Lepton Anomalous Magnetic Moment

The anomalous magnetic moment, $g-2$, of the electron is one of the crowning achievements of QED. Experimental measurements and theoretical calculations agree to one part in a trillion! The $g-2$ of the muon is a strong test of the SM as it is sensitive to states beyond the SM. Measuring $g-2$ of the muon provides a probe into new BSM physics.

Here, lets focus on the first radiative correction of the lepton $g-2$.

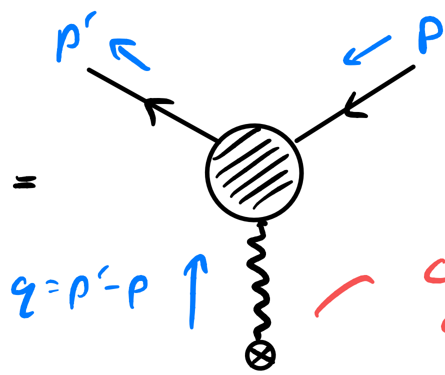
Recall that g is a measure of a leptons susceptibility to magnetic fields,

$$\vec{\mu} = \underbrace{g}_{\text{gyromagnetic ratio}} \frac{e}{2m} \underbrace{\vec{S}}_{\text{spin operator}}$$

To calculate g in QED, consider a lepton (here first focus on electron) in a <u>classical</u> (or background) EM field

$$\mathcal{L}_{int} = -e\bar{\psi}\gamma^\mu\psi(\underbrace{A_\mu}_{\text{quantum field}} + \underbrace{A_\mu^{cl.}}_{\text{classical field}})$$

We want to compute the one-body scattering amplitude and connect it to the non-relativistic potential $V = -\vec{\mu}\cdot\vec{B}$.

$iM =$ 

classical EM field
e.g., from static nucleus

One-body amplitude has generic structure

Momentum not conserved in background field.

$$\langle p', s' | iT | p, s \rangle \equiv 2\pi\,\delta(E'-E)\, iM$$
$$= \bar{u}(p',s')\left[-ie\Gamma^\mu(p',p)\right]u(p,s)\,\tilde{A}^{cl.}_\mu(q)$$

$$\tilde{A}^{cl.}_\mu(q) = \int d^4x\, e^{iq\cdot x} A^{cl.}_\mu(x)$$

The <u>vertex function, Γ^μ,</u> in general contains 12 tensors formed from momenta and gamma matrices. We can simplify things by considering on-shell leptons only, that is we use the Dirac eqn.

$$(\not{p} - m)u(p,s) = 0 \qquad \text{with} \qquad p^2 = m^2$$
$$\bar{u}(p',s')(\not{p}' - m) = 0 \qquad \text{with} \qquad p'^2 = m^2$$

This reduces the number of terms. From Lorentz invariance and C,P,T symmetry (recall QED is invariant under C,P,T), we can write generally

$$\Gamma^\mu = A\gamma^\mu + B(p'+p)^\mu + C(p'-p)^\mu$$

where A, B, C are scalar functions of

$$Q^2 \equiv -q^2 = -(p'-p)^2 = 2p'\cdot p - 2m^2$$

The EM current is conserved, so from Ward identity

$$\begin{aligned} 0 &= q_\mu \Gamma^\mu \\ &= q_\mu \left(A\gamma^\mu + B(p'+p)^\mu + C(p'-p)^\mu \right) \\ &= A\not{q} + B\, q\cdot p + C q^2 \end{aligned}$$

$\bar{u}(p')\not{q}u(p) = 0$ $\qquad$ $q\cdot p = p'^2 - p^2 = 0$

$$= Cq^2$$

$$\Rightarrow \quad C = 0$$

Also, by convention, we use the Gordon Identity,

$$\bar{u}\gamma^\mu u = \bar{u}\left[\frac{(p'+p)^\mu}{2m} + \frac{i\sigma^{\mu\nu}}{2m} q_\nu \right] u$$

where $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$

to write the vertex function Γ^μ in the form

$$\boxed{\Gamma^\mu(p',p) = \gamma^\mu F_1(Q^2) + \frac{i\sigma^{\mu\nu}}{2m} q_\nu F_2(Q^2)}$$

F_1 is the Dirac Form-factor

F_2 is the Pauli Form-factor

The Form-factors contain complete information about the EM fields influence on the lepton.

To gain an understanding of their physical meaning, let us consider time-independent field configurations

$$A_\mu^{cl.}(x) = A_\mu^{cl.}(\vec{x})$$

or,
$$\tilde{A}_\mu^{cl.}(q) = 2\pi\,\delta(q^0)\,\tilde{A}_\mu^{cl.}(\vec{q})$$

Electric Coupling

Consider a static electric source $A_\mu^{cl.}(x) = (\varphi(\vec{x}), \vec{0})$

$$\Rightarrow \tilde{A}_\mu^{cl.}(\vec{q}) = (\tilde{\varphi}(\vec{q}), \vec{0})$$

Must recover a non-relativistic limit

$$V(\vec{x}) = e\,\varphi(\vec{x})$$

So,

$$i\mathcal{M} = -ie\,\bar{u}(p',s')\,\Gamma^0(p',p)\,u(p,s)\,\tilde{\varphi}(\vec{q})$$

Suppressing $2\pi\,\delta(E'-E)$

$$= -ie\,\bar{u}(p',s')\left\{\gamma^0 F_1 + \frac{i\sigma^{0\nu}}{2m} q_\nu F_2\right\} u(p,s)\,\tilde{\varphi}(\vec{q})$$

We want to examine the non-relativistic limit,

$\vec{q} = (\vec{p}\,' - \vec{p}) \to 0$ and $\vec{p} \to \vec{0}$.

