Dirac (fermion) generating function is

$$Z_D[\eta,\bar\eta] = \int \mathcal{D}\psi\, \mathcal{D}\bar\psi \; e^{i\int_{x,y} \bar\psi(x)\, i S^{-1}(x,y)\, \psi(y) + i\int_x \bar\eta(x)\psi(x) + \bar\psi(x)\eta(x)}$$

$$= e^{i\int_x\int_y \bar\eta(x)\, i S(x,y)\, \eta(y)}$$

with $iS(x,y) = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot(x-y)}\, iS(p)$

and $iS(p) = \frac{i}{\not p - m + i\epsilon} = \frac{i(\not p + m)}{p^2 - m^2 + i\epsilon}$

and finally, N^{ξ}_{vac} is the vacuum-to-vacuum transition amplitude

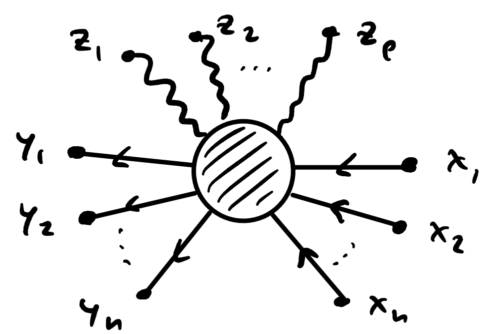
$$N^{\xi}_{vac} \equiv e^{-iq\int d^4x \frac{\delta}{\delta\eta}\gamma^\mu \frac{\delta}{\delta\bar\eta}\frac{\delta}{\delta J^\mu}} Z^{\xi}_{FP}[J^\mu]\, Z_D[\eta,\bar\eta] \Big|_{J^\mu=\eta=\bar\eta=0}$$

A generic correlation function with n-fermions, n-antifermions, and p-photon fields is

$$G^{\mu_1 \cdots \mu_p}(x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_p)$$

$$= \langle 0 | T\{ \prod_{j=1}^{n} \psi(x_j) \bar{\psi}(y_j) \prod_{k=1}^{p} A_\mu(z_k) \} | 0 \rangle$$

$$= (-i)^{2n+p} \frac{\delta^{2n+p}}{\delta \eta_1 \cdots \delta \eta_n \, \delta \bar{\eta}_1 \cdots \delta \bar{\eta}_n \, \delta J_1 \cdots \delta J_p} Z^{\xi}_{QED}[J^\mu, \eta, \bar{\eta}]$$

$=$ 

The connected correlation functions are found by taking appropriate functional derivatives of W,

$$W^{\xi}_{QED}[J^\mu, \eta, \bar{\eta}] = -i \ln Z^{\xi}_{QED}[J^\mu, \eta, \bar{\eta}]$$

The LSZ reduction theorem then allows us to get S-matrix elements.

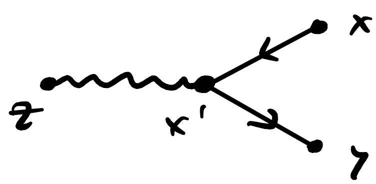
The QED Vertex

To get the QED vertex, look at 3-point function

$$\langle 0 | T\{ \psi(x) \bar{\psi}(y) A^\mu(z) \} | 0 \rangle$$

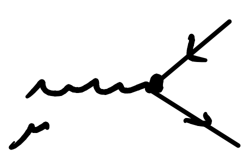
$$= (-i)^3 \frac{\delta^3}{\delta J^\mu \delta \eta \delta \bar{\eta}} W_{QED}^{q}$$

$$= (-i)^4 \cdot (-iq) \int d^4x' \, iS(x-x') \gamma^\nu \, iS(x'-y) \, iD_{\mu\nu}(x'-z)$$

$=$ 

So, momentum-space vertex is

$$\boxed{i\Gamma^\mu = -iq\gamma^\mu}$$

$=$ 

Beyond QED

Quantum electrodynamics (QED) is our first example of a quantum gauge field theory. It is an Abelian gauge theory, with A_μ being an element of $u(1) \sim \mathbb{R}$. Notice that we cannot add a polynomial potential $V(\bar{\psi}\psi)$ if we desire a renormalizable theory.

Quantizing with ψ identified as e^- and e^+ field with $q = e < 0$. QED, "the theory of photons and electrons", is the most accurate theory to date. The coupling $-q\not{A}$ gives the correct gyromagnetic ratio for the electron. QED gives the usual Maxwell eqns., & predicts that the photon is massless. This all comes from imposing local gauge invariance.

We can extend QED to theory with more than one fermion species. Suppose we want e^-, μ^-, τ^-, so each species has field ψ_j, $j = 1, 2, 3$.

$$\Rightarrow \quad \mathcal{L} = \frac{1}{2} i \sum_j \bar{\psi}_j \gamma^\mu D_\mu \psi_j + \text{h.c.}$$

$$- M_{jk} \bar{\psi}_j \psi_k - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

Can take this diagonal in j,k space.

Mass mixing term

$M_{jk} \bar{\psi}_j \psi_k$ form looks (possibly) like flavor oscillations, but in fact no flavor-changing mass terms are physical. To see this, define new fields

$$\begin{pmatrix} \psi_e \\ \psi_\mu \\ \psi_\tau \end{pmatrix} = U \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}$$

unitary matrix, $U^\dagger U = \mathbb{1}$

and set

$$U^\dagger M U = \begin{pmatrix} m_e & & \\ & m_\mu & \\ & & m_\tau \end{pmatrix}$$

$$\Rightarrow \quad \bar{\psi}_j M_{jk} \psi_k \rightarrow \bar{\psi}_{f'} [U^\dagger M U]_{f'f} \psi_f \quad , f = e, \mu, \tau$$

and $[U^\dagger M U]_{f'f} = \delta_{f'f}\, m_f$

Diagonal mass term
$\Rightarrow$ No mixing!

For this to be true, need M Hermitian. But,

$\mathcal{L}$ is Hermitian $\Rightarrow$ $\bar{\psi} M \psi$ is Hermitian

check:

$$\boxed{\begin{aligned}(\bar{\psi}M\psi)^{\dagger} &= (\psi^{\dagger}\gamma^{0}M\psi)^{\dagger} \\ &= \psi^{\dagger}M^{\dagger}\gamma^{0\dagger}\psi \\ &= \psi^{\dagger}\gamma^{0}M^{\dagger}\psi \\ &= \bar{\psi}M^{\dagger}\psi \qquad \Rightarrow \qquad M^{\dagger}=M \quad \blacksquare\end{aligned}}$$

Kinetic terms are unaffected since $U^{\dagger}U = \mathbb{1}$

$\Rightarrow$ Can eliminate mass mixing terms as unphysical

So,

$$\mathcal{L} = \frac{1}{2} i \sum_{f=e,\mu,\tau} \bar{\psi}_f \not{D} \psi_f + h.c.$$

$$- \sum_{f=e,\mu,\tau} m_f \bar{\psi}_f \psi_f \quad - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

Notice that even though $\mathcal{L}$ is diagonal in flavour, there are still possible "flavour transitions", e.g.,

$$e^- e^+ \to \gamma^* \to \mu^- \mu^+$$

What we have is a simple overview of how gauge theory works. Why do we bother with such a theory? We have found that the interactions of the SM can be consistently described by such theories, and they are exceedingly predictive and explain all* the phenomena we observe.

We now explore QED consequences by examining select processes.

* Except gravity, Dark matter, universe expansion, ...