Nonabelian Gauge Theory

We have seen how to construct QED from considering local $U(1)$ gauge symmetries. For strong, and weak interactions, we postulate a similar construction can be made. For example, we saw that in order to have a consistent construction of hadrons from the quark model, we introduced $SU(3)_c$, the color degrees of freedom. So, let us consider a class of non-abelian gauge theories based on $SU(N)$.

First, let us extend $U(1)$ scalar QED. Consider a classical theory of n-complex scalars φ_j, $j = 1, 2, \dots, n$.

assume same mass

$$\mathcal{L} = \partial_\mu \varphi_j^* \partial^\mu \varphi_j - m^2 \varphi_j^* \varphi_j - V(\varphi_j)$$

Think of φ_j as n-component object which transforms as an n-dim rep. $\underset{\sim}{n}$ of $SU(N)$

$$U(\alpha)$$

group element

set of parameters

i.e.,

$$\varphi_j(x) \to \varphi_j'(x) = U_{jk}(\alpha^a)\, \varphi_k(x)$$

$a = 1, \ldots, N^2-1$ params of $SU(N)$

$\underset{\sim}{n}$ matrix rep $(n \times n)$

with $U_{jk}(\alpha^a) = \left[\exp(i\alpha^a T_a)\right]_{jk}$

T_a = generators of $SU(N)$

$\Rightarrow \quad [T_a, T_b] = i\, C_{ab}{}^c\, T_c$

$\hookrightarrow$ structure constants of $SU(N)$

Example: $n=3$, $\cup$ the $\underset{\sim}{3}$ of $SU(3)$

then,

$$U_{jk}(\alpha^a) = \left[\exp\left(\tfrac{1}{2} i \alpha^a \lambda_a\right)\right]_{jk} \quad a=1,\ldots,8$$

Gell-Mann matrices

$\frac{1}{2}\lambda^a_{jk}$ are generators of $su(3)$ in the $\underset{\sim}{3}$ rep.

Example: $n=3$, $\cup$ the $\underset{\sim}{3}$ of $SU(2)$

then,

$$U_{jk}(\alpha^\ell) = \left[\exp(i\alpha^\ell T_\ell)\right]_{jk} \quad \text{w/ 3 params } \alpha^\ell$$

3 $SU(2)$ generators, $T_\ell \Rightarrow \underset{\sim}{3}$ = adjoint of $su(2)$

$$\Rightarrow \quad (T_\ell)_{jk} = -i\, \epsilon_{\ell jk}$$

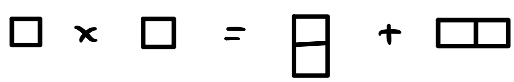
So, Lagrange density $\mathcal{L}$ for n-scalars doesn't tell you the group structure (many possibilities).

For global symmetries ($\alpha^a =$ constant), the potential $V(\varphi_j)$ must be invariant (transforms like a singlet).

Its allowed form depends on the group.

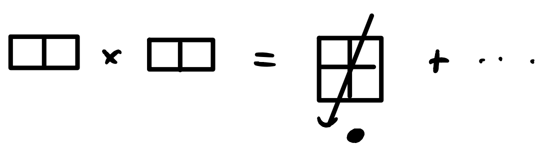
e.g., $\underset{\sim}{3}$ of $su(3)$

$$\underset{\sim}{3} \times \underset{\sim}{3} = \underset{\sim}{3}^* + \underset{\sim}{6} \not\ni \underset{\sim}{1}$$



$\underset{\sim}{3}$ of $su(2)$

$$3 \times 3 = 1 + \cdots$$



$\Rightarrow \mathcal{L}$ is invariant under global transformations (assuming $V(\varphi)$ is)

$$\varphi \to U\varphi \;,\; \varphi^\dagger \to \varphi^\dagger U^\dagger$$

since $U^\dagger U = \mathbb{1}$

$$\Rightarrow \varphi^\dagger \varphi \to \varphi^\dagger U^\dagger U \varphi = \varphi^\dagger \varphi$$

and $\partial_\mu \varphi$ transforms covariantly under global transformations because α^a are constants

$$\partial_\mu \varphi \to \partial(U\varphi) = U \partial_\mu \varphi$$

How to make this a local transformation?

$$\alpha^a = \alpha^a(x)$$

Now, $\partial_\mu \varphi$ is <u>not</u> covariant since

$$\partial_\mu \varphi \to \partial_\mu (U\varphi) = U\partial_\mu \varphi + (\partial_\mu U)\varphi$$

covariant piece

not covariant

Repeat idea used for QED

- Define new covariant derivative & $\mathcal{L}'$ such that it is invariant under local gauge transformations.

