Phenomenology I - QED

It is useful to examine some common QED processes. There are many important processes, we will concentrate only on a few. Let us first consider high-energy $\mu^-\mu^+$ pair production from e^-e^+ annihilation.

$e^-e^+ \to \mu^-\mu^+$

We are interested in the unpolarized differential and total cross-sections. This is often the desired quantity, given that the spins of the incoming particles are arbitrary (beams are unpolarized), and one does not measure the spins of the final state.

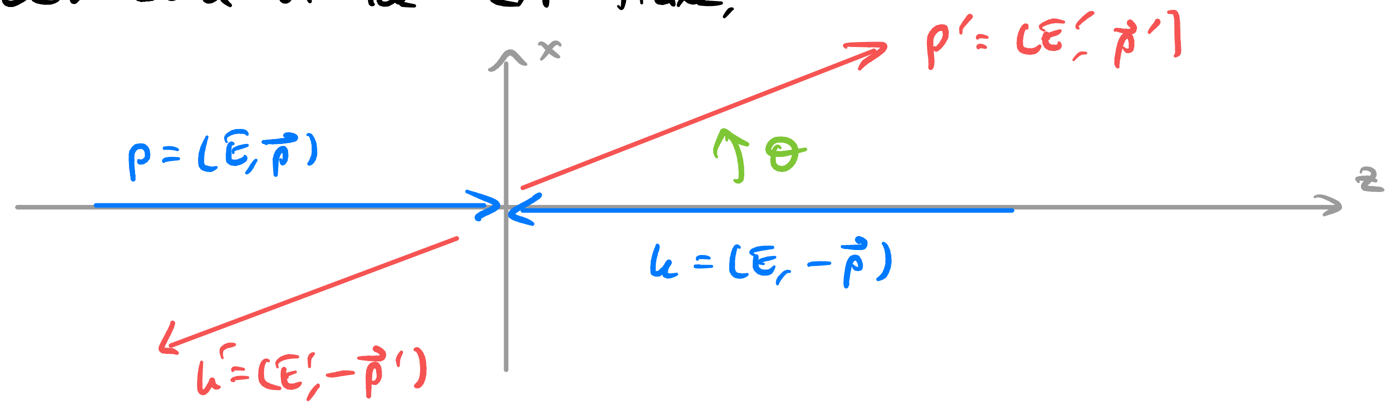
Let us define some kinematics

momentum, $p = (E, \vec{p})$

$$e^-_s(p) + e^+_r(k) \to \mu^-_{s'}(p') + \mu^+_{r'}(k')$$

spin projection

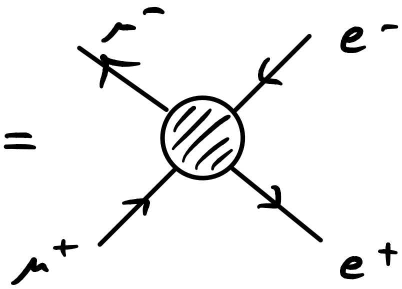
Let's work in the CM frame,

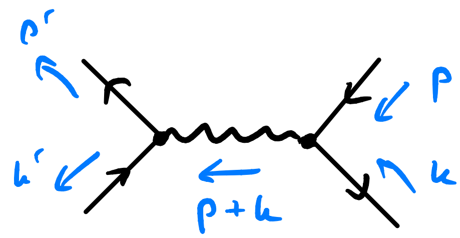


To simplify the analysis, let's consider the ultrarelativistic limit, $\sqrt{s} \gg m_\mu \gg m_e$. So, we have (exercise)

$$E = E' = \frac{\sqrt{s}}{2}, \quad |\vec{p}| = |\vec{p}\,'| \sim \frac{\sqrt{s}}{2}.$$