So,
$$\bar{u}(p',s')\,\Gamma^0(p',p)\,u(p,s) = \bar{u}(p',s')\,\gamma^0 u(p,s)\,F_1$$
$$= u^\dagger(p',s')\,u(p,s)\,F_1$$

Recall the Dirac spinors in Chiral basis

$$u(p,s) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi_s \\ \sqrt{p\cdot\bar{\sigma}}\,\xi_s \end{pmatrix} \qquad \text{with} \quad \begin{aligned} \sigma &= (\mathbb{1}, \vec{\sigma}) \\ \bar{\sigma} &= (\mathbb{1}, -\vec{\sigma}) \end{aligned}$$

and $\xi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\xi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

In the non-relativistic limit
$$\sqrt{p\cdot\sigma} = \sqrt{m - \vec{p}\cdot\vec{\sigma}}$$
$$\simeq \sqrt{m}\left(1 - \frac{\vec{p}\cdot\vec{\sigma}}{2m}\right)$$
$$\sqrt{p\cdot\bar{\sigma}} = \sqrt{m + \vec{p}\cdot\vec{\sigma}}$$
$$\simeq \sqrt{m}\left(1 + \frac{\vec{p}\cdot\vec{\sigma}}{2m}\right)$$

$$\Rightarrow u^\dagger(p',s')\,u(p,s) = m\,(\xi_{s'}^\dagger, \xi_{s'}^\dagger)\begin{pmatrix} \xi_s \\ \xi_s \end{pmatrix} + \mathcal{O}(\vec{p}, \vec{p}\,')$$
$$= 2m\,\xi_{s'}^\dagger \xi_s + \mathcal{O}(\vec{p}, \vec{p}\,')$$
$$= 2m\,\delta_{s's} + \mathcal{O}(\vec{p}, \vec{p}\,')$$

spin preserving

Therefore, the T-matrix element is

$$i\mathcal{M} \simeq -ieF_1(0)\,\tilde{\varphi}(\vec{q})\cdot 2m\,\delta_{s's}$$

Let us compare to the Born amplitude in non-relativistic quantum mechanics

$$\langle \vec{p}'|iT|\vec{p}\rangle = -i\tilde{V}(\vec{q})\cdot 2\pi\delta(E'-E)$$

states in NRQM

normalized as $\langle \vec{p}'|\vec{p}\rangle = (2\pi)^3\delta^{(3)}(\vec{p}'-\vec{p})$

So, conclude

$$\tilde{V}(\vec{q}) = -\frac{1}{2m}\mathcal{M}$$

$$= eF_1(0)\,\tilde{\varphi}(\vec{q})\cdot\delta_{s's}$$ keep implicit in $V(\vec{x})$

Fourier transform

$$V(\vec{x}) = eF_1(0)\,\varphi(\vec{x})$$

$$\Rightarrow \boxed{F_1(0) = 1}$$ to all orders in perturbation theory!

At zero momentum transfer, the Dirac form-factor is fixed to 1. This is known as the charge renormalization condition. In other words, "e" is a free parameter in QED, and we fix it by requiring $F_1(0) = 1$.

Magnetic Coupling

Let us repeat the previous analysis for a static magnetic field. The vector potential of a static magnetic field is $A_\mu = (0, \vec{A})$ with $\vec{B} = \vec{\nabla} \times \vec{A}$

Consider the k^{th}-component

$$B^k = (\vec{\nabla} \times \vec{A})^k = \epsilon^{kij} \partial_i A_j$$

Now,

$$B^k(\vec{x}) = \int \frac{d^3\vec{q}}{(2\pi)^3} e^{i\vec{q}\cdot\vec{x}} \tilde{B}^k(\vec{q})$$

$$= \epsilon^{kij} \frac{\partial}{\partial x^i} \int \frac{d^3\vec{q}}{(2\pi)^3} e^{i\vec{q}\cdot\vec{x}} \tilde{A}_j(\vec{q})$$

$$= \int \frac{d^3\vec{q}}{(2\pi)^3} e^{i\vec{q}\cdot\vec{x}} \epsilon^{kij} \frac{\partial}{\partial x^i} (i q^\ell x^\ell) \tilde{A}_j(\vec{q})$$

So,

$$\tilde{B}^k(\vec{q}) = \epsilon^{kij} \cdot i q^\ell \delta_{i\ell} \tilde{A}_j(\vec{q})$$

$$= i\epsilon^{kij} q_i \tilde{A}_j(\vec{q}) = i\epsilon^{kij} q_i \tilde{A}_j(\vec{q})$$

$$= + i\epsilon^{ijk} q_i \tilde{A}_j(\vec{q}) \quad \checkmark \quad \epsilon^{kij} = -\epsilon^{ikj} = +\epsilon^{ijk}$$

$$\Rightarrow \quad \tilde{B}^k(\vec{q}) = i\epsilon^{ijk} q_i \tilde{A}_j^{\text{cl.}}(\vec{q}) \; ^*$$

we want to keep terms proportional to $\vec{q}$

Note: Peskin & Schroeder report

$$\tilde{B}_k(\vec{q}) = -i\epsilon^{ijk} q_i \tilde{A}_j^{\text{cl.}}(\vec{q})$$

The scattering amplitude is then

$$i\mathcal{M} = -ie\,\bar{u}(p',s')\,\Gamma^{\mu}(p',p)\,u(p,s)\,\tilde{A}^{cl.}_{\mu}(\vec{q})$$

$$= +ie\,\bar{u}(p',s')\,\Gamma^{k}(p',p)\,u(p,s)\,\tilde{A}^{cl.}_{k}(\vec{q})$$

$$= +ie\,\bar{u}(p',s')\left[\gamma^{k}F_1 + \frac{i\sigma^{k\nu}q_\nu}{2m}F_2\right]u(p,s)\,\tilde{A}^{cl.}_{k}(\vec{q})$$

It is useful to use the Gordon Identity

$$\bar{u}\gamma^{\mu}u = \bar{u}\left[\frac{(p'+p)^{\mu}}{2m} + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\right]u$$

such that

spin-dependent term

$$i\mathcal{M} = ie\,\bar{u}(p',s')\left[\frac{(p'+p)^{\mu}}{2m}F_1 + \frac{i\sigma^{k\nu}}{2m}q_\nu\,(F_1+F_2)\right]u(p,s)\,\tilde{A}^{cl.}_{k}(\vec{q})$$

spin-independent
$\Rightarrow$ contributes only to kinetic energy $\sim \vec{p}\cdot\vec{A}$

Keeping only the spin-dependent piece

$$i\mathcal{M} = ie\,[F_1+F_2]\,\bar{u}(p',s')\,\frac{i\sigma^{k\nu}}{2m}q_\nu\,u(p,s)\,\tilde{A}^{cl.}_{k}(\vec{q})$$

We now take the non-relativistic limit.