Define:

convention

$$\boxed{(D_\mu)_{jk} = \delta_{jk}\partial_\mu + igA_\mu^a (T_a)_{jk}}$$

$n \times n$ operator

for $e^{i\alpha^a T_a}$

Hermitian generators in rep. for object on which D_μ acts.

g = coupling

$a = 1, \ldots, N^2 - 1$ gauge fields

Also, <u>require</u>

$$\boxed{A_\mu^a(x) T_a \to U A_\mu^a(x) T_a U^{-1} + \frac{i}{g}(\partial_\mu U)U^{-1}}$$

Note that these are matrix eqns. Also, D_μ contains T_a $\Rightarrow$ Form of D_μ depends on what it is acting on.

Quite often, one defines an $n \times n$ gauge field

$$\boxed{(A_\mu(x))_{jk} \equiv A_\mu^a(x)\,(T_a)_{jk}}$$

Prove that $D_\mu X$ is covariant if X is covariant.

Let $X \in \underline{n}$ of $SU(N)$, s.t. $X \to X' = U X$

In this notation,

$n \times n$ matrix

$$D_\mu X = \partial_\mu X + ig A_\mu X .$$

Under local gauge transformations,

$$D_\mu X \to \partial_\mu (U X) + ig\left[U A_\mu U^{-1} + \frac{i}{g}(\partial_\mu U) U^{-1}\right](U X)$$

$$= U \partial_\mu X + (\partial_\mu U) X$$

$$+ ig\, U A_\mu \underline{U^{-1}(U} X) + ig\left(+\frac{i}{g}\right)(\partial_\mu U)\underline{U^{-1}(U} X)$$

$$= U \partial_\mu X + \cancel{(\partial_\mu U)} X$$

$$+ ig\, U A_\mu X - \cancel{(\partial_\mu U)} X$$

$$= U(\partial_\mu X + ig A_\mu X) = U D_\mu X \qquad \blacksquare$$

So, if $\chi \to U\chi \Rightarrow \boxed{D_\mu \to U D_\mu U^{-1}}$

Note the Abelian limit,

$$U = e^{i\alpha(x)} \Rightarrow T = 1$$

So,

$$A_\mu^a T_a \to U A_\mu^a T_a U^{-1} + \frac{i}{g}(\partial_\mu U)U^{-1}$$

$$A_\mu \to e^{i\alpha} A_\mu e^{-i\alpha} + \frac{i}{g}(i\partial_\mu \alpha)\, e^{i\alpha} e^{-i\alpha}$$

$$\Rightarrow \quad A_\mu \to A_\mu - \frac{1}{g}\partial_\mu \alpha \quad ✓$$

Therefore, we have a new Lagrangian $\mathcal{L}'$

$$\mathcal{L}' = (D_\mu \varphi)^\dagger D^\mu \varphi - m^2 \varphi^\dagger \varphi - V(\varphi)$$

which is invariant under local SU(N) gauge transformations that includes N^2-1 gauge fields A_μ^a.

We now want to include a kinetic term for A_μ^a.
We require that it is Lorentz invariant, locally gauge invariant, 2nd order in derivatives, & generalizes Maxwell.

There is a useful "trick", with profound connections to geometry, to find the field-strength tensor for an SU(N) field.

Consider the U(1) case,

$$[D_\mu, D_\nu] = [\partial_\mu + igA_\mu, \partial_\nu + igA_\nu]$$
$$= ig\,\partial_\mu A_\nu - ig\,\partial_\nu A_\mu$$
$$= ig\,F_{\mu\nu}$$

$$\Rightarrow \boxed{igF_{\mu\nu} = [D_\mu, D_\nu]}$$

For SU(N) fields,

Matrices

$$[D_\mu, D_\nu] = [\partial_\mu + igA_\mu, \partial_\nu + igA_\nu]$$
$$= ig\,\partial_\mu A_\nu - ig\,\partial_\nu A_\mu + (ig)^2[A_\mu, A_\nu]$$
$$= ig\,F_{\mu\nu}$$

NOT zero for SU(N)

$$\Rightarrow \boxed{F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]}$$

This is the SU(N) field strength tensor.