At leading order in α, there is a single Feynman diagram

$i\mathcal{M} =$ 

$=$ 

$+ \mathcal{O}(\alpha^2)$

$$= \underbrace{\bar{u}_{s'}(p')(-iq\gamma^\mu)v_{r'}(k')}_{\text{muon vertex}} \underbrace{\frac{-ig_{\mu\nu}}{(p+k)^2+i\epsilon}}_{\text{photon propagator}} \underbrace{\bar{v}_r(k)(-iq\gamma^\nu)u_s(p)}_{\text{electron vertex}}$$

$$= -i\frac{q^2}{s} a^\mu b_\mu$$

$$= -i\frac{4\pi\alpha}{s} a^\mu b_\mu$$

Now, the differential cross-section is

$$\frac{d\sigma}{d\Omega} = \int d\Phi_2 \langle|\mathcal{M}|^2\rangle$$

where $\langle|\mathcal{M}|^2\rangle = \frac{1}{4}\sum_{r,s}\sum_{r',s'} |\mathcal{M}|^2$

So,
$$|\mathcal{M}|^2 = \mathcal{M}^* \mathcal{M} = \mathcal{M}^\dagger \mathcal{M}$$
$$= \left(\frac{4\pi\alpha}{s}\right)^2 [a^\mu a^{\nu\dagger}][b_\mu b_\nu^\dagger]$$

Now,
$$a^\mu = \bar{u}\gamma^\mu v \Rightarrow a^{\mu\dagger} = (\bar{u}\gamma^\mu v)^\dagger \qquad \bar{u} = u^\dagger\gamma^0$$
$$= v^\dagger \gamma^{\mu\dagger} \bar{u}^\dagger = v^\dagger \gamma^0\gamma^\mu\gamma^0 (u^\dagger\gamma^0)^\dagger$$
$$\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$$
$$= \bar{v}\gamma^\mu \underbrace{\gamma^0\gamma^0}_{\gamma^{0\dagger} = \gamma^0 \ \& \ \gamma^{0\,2} = \mathbb{1}} u = \bar{v}\gamma^\mu u$$

$$\& \quad b^\mu = \bar{v}\gamma^\mu u \Rightarrow b^{\mu\dagger} = (\bar{v}\gamma^\mu u)^\dagger = \bar{u}\gamma^\mu v$$

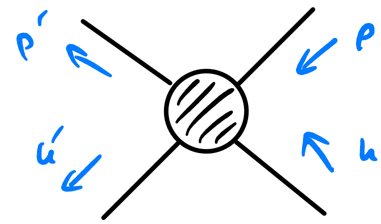
So, we find
$$\sum_{s,r} b_\mu b_\nu^\dagger = \sum_{s,r} \bar{v}_r(k)\gamma_\mu u_s(p)\,\bar{u}_s(p)\gamma_\nu v_r(k)$$
$$= \sum_{s,r} \mathrm{tr}\left[v_r(k)\bar{v}_r(k)\gamma_\mu u_s(p)\bar{u}_s(p)\gamma_\nu\right]$$
$$= \mathrm{tr}\Big[\underbrace{\sum_r v_r(k)\bar{v}_r(k)}_{= \not{k} - m_e \to \not{k}}\gamma_\mu \underbrace{\sum_s u_s(p)\bar{u}_s(p)}_{= \not{p} + m_e \to \not{p}}\gamma_\nu\Big]$$
$$= \mathrm{tr}[\not{k}\gamma_\mu\not{p}\gamma_\nu] + \mathcal{O}(m_e^2/s)$$
$$= 4(k_\mu p_\nu + k_\nu p_\mu - g_{\mu\nu}\, k\cdot p)$$

Recall the Mandelstam variables

$$s = (p+k)^2 = (p'+k')^2$$

$$t = (p-p')^2 = (k-k')^2$$

$$u = (p-k')^2 = (k-p')^2$$



For $s \gg m_\mu^2 \gg m_e^2 \Rightarrow \quad s \simeq 2p\cdot k, \quad t \simeq -2p\cdot p'$

So,
$$\sum_{s,r} b_\mu b_\nu^\dagger = 4\left(k_\mu p_\nu + k_\nu p_\mu - g_{\mu\nu}\, k\cdot p\right)$$
$$= 4\left(k_\mu p_\nu + k_\nu p_\mu - g_{\mu\nu}\, s/2\right)$$