Recall

$$u(p,s) \simeq \sqrt{m}\begin{pmatrix}\left(1-\frac{\vec{p}\cdot\vec{\sigma}}{2m}\right)\xi_s \\ \left(1+\frac{\vec{p}\cdot\vec{\sigma}}{2m}\right)\xi_s\end{pmatrix} = \sqrt{m}\begin{pmatrix}\xi_s \\ \xi_s\end{pmatrix}$$

Since we are already working linear in q in the vertex function.

So, $i\mathcal{M} = ie[F_1(0) + F_2(0)]\, u^\dagger(p',s')\, i\frac{\gamma^0\sigma^{k\nu}}{2m} q_\nu\, u(p,s)\, \tilde{A}^{cl.}_k(\vec{q})$

$$\simeq ie[F_1(0) + F_2(0)]\, m(\xi_{s'}^\dagger, \xi_{s'}^\dagger)\, i\frac{\gamma^0\sigma^{k\nu}}{2m} q_\nu \begin{pmatrix} \xi_s \\ \xi_s \end{pmatrix} \tilde{A}^{cl.}_k(\vec{q})$$

Now, $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$

$$= \frac{i}{2}(\gamma^\mu\gamma^\nu - \underbrace{\gamma^\nu\gamma^\mu})$$

$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$

$$= i\gamma^\mu\gamma^\nu - ig^{\mu\nu}$$

So, $\gamma^0\sigma^{k\nu}q_\nu = i\gamma^0(\gamma^k\gamma^\nu - g^{k\nu})q_\nu$

$$= i\gamma^0 \underbrace{\gamma^k\gamma^0}_{= -\gamma^0\gamma^k} q_0 - i\gamma^0\gamma^k\gamma^j q_j + i\gamma^0 q^k$$

$$= -i\underbrace{\cancel{\gamma^0\gamma^0}}_{\mathbb{1}}\gamma^k q_0 + i\gamma^0 q^k - i\gamma^0\gamma^k\gamma^j q_j$$

Drop as $\vec{q} \to 0$

In non-relativistic limit, $q_0 \to 0$

$$\Rightarrow \gamma^0\sigma^{k\nu}q_\nu = -i\gamma^0\gamma^k\gamma^j q_j$$

$$= -i\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} q_j$$

$$= -i\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}\begin{pmatrix} -\sigma^k\sigma^j & 0 \\ 0 & -\sigma^k\sigma^j \end{pmatrix} q_j$$

$$= i\begin{pmatrix} 0 & \sigma^k\sigma^j \\ \sigma^k\sigma^j & 0 \end{pmatrix} q_j$$

So, the T-matrix element is

$$i\mathcal{M} \simeq ie\left[F_1(0)+F_2(0)\right] m \left(\xi_{s'}^{\dagger}, \xi_{s'}^{\dagger}\right) \frac{i\gamma^0\sigma^{k\nu}q_\nu}{2m} \begin{pmatrix}\xi_s\\ \xi_s\end{pmatrix} \tilde{A}_k^{cl.}(\vec{q})$$

$$= ie\left[F_1(0)+F_2(0)\right] m \left(\frac{\cancel{i}}{2m}\right)$$

$$\times \left(\xi_{s'}^{\dagger}, \xi_{s'}^{\dagger}\right) \cancel{i} \begin{pmatrix} 0 & \sigma^k\sigma^j \\ \sigma^k\sigma^j & 0 \end{pmatrix} \begin{pmatrix}\xi_s\\ \xi_s\end{pmatrix} q_j \tilde{A}_k^{cl.}(\vec{q})$$

$$= -i\left(\frac{e}{2m}\right) m \left[F_1(0)+F_2(0)\right] \xi_{s'}^{\dagger}\left(\sigma^k\sigma^j + \sigma^k\sigma^j\right)\xi_s\, q_j \tilde{A}_k(\vec{q})$$

$$= -i\left(\frac{e}{m}\right) 2m \left[F_1(0)+F_2(0)\right] \xi_{s'}^{\dagger}\sigma^k\sigma^j\xi_s\, q_j \tilde{A}_k^{cl.}(\vec{q})$$

Recall $\sigma^k\sigma^j = \delta^{jk}\mathbb{1} + i\epsilon^{kjl}\sigma_l$

$\epsilon^{kjl} = -\epsilon^{jkl}$

therefore $\sigma^k\sigma^j q_j = \underbrace{q^k}_{\text{Drop as } \vec{q}\to 0}\mathbb{1} + i\epsilon^{kjl}\sigma_l q_j$

$$\Rightarrow i\mathcal{M} = +i\left(\frac{e}{2m}\right) 2m\left[F_1(0)+F_2(0)\right] \underbrace{i\epsilon^{jkl} q_j \tilde{A}_k^{cl.}(\vec{q})}_{\tilde{B}^l(\vec{q})} \xi_{s'}^{\dagger}\sigma_l\xi_s$$

Recall $\langle\vec{S}\rangle = \xi_{s'}^{\dagger}\dfrac{\vec{\sigma}}{2}\xi_s$

Therefore, we find

$$i\mathcal{M} = i(2m)\cdot 2\left[F_1(0)+F_2(0)\right]\left(\frac{e}{2m}\right)\langle\vec{S}\rangle\cdot\vec{\tilde{B}}(\vec{q})$$

Compare with Born approximation

$$\tilde{V}(\vec{q}) = -\frac{1}{2m}\mathcal{M}$$

$$= -\frac{e}{2m}\cdot 2[F_1(0)+F_2(0)]\langle\vec{S}\rangle\cdot\tilde{\vec{B}}(\vec{q})$$

Fourier transform

$$V(x) = -\underbrace{\frac{e}{2m}\cdot 2[F_1(0)+F_2(0)]\langle\vec{S}\rangle}_{\langle\vec{\mu}\rangle}\cdot\vec{B}(\vec{x})$$

$$= -\langle\vec{\mu}\rangle\cdot\vec{B}(\vec{x})$$

Lepton magnetic moment

$$\boxed{\langle\vec{\mu}\rangle = 2[F_1(0)+F_2(0)]\frac{e}{2m}\langle\vec{S}\rangle}$$

Generally, $\vec{\mu} = g\frac{e}{2m}\vec{S}$

Landé g-factor

so, $g = 2[F_1(0)+F_2(0)]$

$= 2[1 + F_2(0)]$

Charge renormalization

The Dirac eqn. predicts $g=2$, i.e., to leading order in QED

$$g = 2 + \mathcal{O}(\alpha)$$

$$\Rightarrow F_2(0) = \mathcal{O}(\alpha)$$

Unlike the electric charge, g is not a parameter of QED $\Rightarrow$ g is a pure prediction!