Notice that we can pull out the SU(N) generators

$$F_{\mu\nu} = \partial_\mu A_\nu^a T_a - \partial_\nu A_\mu^a T_a + ig A_\mu^b A_\nu^c \underbrace{[T_b, T_c]}_{= i C_{bc}{}^a T_a}$$

$$= \left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g C_{bc}{}^a A_\mu^b A_\nu^c\right) T_a$$

$$\equiv F_{\mu\nu}^a T_a$$

where $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g C_{bc}{}^a A_\mu^b A_\nu^c$

Note that $C_{bc}{}^a = C_{abc}$ for N^2-1 gauge fields

(Assume killing metric)

So,

$$\boxed{F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu - g C_{abc} A_\mu^b A_\nu^c}$$

Recall that $D_\mu \to U D_\mu U^{-1}$ under local transformations

$\Rightarrow$ since $F_{\mu\nu} = \frac{1}{ig}[D_\mu, D_\nu] \Rightarrow F_{\mu\nu} \to U F_{\mu\nu} U^{-1}$

So, $F_{\mu\nu}$ is covariant, not invariant (unlike QED).

Therefore, to get an invariant term in the Lagrange density we take the trace,

i.e., $\mathcal{L}_{KE} \sim tr[F_{\mu\nu} F^{\mu\nu}]$ trace in matrix space of U

$$\longrightarrow tr[U F_{\mu\nu} U^{-1} U F^{\mu\nu} U^{-1}] = tr[F_{\mu\nu} F^{\mu\nu}]$$

This is invariant for any T_a. From group theory, $tr[T_a, T_b] \propto \delta_{ab}$ for any rep. of $SU(N)$.

Consider $\underset{\sim}{3}$ of $SU(3)$

$$(T_a)_{jk} = \frac{1}{2}(\lambda_a)_{jk}$$

$$\Rightarrow \quad tr[T_a \, T_b] = \frac{1}{4} tr[\lambda_a \, \lambda_b] = \frac{1}{4} \cdot 2\delta_{ab} = \frac{1}{2}\delta_{ab}$$

This normalization can be chosen for $\underset{\sim}{N}$ of $SU(N)$.

With this choice, trace in the vector rep.

$$\boxed{\begin{aligned} \mathcal{L}_{KE} &= -\frac{1}{2} tr[F_{\mu\nu} F^{\mu\nu}] \\ &= -\frac{1}{4} F_{\mu\nu}{}^{a} F^{\mu\nu, a} \end{aligned}}$$

Reduces to $-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ for $U(1)$ case correctly.

Therefore, the complete "scalar non-abelian gauge theory" is

$$\boxed{\mathcal{L}' = (D_\mu \varphi)^\dagger (D^\mu \varphi) - m^2 \varphi^\dagger \varphi - V(\varphi) - \frac{1}{2} tr[F_{\mu\nu} F^{\mu\nu}]}$$

Notice that A_μ^a must be massless to maintain gauge invariance, $\to -\frac{1}{2}m^2 A_\mu^a A^{\mu a}$ is not allowed in $\mathcal{L}$. t'Hooft (1973) showed that non-abelian gauge theories are renormalizable with suitable $V(\varphi)$.

The interactions are more complicated than scalar QED. Notice that in the gauge field kinetic term is

$$\mathcal{L}_{KE} = -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a}$$

$$\supset \underbrace{g\, C_{abc} (\partial_\mu A_\nu)^a A_\mu^b A_\nu^c}_{\text{triple gauge self-interaction}}, \quad \underbrace{g^2 C_{abe} C_{cde} A_\mu^b A_\nu^c A_\mu^d A_\nu^e}_{\text{quartic gauge self-int.}}$$

This gives interesting dynamics, e.g., bound states of gluons in $SU(3)_c$. This type of theory is generally called Yang-Mills theory (1954)

$$\boxed{\mathcal{L}_{YM} = -\frac{1}{2} \mathrm{tr}\left[F_{\mu\nu} F^{\mu\nu}\right]}$$