Similarly,
$$\sum_{s',r'} a^\mu a^{\nu\dagger} = \mathrm{tr}\left[\not{p}'\gamma^\mu \not{k}'\gamma^\nu\right]$$
$$= 4\left(p'^\mu k'^\nu + p'^\nu k'^\mu - g^{\mu\nu} s/2\right)$$

So,
$$\langle |M|^2\rangle = \left(\frac{4\pi\alpha}{s}\right)^2 \overset{\text{Spin avg.}}{\frac{1}{4}} \sum_{s,r} b_\mu b_\nu^\dagger \sum_{s',r'} a^\mu a^{\nu\dagger}$$

$$= 4\left(\frac{4\pi\alpha}{s}\right)^2 \left(k_\mu p_\nu + k_\nu p_\mu - g_{\mu\nu} s/2\right)\left(p'^\mu k'^\nu + p'^\nu k'^\mu - g^{\mu\nu} s/2\right)$$

$$= 4\left(\frac{4\pi\alpha}{s}\right)^2 \left[2\, k\cdot p'\, p\cdot k' + 2\, k\cdot k'\, p\cdot p' \underbrace{- s\, k\cdot p}_{\cancel{-s\left(\frac{s}{2}\right)}} \underbrace{- s\, k'\cdot p'}_{\cancel{-s\left(\frac{s}{2}\right)}} \cancel{+ s^2}\right]$$

$$= 2\left(\frac{4\pi\alpha}{s}\right)^2 \left(u^2 + t^2\right)$$

but, $s + t + u = 0 \Rightarrow u = -s - t$

$$\Rightarrow \langle |M|^2\rangle = 2\left(\frac{4\pi\alpha}{s}\right)^2 \left(t^2 + (s+t)^2\right)$$

So,

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s} \cancelto{1}{\frac{|\vec{p}\,'|}{|\vec{p}|}} \langle |M|^2 \rangle$$

since $s \to \infty$

$$= \frac{1}{64\pi^2 s} \cdot 2 \left(\frac{4\pi\alpha}{s} \right)^2 \left(t^2 + (s+t)^2 \right)$$

$$= \frac{\alpha^2}{2s^3} \left(t^2 + (s+t)^2 \right)$$

$$= \frac{\alpha^2}{2s} \left[\left(\frac{t}{s}\right)^2 + \left(1 + \frac{t}{s}\right)^2 \right]$$

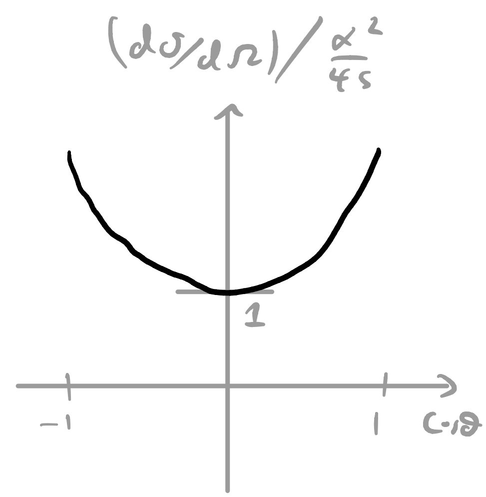
Now, $t = -2|\vec{p}\,'||\vec{p}|(1-\cos\theta)$

$$= -\frac{s}{2}(1-\cos\theta) \Rightarrow \frac{t}{s} = -\frac{1}{2} + \frac{1}{2}\cos\theta$$

So, $$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2s} \left[\frac{1}{4}(1-\cos\theta)^2 + \frac{1}{4}(1+\cos\theta)^2 \right]$$

$$= \frac{\alpha^2}{8s} \left[1 + \cos^2\theta - \cancel{\cos\theta} + 1 + \cos^2\theta + \cancel{\cos\theta} \right]$$