Therefore, we can compute the form-factor order-by-order in $\alpha = e^2/4\pi$.

$$F_2(0) = \underbrace{F_2^{(1)}(0)}_{\mathcal{O}(\alpha)} + \underbrace{F_2^{(2)}(0)}_{\mathcal{O}(\alpha^2)} + \cdots$$

It is convention to define $F_2(0) \equiv a_\ell$, the anomalous magnetic moment of the lepton ℓ

$$\boxed{\begin{aligned} a_\ell &= F_2(0) \\ &= \frac{g-2}{2} \end{aligned}}$$

A comment on the UV behavior of a_ℓ. Since g is NOT a parameter of the theory, it cannot be used to absorb UV divergences of radiative corrections. We would require an operator of the form

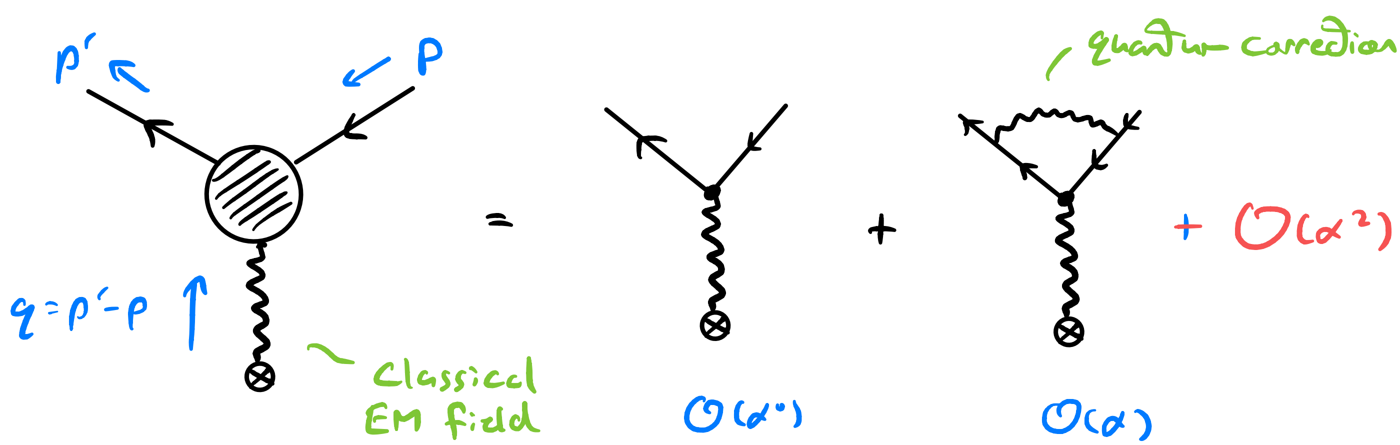
$$\mathcal{L} = g \cdot e \bar{\psi} \frac{i\sigma^{\mu\nu}}{2m} F_{\mu\nu} \psi$$

this can cancel a divergence.

However, if QED is to be a renormalizable QFT, this operator is NOT allowed as it is a dimension-5 operator. Without such an operator, there can be no UV divergence, or the theory is not correct or not "renormalizable".

Perturbative corrections to "g−2"

Let us begin the computation of the anomalous magnetic moment of the electron in QED. We will concentrate on the leading $\mathcal{O}(\alpha^0)$ and next-to-leading $\mathcal{O}(\alpha)$ perturbative corrections in $\alpha = e^2/2m$. From the Feynman rules for an electron in a classical EM field



$$-ie\,\Gamma^\mu(p',p) = -ie \sum_{n=0}^{\infty} \Gamma_n^\mu(p',p)$$

coupling to classical field

The quantum corrections have the form

$$\Gamma_n^\mu = \left(\frac{\alpha}{\pi}\right)^n \bar{\Gamma}_n^\mu$$

no α-dependence

conventional

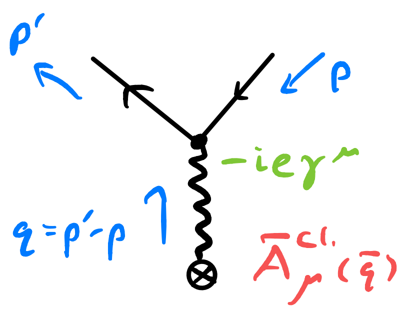
The corresponding Form-factors have the expansion

$$F_j = \sum_{n=0}^{\infty} F_j^{(n)} \qquad \text{for } j = 1, 2 .$$

Leading order

At leading order,

$$i\mathcal{M} =$$



$$+ \mathcal{O}(\alpha)$$

$$= -ie\,\bar{u}(p',s')\left[\gamma^\mu + \mathcal{O}(\alpha)\right]u(p,s)\,\tilde{A}^{cl.}_\mu(\bar{q})$$

So, we find $\Gamma_0^\mu = \gamma^\mu$. Comparing to the generic Lorentz decomposition, we conclude

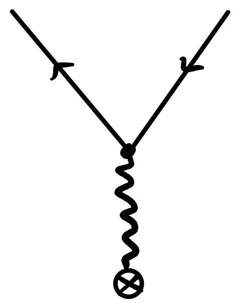
$$F_1^{(0)} = 1 \quad , \quad F_2^{(0)} = 0 .$$

So, $\boxed{g = 2 + \mathcal{O}(\alpha)}$ Dirac's triumph!