Functional Quantization of Yang-Mills Theory

Yang Mills theory is a self-interacting gauge-field theory invariant under SU(N) local gauge transformations. Let us look at its quantization. The Yang-Mills action is

$$S_{YM} = -\frac{i}{2}\int d^4x \; tr[F_{\mu\nu}F^{\mu\nu}]$$

$$= -\frac{i}{4}\int d^4x \; F_{\mu\nu}^{a} F^{\mu\nu a}$$

Like Abelian gauge theory, Yang-Mills suffers from the same issues upon quantization. Let us use the Faddeev-Popov procedure, anticipating a perturbative QFT in small coupling g.

$$\int \mathcal{D}A \; e^{i\int d^4x \, \mathcal{L}_{YM}}$$

↑ insert Faddeev-Popov unity

Let us focus on Lorentz ξ-gauges (R_ξ)

$$\boxed{G_\omega(A_\mu^a) = \partial_\mu A^{\mu a,\omega} - \omega^a = 0}$$

Following the same steps as in QED, we find

$$\int \mathcal{D}A \, e^{i\int d^4x \, \mathcal{L}_{YM}}$$

$$= \left(\int \mathcal{D}\alpha\right) \int \mathcal{D}A_\mu^a \, \Delta_{FP}^{YM}(A_\mu^a) \, \delta(G(A_\mu^a)) \, e^{i\int d^4x \, \mathcal{L}_{YM}}$$

$$= N_\xi \left(\int \mathcal{D}\alpha\right) \int \mathcal{D}A_\mu^a \, \Delta_{FP}^{YM}(A_\mu^a) \, e^{i\int d^4x \left(\mathcal{L}_{YM} - \frac{1}{2\xi}(\partial^\mu A_\mu^a)^2 \right)}$$

gauge transformation parameter

Result of Gaussian weighted integral over all possible gauge configurations.

The essential difference between this and QED is that here Δ_{FP}^{YM} is NOT a constant, and cannot be removed from the integral.

$$\Delta_{FP}^{-1}(A_\mu^a) = \int \mathcal{D}\alpha \, \delta(G(A_\mu^{a,\alpha}))$$

$$= \det{}^{-1}\left(\frac{\delta G}{\delta \alpha}\right)_{\alpha=0}$$

$$= \det{}^{-1}\left(\partial^\mu \frac{\delta A_\mu^{a,\alpha}(x)}{\delta \alpha^b(y)}\right)$$

To evaluate this, consider infinitesimal transformation

$$A_\mu \to U A_\mu U^{-1} + \frac{i}{g}(\partial_\mu U)U^{-1} \quad \text{with} \quad U = e^{i\alpha^a T_a}$$

if $\alpha^a \ll 1 \quad \Rightarrow \quad U = \mathbb{1} + i\alpha^a T_a$

$T_a^+ = T_a$

$$\Rightarrow A_\mu = A_\mu^a T_a \to (\mathbb{1} + i\alpha^a T_a) A_\mu^b T_b (\mathbb{1} - i\alpha^c T_c^+)$$
$$+ \frac{i}{g}(i\partial_\mu \alpha^a T_a)(\mathbb{1} - i\alpha^b T_b^+) + O(\alpha^2)$$
$$= A_\mu^a T_a + i\alpha^b A_\mu^c (T_b T_c - T_c T_b)$$
$$- \frac{1}{g}\partial_\mu \alpha^a T_a + O(\alpha^2)$$
$$= \left(A_\mu^a + i\alpha^b A_\mu^c (iC_{bca}) - \frac{1}{g}\partial_\mu \alpha^a\right) T_a$$

So
$$\boxed{\delta A_\mu^a = -\frac{1}{g}\partial_\mu \alpha^a - C_{abc}\,\alpha^b A_\mu^c + O(\alpha^2)}$$

So
$$\Delta_{FP}^{YM}(A_\mu^a) = \det\left(\frac{\delta}{\delta\alpha^b}\left(\partial^\mu\left(-\frac{1}{g}\partial_\mu \alpha^a(x) - C_{acd}\,\alpha^c A_\mu^d\right)\right)\right)$$
$$= \det\left(\partial^\mu\left[-\frac{1}{g}\delta^{ab}\partial_\mu - C_{abc}A_\mu^c\right]\delta^{(4)}(x-y)\right)$$

depends on A_μ^a !