$$= \frac{\alpha^2}{4s}(1+\cos^2\theta)$$



Therefore,

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}(1+\cos^2\theta)}$$

Notice: Cross-section is symmetric in $\cos\theta$

The cross-section is then

$$
\begin{aligned}
\sigma &= \int d\Omega \, \frac{d\sigma}{d\Omega} \\
&= 2\pi \int_{-1}^{1} d\cos\theta \left[\frac{\alpha^2}{4s} \left(1 + \cos^2\theta \right) \right] \\
&= \frac{\pi\alpha^2}{2s} \left(\cos\theta \Big|_{-1}^{1} + \frac{\cos^3\theta}{3} \Big|_{-1}^{1} \right) \\
&= \frac{\pi\alpha^2}{2s} \left(2 + \frac{2}{3} \right) \\
&= \frac{4}{3} \frac{\pi\alpha^2}{s} \quad \Rightarrow \quad \boxed{\sigma = \frac{4}{3} \frac{\pi\alpha^2}{s}}
\end{aligned}
$$

This reaction ($e^-e^+ \to \mu^-\mu^+$) sets the scale for many high-energy e^-e^+-reactions. Let us define R as

$$
\boxed{R = \frac{\sigma(e^-e^+ \to \text{hadrons})}{\sigma(e^-e^+ \to \mu^-\mu^+)}}
$$

So, $\sigma(e^-e^+ \to \mu^-\mu^+)$ sets the scale.

$$
\begin{aligned}
\sigma(e^-e^+ \to \mu^-\mu^+) &= \frac{4\pi\alpha^2}{3s} (\hbar c)^2 \\
&= \frac{4\pi}{3s} \left(\frac{1}{137} \right)^2 (0.197 \text{ GeV} \cdot \text{fm})^2 \underbrace{\left(\frac{1 \text{ b}}{100 \text{ fm}^2} \right)}_{=1} \\
&\simeq 86.8 \text{ nb} \left(\frac{\text{GeV}^2}{s} \right)
\end{aligned}
$$

Quark - Antiquark production

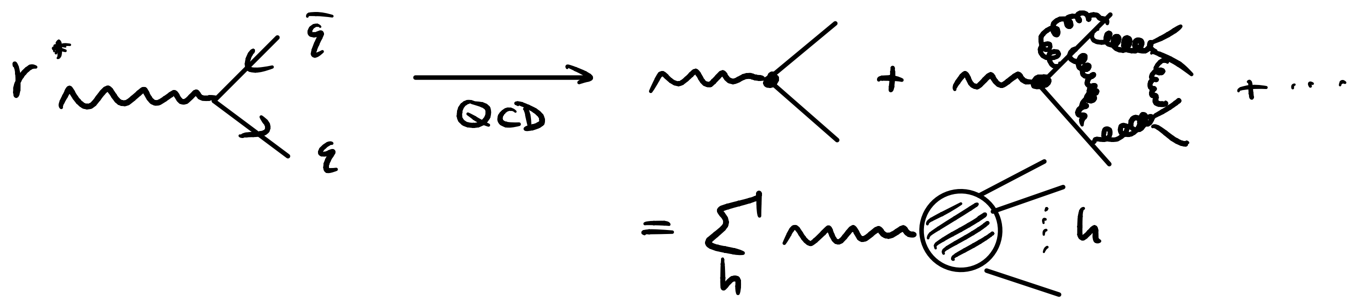
From the previous analysis, we get directly the cross-section for $q\bar{q}$ - final states. We only need to replace

$$e \rightarrow Q_q e$$

along with multiplying by N_c to account for $N_c = 3$ colors.

$$\Rightarrow \quad \sigma_{e^-e^+ \rightarrow q\bar{q}} = 3\, Q_q^2\, \sigma_{e^-e^+ \rightarrow \mu^-\mu^+}$$

Of course, free quarks are not observed, but hadrons



In experiment, the $q\bar{q}$ final state is observed as two back-to-back jets. The fact that the cross-section for high-energies can be computed from

$$\sigma_{e^-e^+ \rightarrow \text{hadrons}} \xrightarrow[\sqrt{s} \rightarrow \infty]{} \left(3 \sum_{f=1}^{N_f} Q_f^2 \right) \sigma_{e^-e^+ \rightarrow \mu^-\mu^+}$$

We will see that this is a consequence of an important property of QCD – Asymptotic Freedom.

At high energies, the photons "see" through the cloud of QCD virtual states surrounding each quark.