Next-to-Leading Order

At next-to-leading order (NLO), we find four diagrams contributing to the amplitude at $\mathcal{O}(\alpha)$

$i\mathcal{M}$ =



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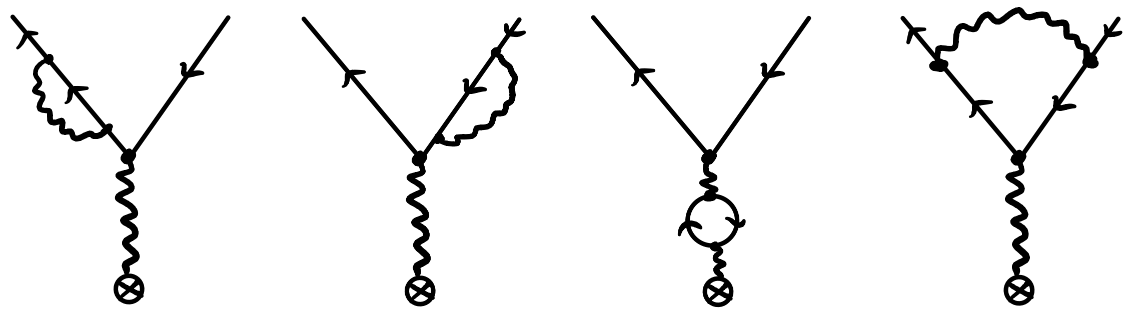


$+ \mathcal{O}(\alpha^2)$

The first two terms at $\mathcal{O}(\alpha)$ contribute to the mass and wavefunction renormalization of the electron. The third term contributes to the vacuum polarization of the EM field. The last diagram is the only correction to the vertex function

It is convenient to define $\Gamma^\mu = \gamma^\mu + \Lambda^\mu$,

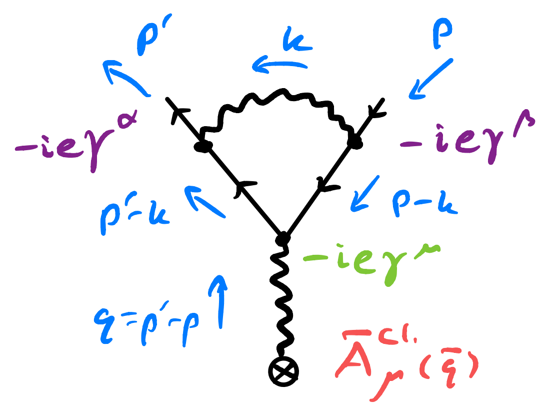
with the expansion

$$\Lambda^\mu = \sum_{n=0}^{\infty} \Lambda^\mu_n$$

Since $\Gamma^\mu_0 = \gamma^\mu$, $\Rightarrow \Lambda^\mu_0 = 0$.

From the QED Feynman rules, the vertex correction is

$$-ie\,\bar{u}(p',s')\,\Lambda^\mu_1(p',p)\,u(p,s)\,\tilde{A}^{cl.}_\mu(\vec{q}) =$$



where Λ^μ_1 is

$$\Lambda^\mu_1(p',p) = (-ie)^2 \int \frac{d^4k}{(2\pi)^4} \frac{-ig_{\alpha\beta}}{k^2} \gamma^\alpha \frac{i}{\not{p}'-\not{k}-m} \gamma^\mu \frac{i}{\not{p}-\not{k}-m} \gamma^\beta$$

$$= -e^2 \int \frac{d^4k}{(2\pi)^4} \frac{-i^3\, N^\mu(p',p,k)}{k^2[(p'-k)^2-m^2][(p-k)^2-m^2]}$$

with $N^\mu = \gamma_\nu [(\not{p}'-\not{k})+m] \gamma^\mu [(\not{p}-\not{k})+m] \gamma^\nu$

Note that we will always be interested in the on-shell case, $p'^2 = p^2 = m^2$, and the Dirac spinors acting on Λ^μ_1.

Form-Factor Extraction

Our task is now to compute Λ_1^μ. In general, loop integrals are divergent and require regularization. We know that g is a finite prediction, which means that the contribution to F_2 must be finite and does not need regularization. It is convenient to isolate the contribution to F_2, such that we can avoid the complications of regularization. Let us then construct projectors for the form-factors.

Recall

$$\Gamma^\mu = \gamma^\mu F_1 + \frac{i\sigma^{\mu\nu}}{2m} q_\nu F_2$$

The vertex is evaluated on-mass-shell, $\bar{u}(p',s')\Gamma^\mu u(p,s)$.

The Dirac spinors themselves satisfy Dirac's eqn.

$$(\not{p}-m)u = 0 \quad \text{and} \quad \bar{u}(\not{p}-m) = 0$$

Since completeness relation is $\sum_s' u(p,s)\bar{u}(p,s) = \not{p}+m$ we expect that $(\not{p}'+m)\Gamma^\mu(\not{p}+m)$ has the same decomposition

$$(\not{p}'+m)\Gamma^\mu(\not{p}+m) = (\not{p}'+m)\left[\gamma^\mu F_1 + \frac{i\sigma^{\mu\nu}}{2m} q_\nu F_2\right](\not{p}+m)$$

provided $p^2 = p'^2 = m^2$.

We now multiply on the left and contract with both $P_\mu = (p'+p)_\mu^*$ and γ_μ, and then take the trace

- $$\mathrm{tr}[(\not{p}'+m)\not{P}_\mu \Gamma^\mu(\not{p}+m)] = \mathrm{tr}[(\not{p}'+m)\not{P}(\not{p}+m)]F_1 + \mathrm{tr}[(\not{p}'+m)P_\mu \sigma^{\mu\nu}q_\nu(\not{p}+m)]\frac{iF_2}{2m}$$

- $$\mathrm{tr}[\gamma_\mu(\not{p}'+m)\Gamma^\mu(\not{p}+m)] = \mathrm{tr}[\gamma_\mu(\not{p}'+m)\gamma^\mu(\not{p}+m)]F_1 + \mathrm{tr}[\gamma_\mu(\not{p}'+m)\sigma^{\mu\nu}q_\nu(\not{p}+m)]\frac{iF_2}{2m}$$

Recall the trace identities

$$\mathrm{tr}[\mathbb{1}] = 4$$
$$\mathrm{tr}[\gamma_{\alpha_1}\cdots\gamma_{\alpha_{2n+1}}] = 0$$
$$\mathrm{tr}[\gamma_\mu\gamma_\nu] = 4g_{\mu\nu}$$
$$\mathrm{tr}[\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma] = 4(g_{\mu\nu}g_{\rho\sigma} - g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho})$$

$$\gamma_\mu\gamma^\mu = 4\mathbb{1}$$
$$\gamma_\mu\gamma^\nu\gamma^\mu = -2\gamma^\nu$$
$$\gamma_\mu\gamma^\nu\gamma^\rho\gamma^\mu = 4g^{\nu\rho}$$
$$\gamma_\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\mu = -2\gamma^\sigma\gamma^\rho\gamma^\nu$$