$$= \det\left[M_{ab}(x-y)\right]$$

Such a determinant can be rewritten as a Gaussian integral over fictitious fermionic fields $\bar{c}^a(x)$ and $c^b(x)$

$$\det(M_{ab}) = \int \mathcal{D}\bar{c}\,\mathcal{D}c\; e^{i\int_{x,y} \bar{c}^a(x)\, M_{ab}(x-y)\, c^b(y)}$$

$$= \int \mathcal{D}\bar{c}\,\mathcal{D}c\; e^{i\int d^4x\, \bar{c}^a \partial^\mu [\delta^{ab}\partial_\mu + g C_{abc} A_\mu^c] c^b}$$

kinetic term of a massless bosonic field (yet, c's are fermions?)

coupling to gauge fields

The new (unphysical) degrees of freedom are called <u>Faddeev-Popov ghosts</u>

↳ (Have wrong statistics)

The entire quantum Yang-Mills Lagrange density is then

$$\boxed{\begin{aligned}\mathcal{L}_{YM}^{\xi} &= -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a} - \frac{1}{2\xi}(\partial^\mu A_\mu^a)^2 + \bar{c}^a \partial^\mu [\delta^{ab}\partial_\mu + g C_{abc} A_\mu^c] c^b \\ &= -\frac{1}{4}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)^2 - \frac{1}{2\xi}(\partial^\mu A_\mu^a)^2 + \bar{c}^a \partial^2 c^a + \mathcal{L}_I\end{aligned}}$$

where interaction term contains ghost-gauge and gauge self-interactions:

$$\boxed{\begin{aligned}\mathcal{L}_I = &- g\, C_{abc} (\partial_\mu \bar{c}^a) A_\mu^c c^b \\ &- \frac{1}{2} g\, C_{abc} (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) A^{b,\mu} A^{c,\nu} \\ &+ \frac{1}{4} g^2 C_{abc} C_{ade} A_\mu^b A_\nu^c A^{d,\mu} A^{e,\nu}\end{aligned}}$$

Note that physical observables are independent of gauge choice and ghost. But, these are needed to ensure proper features of the theory order-by-order in perturbation theory, e.g., unitarity. Some gauge choices can get rid of ghosts (see EW theory & unitarity gauge)

Feynman Rules

Assuming the coupling is small, we can organize observables as a perturbation series in g.

$$Z[J^a_\mu, \bar{\eta}^b_c, \eta^c_c]_{YM}$$

pure Yang-Mills ghost fields

$$= e^{i\int d^4x\, \mathcal{L}_I\left(-i\frac{\delta}{\delta J^a_\mu}, -i\frac{\delta}{\delta \bar{\eta}^b_c}, -i\frac{\delta}{\delta \eta^c_c}\right)} Z^0_{YM}[J^a_\mu]\, Z_g[\bar{\eta}^b_c, \eta^c_c]$$

w/

$$Z_g[\bar{\eta}^b_c, \eta^c_c] = e^{-i\int_{x,y} \bar{\eta}^a_c(x)\, iG^{ab}(x-y)\, \eta^b_c(y)}$$

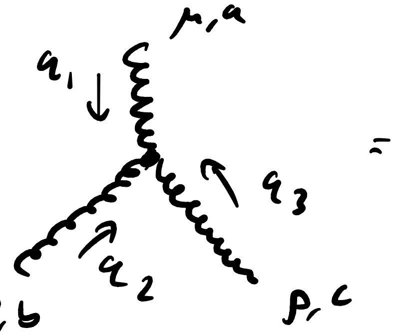
The propagators are

μ, a ~~~~~~~~ ν, b $\quad = -i\dfrac{\delta_{ab}}{k^2}\left[g_{\mu\nu} - (1-\xi)\dfrac{k_\mu k_\nu}{k^2}\right]$ (gauge)

a .... b $\quad = -i\dfrac{\delta_{ab}}{k^2} \equiv iG^{ab}(k)$ (ghost)

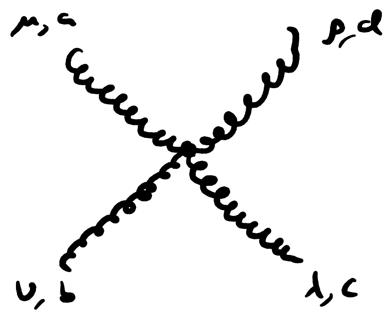
Notice

The vertices can also be derived

$i\Gamma^{(3)} =$ 

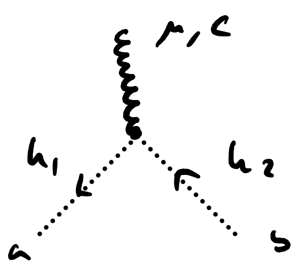
$$= -ig\, C_{abc}\Big[(q_1-q_2)_\rho\, g_{\mu\nu} + (q_2-q_3)_\mu\, g_{\nu\rho} + (q_3-q_1)_\nu\, g_{\mu\rho}\Big]$$

$$q_1+q_2+q_3=0$$

$i\Gamma^{(4)} =$ 

$$q_1+q_2+q_3+q_4=0$$

$$= ig^2\Big[C_{abe}C_{cde}\,(g_{\mu\lambda}g_{\nu\rho} - g_{\nu\lambda}g_{\mu\rho}) + C_{ace}C_{bde}\,(g_{\mu\nu}g_{\lambda\rho} - g_{\lambda\nu}g_{\mu\rho}) + C_{ade}C_{cbe}\,(g_{\mu\lambda}g_{\nu\rho} - g_{\rho\lambda}g_{\mu\nu})\Big]$$

$i\Gamma_g =$ 

$$= -g\, C_{abc}\, k_{1\mu}$$

Nonabelian Spinor Field theory

Repeat previous for spinor fields. Start with globally invariant theory

$$\mathcal{L} = \frac{1}{2} i \bar{\psi}_j \not{\partial} \psi_j + h.c. - m \bar{\psi}_j \psi_j$$

with $\psi_j \in \underset{\sim}{n}$ of $SU(N)$ $\Rightarrow$ $\psi_j \to U_{jk} \psi_k$

Already know $D_\mu X$ is covariant for all $X \to U X$

$\Rightarrow$ Immediately get nonabelian spinor gauge theory

$$\boxed{\mathcal{L} = \frac{1}{2} i \bar{\psi} \not{D} \psi + h.c. - m \bar{\psi} \psi - \frac{1}{2} \mathrm{tr}[F_{\mu\nu} F^{\mu\nu}]}$$

Warning: Many suppressed indices!

ψ has n components ψ_j, but also each one is a 4-component spinor.

So, really ψ_j^α $\longrightarrow \alpha = 1, \dots, 4$

$\longrightarrow j = 1, \dots, n$

e.g., $\not{D}$ is really

$$\not{D}_{jk}^{\alpha\beta} = (\gamma^\mu)^{\alpha\beta} \delta_{jk} \partial_\mu + ig (\gamma^\mu)^{\alpha\beta} A_\mu^a(x) (T_a)_{jk}$$

Quantum Chromodynamics

Consider fixed single quark flavor, u.

This comes in 3 colors, RGB. Label colors as $j = 1, 2, 3$.

$\Rightarrow$ quark field is ψ_{u_j} , $j = 1, 2, 3$ — suppressed spinor indices

Suppose u_j transforms as $\underset{\sim}{3}$ of $SU(3)_c$

$$\psi_{u_j} \to \psi'_{u_j} = U_{jk} \psi_{u_k}$$

where $U_{jk} = \left[\exp\left(\frac{1}{2} i \alpha^a \lambda_a \right) \right]_{jk}$ — Gell-Mann matrices

This theory has a <u>global</u> $SU(3)_c$ invariance

$$\mathcal{L} = \frac{1}{2} i \bar{\psi}_{u_j} \not{\partial} \psi_{u_j} + \text{h.c.} - m_u \bar{\psi}_{u_j} \psi_{u_j}$$

Promote to local invariance

$$\Rightarrow \delta_{jk} \partial_\mu \to (D_\mu)_{jk} = \delta_{jk} \partial_\mu + i g_s A_\mu^a \left(\tfrac{1}{2} \lambda_a \right)_{jk}$$

strong coupling (g_s)

$a = 1, \ldots, 8$ for $SU(3)_c$ $\Rightarrow$ 8 gauge fields "<u>gluons</u>"

and

$$G_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g_s f_{abc} A_\mu^b A_\nu^c$$

$su(3)$ structure constants

So, "one-flavor QCD" is

$$\mathcal{L} = \frac{1}{2} i \bar{\psi}_u \not{D} \psi_u + \text{h.c.} - m_u \bar{\psi}_u \psi_u - \frac{1}{2} \text{tr}[G_{\mu\nu} G^{\mu\nu}]$$

For full QCD (6 flavors), introduce flavor index

$$f = u, d, s, c, b, t$$

We then have 6 quark fields ψ_f with masses m_f.