We note some useful kinematic relations, with $p'^2 = p^2 = m^2$

$$\left.\begin{aligned} P^2 &= (p'+p)^2 = 2m^2 + 2p'\cdot p \\ q^2 &= (p'-p)^2 = 2m^2 - 2p'\cdot p \end{aligned}\right\} \quad P^2 + q^2 = 4m^2,$$

$$P\cdot q = (p'+p)\cdot(p'-p) = p'^2 - p^2 = 0$$

$$p'\cdot P = p\cdot P = 2m^2 - \frac{q^2}{2}$$

$$p'\cdot q = -p\cdot q = \frac{q^2}{2}$$

Evaluating the traces

$$\mathrm{tr}[(\not{p}'+m)\not{P}(\not{p}+m)] = m\,\mathrm{tr}[\not{p}'\not{P}] + m\,\mathrm{tr}[\not{P}\not{p}]$$

$$= 4m\,(p'\cdot P + p\cdot P)$$

$$= 4m(4m^2 - q^2)$$

want to express scalars in terms of momentum transfer q^2

$$\mathrm{tr}[(\not{p}'+m)P_\mu \sigma^{\mu\nu} q_\nu(\not{p}+m)] = i\,\mathrm{tr}[(\not{p}'+m)\not{P}\not{q}(\not{p}+m)] - \cancel{i\,\mathrm{tr}[(\not{p}'+m)P\cdot q(\not{p}+m)]}$$

recall $\sigma^{\mu\nu} = i\gamma^\mu\gamma^\nu - ig^{\mu\nu}$

$P\cdot q = 0$

with,

$$i\,\mathrm{tr}[(\not{p}'+m)\not{P}\not{q}(\not{p}+m)] = i\,\mathrm{tr}[\not{p}'\not{P}\not{q}\not{p}] + im^2\,\mathrm{tr}[\not{P}\not{q}]$$

$$= 4i(p'\cdot P\, q\cdot p - p'\cdot q\, P\cdot p + p'\cdot p\, \cancel{P\cdot q}^{\,0}) + 4im^2\,\cancel{P\cdot q}^{\,0}$$

$$= -4i\frac{q^2}{2}\left[P\cdot p' + P\cdot p\right]$$

$$= -2iq^2(4m^2 - q^2)$$

$$\Rightarrow \mathrm{tr}[(\not{p}'+m)P_\mu \sigma^{\mu\nu} q_\nu(\not{p}+m)] = -2iq^2(4m^2 - q^2)$$

$$\mathrm{tr}[\gamma_\mu(\not{p}'+m)\gamma^\mu(\not{p}+m)] = \mathrm{tr}[\gamma_\mu\not{p}'\gamma^\mu\not{p}] + m^2\,\mathrm{tr}[\gamma_\mu\gamma^\mu]$$
$$= -2\,\mathrm{tr}[\not{p}'\not{p}] + 4m^2\,\mathrm{tr}[\mathbb{1}]$$
$$= -8\,p'\cdot p + 16m^2$$
$$= 4(q^2+2m^2)$$

$$\mathrm{tr}[\gamma_\mu(\not{p}'+m)\sigma^{\mu\nu}q_\nu(\not{p}+m)] = i\,\mathrm{tr}[\gamma_\mu(\not{p}'+m)\gamma^\mu\not{q}(\not{p}+m)]$$

$\sigma^{\mu\nu} = i\gamma^\mu\gamma^\nu - ig^{\mu\nu}$

$$-i\,\mathrm{tr}[\not{q}(\not{p}'+m)(\not{p}+m)]$$
$$= im\,\mathrm{tr}[\gamma_\mu\not{p}'\gamma^\mu\not{q}] + im\,\mathrm{tr}[\gamma_\mu\gamma^\mu\not{q}\not{p}]$$
$$-im\,\mathrm{tr}[\not{q}\not{p}] - im\,\mathrm{tr}[\not{q}\not{p}']$$
$$= -2im\,\mathrm{tr}[\not{p}'\not{q}] + 4im\,\mathrm{tr}[\not{q}\not{p}]$$
$$-im\,\mathrm{tr}[\not{q}\not{p}] - im\,\mathrm{tr}[\not{q}\not{p}']$$
$$= 3im[\not{p}\not{q}] - 3im\,\mathrm{tr}[\not{p}'\not{q}]$$
$$= 12im\,(p-p')\cdot q$$
$$= -12im\,q^2$$

Therefore, the relations

- $$\mathrm{tr}[(p\!\!\!/'+m)P_\mu\Gamma^\mu(p\!\!\!/+m)] = \mathrm{tr}[(p\!\!\!/'+m)P\!\!\!\!/\,(p\!\!\!/+m)]F_1 + \mathrm{tr}[(p\!\!\!/'+m)P_\mu\sigma^{\mu\nu}q_\nu(p\!\!\!/+m)]\frac{iF_2}{2m}$$

- $$\mathrm{tr}[\gamma_\mu(p\!\!\!/'+m)\Gamma^\mu(p\!\!\!/+m)] = \mathrm{tr}[\gamma_\mu(p\!\!\!/'+m)\gamma^\mu(p\!\!\!/+m)]F_1 + \mathrm{tr}[\gamma_\mu(p\!\!\!/'+m)\sigma^{\mu\nu}q_\nu(p\!\!\!/+m)]\frac{iF_2}{2m}$$

simplify to

- $$\mathrm{tr}[(p\!\!\!/'+m)P_\mu\Gamma^\mu(p\!\!\!/+m)] = 4m(4m^2-q^2)F_1 - 2iq^2(4m^2-q^2)\left(\frac{iF_2}{2m}\right)$$

- $$\mathrm{tr}[\gamma_\mu(p\!\!\!/'+m)\Gamma^\mu(p\!\!\!/+m)] = 4(q^2+2m^2)F_1 - 12imq^2\left(\frac{iF_2}{2m}\right)$$

or,

- $$\mathrm{tr}[(p\!\!\!/'+m)P_\mu\Gamma^\mu(p\!\!\!/+m)] = 4m(4m^2-q^2)\left[F_1 + \frac{q^2}{4m^2}F_2\right]$$

- $$\mathrm{tr}[\gamma_\mu(p\!\!\!/'+m)\Gamma^\mu(p\!\!\!/+m)] = 4(q^2+2m^2)F_1 + 6q^2F_2$$

We then solve the resulting 2×2 system for F_1 and F_2

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \frac{1}{AD-BC}\begin{pmatrix} D & -B \\ -C & A \end{pmatrix}\begin{pmatrix} T_1 \\ T_2 \end{pmatrix}$$

$$= \frac{1}{AD-BC}\begin{pmatrix} DT_1 - BT_2 \\ -CT_1 + AT_2 \end{pmatrix}$$

The determinant,

$$AD-BC = 4m\,q^2(4m^2-q^2)\left[6 - \frac{1}{4m^2}\cdot 4(q^2+2m^2)\right]$$

$$= 4m\,q^2(4m^2-q^2)(4 - q^2/m^2)$$

$$= 4\frac{q^2}{m}(4m^2-q^2)^2$$

F_1 form-factor

$$F_1 = \frac{1}{AD-BC}(DT_1 - BT_2)$$

$$= \frac{m}{4q^2(4m^2-q^2)^2}$$

$$\times\, tr\left[\left(6q^2 P_\mu - 4m(4m^2-q^2)\frac{q^2}{4m^2}\gamma_\mu\right)(\not p' + m)\Gamma^\mu(\not p + m)\right]$$

$$= \frac{1}{4(q^2-4m^2)}tr\left[\left(\gamma_\mu - \frac{6m(p'+p)_\mu}{4m^2-q^2}\right)(\not p' + m)\Gamma^\mu(\not p + m)\right]$$

F_2 form-factor

$$F_2 = \frac{1}{AD-BC}(-CT_1 + AT_2)$$

$$= \frac{m}{4q^2(4m^2-q^2)^2}$$

$$\times\, tr\left[\left(-4(q^2+2m^2)P_\mu + 4m(4m^2-q^2)\gamma_\mu\right)(\not p' + m)\Gamma^\mu(\not p + m)\right]$$

$$= \frac{-m^2}{q^2(q^2-4m^2)}tr\left[\left(\gamma_\mu + \left(\frac{q^2+2m^2}{q^2-4m^2}\right)\frac{(p'+p)_\mu}{m}\right)(\not p' + m)\Gamma^\mu(\not p + m)\right]$$

Therefore, we can project the form-factors F_1 and F_2 from Γ^μ using

$$\boxed{\begin{aligned} F_1 &= \frac{1}{4(q^2-4m^2)} \operatorname{tr}\left[\left(\gamma_\mu - \frac{6m(p'+p)_\mu}{4m^2-q^2}\right)(\not{p}'+m)\Gamma^\mu(\not{p}+m)\right] \\ F_2 &= \frac{-m^2}{q^2(q^2-4m^2)} \operatorname{tr}\left[\left(\gamma_\mu + \left(\frac{q^2+2m^2}{q^2-4m^2}\right)\frac{(p'+p)_\mu}{m}\right)(\not{p}'+m)\Gamma^\mu(\not{p}+m)\right] \end{aligned}}$$

Notice that these projectors are a non-perturbative result, and prove useful when evaluating higher-order diagrams. If we were to consider a regularization procedure, e.g., dimensional regularization, then we would need to derive projectors within that regularization, e.g., defining projectors in d-spacetime dimensions in dimensional regularization. This is especially important for finding F_1.

Here, we are only interested in the magnetic moment anomaly, $a_e = F_2(0)$. Therefore, it is useful to expand the projector about $q^2=0$, or more specifically $q^\mu = 0$.

To do the expansion, it is useful to take P, q as the independent kinematic variables instead of p, p',

$$\left.\begin{aligned} P_\mu &= p'_\mu + p_\mu \\ q_\mu &= p'_\mu - p^\mu \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} p'_\mu &= \tfrac{1}{2}(P+q)_\mu \\ p_\mu &= \tfrac{1}{2}(P-q)_\mu \end{aligned}\right.$$

such that $\Gamma^\mu(p', p) = \Gamma^\mu(P, q)$. The form-factor is then

$$F_2(Q^2) = \frac{-m^2}{q^2(q^2-4m^2)} \mathrm{tr}\Big[\Big(\gamma_\mu + \Big(\frac{q^2+2m^2}{q^2-4m^2}\Big)\frac{P_\mu}{m}\Big)$$
$$\times \Big(\tfrac{1}{2}(\not{P}+\not{q})+m\Big)\Gamma^\mu\Big(\tfrac{1}{2}(\not{P}-\not{q})+m\Big)\Big]$$

Now, as $q^\mu \to 0$

$$\Gamma^\mu(P,q) = \Gamma^\mu(P,0) + q^\nu \frac{\partial}{\partial q^\nu}\Gamma^\mu(P,q)\Big|_{q^\mu=0} + \mathcal{O}(q^\nu q^\rho)$$
$$\equiv V^\mu(P) + q^\nu \delta V_\nu^\mu(P)$$

Note that at $q^2=0$, $P^2 = 4m^2 - q^2 = 4m^2$. However, we will consider $P^2 \neq 0$ throughout the derivation, and take $P^2 \to 4m^2$ at the end.

Using the cyclic property of the trace, we write the Form-factor as

$$F_2(Q^2) = \frac{-m^2}{q^2(q^2-4m^2)} \operatorname{tr}\left\{ \left[\left(\tfrac{1}{2}(\not{P}-\not{q})+m\right)\gamma_\mu\left(\tfrac{1}{2}(\not{P}+\not{q})+m\right) + \left(\frac{q^2+2m^2}{q^2-4m^2}\right)\frac{P_\mu}{m}\left(\tfrac{1}{2}(\not{P}-\not{q})+m\right)\left(\tfrac{1}{2}(\not{P}+\not{q})+m\right)\right]\Gamma^\mu \right\}$$

Now,

$$\begin{aligned}\left(\tfrac{1}{2}(\not{P}-\not{q})+m\right)\left(\tfrac{1}{2}(\not{P}+\not{q})+m\right) &= \tfrac{1}{4}\left(\not{P}^2-\not{q}^2+[\not{P},\not{q}]\right) \\ &\quad + \tfrac{1}{2}m(\not{P}-\not{q}) + \tfrac{1}{2}m(\not{P}+\not{q}) + m^2 \\ &= \tfrac{1}{4}(P^2-q^2) + \tfrac{1}{4}[\not{P},\not{q}] + m\not{P} + m^2\end{aligned}$$