Each flavor has 3 colors, so really ψ_{f_j}.

Each fermion has 4 spinor components, so really $\psi^{\alpha}_{f_j}$.

Theory of $3 \times 6 = 18$ free quarks is

$$\mathcal{L} = \frac{1}{2} i \sum_f \bar{\psi}_{f_j} \not{\partial} \psi_{f_j} + \text{h.c.} - \sum_f m_f \bar{\psi}_{f_j} \psi_{f_j}$$

Notice that this has global $SU(3)_c$ symmetry. Can also consider $SU(6)_F$ (flavor) transformations. This $SU(6)_F$ is broken if m_f depends on f.

i.e., ψ_{f_j} is a $\underset{\sim}{3}$ of $SU(3)_c$

& a $\underset{\sim}{6}$ of $SU(6)_f$ (broken!)

Can also have mass-mixing terms, which can be removed by diagonalizing the $SU(6)_F$ space.

To get 6-flavor QCD, promote $SU(3)_c$ to local symmetry, leaving $SU(6)_f$ (broken) global.
i.e., acting on ψ_{f_j}

$$\partial_\mu \to (D_\mu)_{jk} = \delta_{jk}\partial_\mu + i g_s A_\mu^a \left(\tfrac{1}{2}\lambda_a\right)_{jk}$$

↑ still only 8 gluons

This gives locally $SU(3)_c$ invariant, globally $SU(6)_f$ broken theory,

$$\boxed{\mathcal{L} = \frac{1}{2} i \sum_f \bar{\psi}_f \not{D} \psi_f + \text{h.c.} - \sum_f m_f \bar{\psi}_f \psi_f - \frac{1}{2} \operatorname{tr}\left[G_{\mu\nu} G^{\mu\nu}\right]}$$

This is QCD for 6 quark flavors. Upon quantization, this is a theory of interacting quarks and gluons, and self-interacting gluons. This is a very complicated theory because g_s is not small. Lattice methods have made it possible to compute low-energy physics.

General nonabelian Gauge theory with Scalars and Spinors

Let us assume a renormalizable, Lorentz invariant SU(N) gauge theory of a scalar and spinor field. The Lagrange density (schematically) takes the form

$$\boxed{\begin{aligned}\mathcal{L} = &(D_\mu \varphi)^\dagger (D^\mu \varphi) - m^2 \varphi^\dagger \varphi - V(\varphi) \\ &+ \tfrac{1}{2} i \bar{\psi} \not{D} \psi + h.c. - m \bar{\psi}\psi + g_\psi \varphi \bar{\psi}\psi \\ &- \tfrac{1}{2} tr[F_{\mu\nu} F^{\mu\nu}]\end{aligned}}$$

That's it!

Comments

- Scalars can be in several reps. of SU(N), $\underset{\sim}{n}, \underset{\sim}{m}', \underset{\sim}{m}'', \ldots$
  fermions " " " " " " " , $\underset{\sim}{n}, \underset{\sim}{n}', \underset{\sim}{n}'', \ldots$

- Thus the covariant derivative $D_\mu = \partial_\mu + ig A_\mu$ means that T_a in A_μ is chosen appropriately for different reps.

  e.g., $(D_\mu \varphi)^\dagger (D^\mu \varphi)$ may be

$$(D_\mu \varphi_{\underset{\sim}{n}})^\dagger (D^\mu \varphi_{\underset{\sim}{n}}) + (D_\mu \varphi_{\underset{\sim}{m}'})^\dagger (D^\mu \varphi_{\underset{\sim}{n}'})$$

- For the Yukawa term, $g_Y \varphi \bar{\psi} \psi$, we must ensure gauge invariance. So, if $\underset{\sim}{\varphi} \in \underset{\sim}{m}$, $\underset{\sim}{\psi} \in \underset{\sim}{n}'$, $\underset{\sim}{\bar{\psi}} \in \underset{\sim}{n}^*$, then the coupling,

$$g_Y \, \varphi_{\underset{\sim}{m}} \, \bar{\psi}_{\underset{\sim}{n}^*} \, \psi_{\underset{\sim}{n}'}$$

must be such that $\underset{\sim}{m} \times \underset{\sim}{n}^* \times \underset{\sim}{n}' \supset \underset{\sim}{1}$

So, $(g_Y)_{jkl}$ must have symmetries that select only $\underset{\sim}{1}$.

e.g., if $\underset{\sim}{m}, \underset{\sim}{n}, \underset{\sim}{n}' = \underset{\sim}{3}$ in $SU(3)$, cannot have Yukawa coupling because $\underset{\sim}{3} \times \underset{\sim}{3}^* \times \underset{\sim}{3} = \square \times \begin{smallmatrix}\square \\ \square\end{smallmatrix} \times \square \not\supset \underset{\sim}{1}$

- Different multiplets $\underset{\sim}{m}, \underset{\sim}{m}', \ldots, \underset{\sim}{n}, \underset{\sim}{n}', \ldots$ can have different masses.

- Also, can have terms $\bar{\psi} \gamma_5 \psi$, or ψ could be chiral,

e.g., ψ could be $\psi_L = \frac{1}{2}(1-\gamma_5)\psi$

or $\psi_R = \frac{1}{2}(1+\gamma_5)\psi$

or Majorana ψ_M satisfying $\psi^c = \psi$

with $\psi^c \equiv C\bar{\psi}^T$

also, possible $\varphi \bar{\psi} \gamma_5 \psi$

BUT, no Lorentz-violating terms, $\bar{\psi}\gamma^\mu\psi$, $\bar{\psi}\gamma_5\gamma^\mu\psi$, $\bar{\psi}\sigma^{\mu\nu}\psi$

- $V(\varphi)$ is a polynomial up to order 4 in φ, constrained by gauge invariance and hermiticity.

e.g., can find

$$(\varphi_i^\dagger \varphi_i)^2, \quad C_{jklm}\,\varphi_j \varphi_k \varphi_l \varphi_m$$

Has symmetries ensuring $\varphi\varphi\varphi\varphi \supset \underset{\sim}{1}$

- The gauge group could have a product structure

e.g.,
$$G = G_1 \otimes G_2$$

N generators

M generators

$\Rightarrow$ Algebra has form $A = A_1 \oplus A_2$

$\Rightarrow$ Two sets of gauge fields

$$\mathcal{L}_{KE} = -\frac{1}{4} F_{\mu\nu}^{a} F^{\mu\nu a} - \frac{1}{4} G_{\mu\nu}^{c'} G^{\mu\nu c'}$$

span one group

span other group

Covariant derivatives are then

$$D_\mu = \partial_\mu + ig_1 A_\mu^a T_a^{(1)} + ig_2 B_\mu^{c'} T_{c'}^{(2)}$$

independent couplings appear

For example, one factor could be $U(1)$, other could be $SU(3)_c$

$\Rightarrow$ Result is nonabelian gauge theory describing QED & QCD

$$\mathcal{L} = \frac{i}{2}\sum_f \bar{\psi}_f \not{D}_\mu \psi_f + h.c. - \sum_f m_f \bar{\psi}_f \psi_f$$
$$- \frac{1}{2} tr[G_{\mu\nu} G^{\mu\nu}] - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$$G_{\mu\nu} = \partial_\mu G_\nu - \partial_\nu G_\mu + i g_s [G_\mu, G_\nu]$$

with $A_\mu \in u(1)$, $G_\mu = G_\mu^a \frac{\lambda_a}{2} \in su(3)$, $a = 1, \dots, 8$

and

$$D_\mu = \partial_\mu + i \underset{\substack{\uparrow \\ \text{EM coupling}}}{Q_f e} A_\mu + i \underset{\substack{\uparrow \\ \text{strong coupling}}}{g_s} G_\mu$$

(Different charges for f)

Note: Not truly unified because 2 coupling constants!

- Could also have remnant global symmetries, "accidental symmetry"
  - Discrete, e.g., if $(\varphi^\dagger \varphi)^2$, $\varphi \to -\varphi$
  - Continuous, baryon number $\psi \to e^{i\alpha} \psi$

- The Standard Model is based on this $\mathcal{L}$.