Recall,

$$P^2+q^2=4m^2 \Rightarrow P^2-q^2 = 4m^2-2q^2$$

and $[\not{P},\not{q}] = \not{P}\not{q}-\not{q}\not{P} = \not{P}\not{q}+\not{P}\not{q} - \underbrace{\cancel{P\cdot q}}_{0} = 2\not{P}\not{q}$

So, $$\begin{aligned}\left(\tfrac{1}{2}(\not{P}-\not{q})+m\right)\left(\tfrac{1}{2}(\not{P}+\not{q})+m\right) &= m^2-\frac{q^2}{2} + \frac{1}{2}\not{P}\not{q} + m\not{P} + m^2 \\ &= 2m^2-\frac{q^2}{2} + \not{P}\left(m+\frac{\not{q}}{2}\right)\end{aligned}$$

also,

$$\left(\frac{1}{2}(\not{P}-\not{q})+m\right)\gamma_\mu\left(\frac{1}{2}(\not{P}+\not{q})+m\right)$$

$$= \frac{1}{4}(\not{P}-\not{q})\gamma_\mu(\not{P}+\not{q}) + \frac{m}{2}(\not{P}-\not{q})\gamma_\mu + \frac{m}{2}\gamma_\mu(\not{P}+\not{q}) + m^2\gamma_\mu$$

$$= \frac{1}{4}(\not{P}-\not{q})\left[2(P+q)_\mu - (\not{P}+\not{q})\gamma_\mu\right] + m^2\gamma_\mu$$

$$+ \frac{m}{2}\left[(\not{P}-\not{q})\gamma_\mu + 2(P+q)_\mu - (\not{P}+\not{q})\gamma_\mu\right]$$

$$= \frac{1}{2}(P+q)_\mu(\not{P}-\not{q}) - \frac{1}{4}(\not{P}-\not{q})(\not{P}+\not{q})\gamma_\mu$$

$$+ m(P+q)_\mu - m\not{q}\gamma_\mu + m^2\gamma_\mu$$

Now, $(\not{P}-\not{q})(\not{P}+\not{q}) = \not{P}^2 - \not{q}^2 + [\not{P},\not{q}]$

$$= P^2 - q^2 + 2\not{P}\not{q} = 4m^2 - 2q^2 + 2\not{P}\not{q}$$

So,

$$\left(\frac{1}{2}(\not{P}-\not{q})+m\right)\gamma_\mu\left(\frac{1}{2}(\not{P}+\not{q})+m\right)$$

$$= \frac{1}{2}(P+q)_\mu(\not{P}-\not{q}) - \cancel{m^2\gamma_\mu} + \frac{1}{2}q^2\gamma_\mu - \frac{1}{2}\not{P}\not{q}\gamma_\mu$$

$$+ m(P+q)_\mu - m\not{q}\gamma_\mu + \cancel{m^2\gamma_\mu}$$

$$= (P+q)_\mu\left[\frac{1}{2}(\not{P}-\not{q}) + m\right] + \left[\frac{1}{2}q^2 - \frac{1}{2}\not{P}\not{q} - m\not{q}\right]\gamma_\mu$$

So, we find the form-factor takes the form

$$F_2(Q^2) = \frac{-m^2}{q^2(q^2-4m^2)}$$

$$\times \operatorname{tr}\Big\{ \Big[\Big[\frac{1}{2}q^2 - \frac{1}{2}\not{P}\not{q} - m\not{q} \Big]\gamma_\mu + (P+q)_\mu \Big[\frac{1}{2}(\not{P}-\not{q}) + m \Big]$$

$$+ \left(\frac{q^2+2m^2}{q^2-4m^2} \right) \frac{P_\mu}{m} \Big[2m^2 - \frac{q^2}{2} + \not{P}\left(m + \frac{\not{q}}{2} \right) \Big] \Big] \left(V^\mu + q^\nu \delta V_\nu^\mu \right) \Big\}$$

Next, we can average over the spatial direction of q^μ, since $\int d\Omega_q F_2(Q^2) = 4\pi F_2(Q^2)$.

For terms linear in q^μ,

$$\int \frac{d\Omega_q}{4\pi} q_\mu = A P_\mu$$

where A is undetermined, and P_μ is the only four-vector remaining. But, $P \cdot q = 0$

$$\Rightarrow 0 = \int \frac{d\Omega_q}{4\pi} P \cdot q = A P^2$$

the only way this works is if $A = 0$

So,

$$\boxed{\int \frac{d\Omega_q}{4\pi} q_\mu = 0}$$

For term proportional to $q_\mu q_\nu$, we have

$$\int \frac{d\Omega_q}{4\pi} q_\mu q_\nu = A g_{\mu\nu} + B P_\mu P_\nu$$

where A and B are undetermind. Now, contract with $g^{\mu\nu}$,

q^2 is scalar

$$g^{\mu\nu} \int \frac{d\Omega_q}{4\pi} q_\mu q_\nu = \int \frac{d\Omega_q}{4\pi} q^2 = q^2$$

$$= 4A + BP^2$$

so, $q^2 = 4A + BP^2$

Also, contract with $P^\mu P^\nu$, recalling $P \cdot q = 0$

$$\Rightarrow \quad 0 = \int \frac{d\Omega_q}{4\pi} P\cdot q \; P\cdot q = AP^2 + BP^2P^2$$

so, $(A + BP^2)P^2 = 0$

$$\Rightarrow \quad B = -\frac{A}{P^2}$$

which gives $q^2 = 4A + BP^2 = 4A - A = 3A$

$$\Rightarrow \quad A = q^2/3$$

therefore,

$$\boxed{\int \frac{d\Omega_q}{4\pi} q_\mu q_\nu = \frac{1}{3} q^2 \left(g_{\mu\nu} - \frac{P_\mu P_\nu}{P^2} \right)}$$

Finally, note that terms with $q_\mu q_\nu q_\rho$ will also give a zero angular average, $\int d\Omega_q \, q_\mu q_\nu q_\rho = 0$

So, averaging over the form factor

$$F_2(Q^2) = \int \frac{d\Omega_q}{4\pi} F_2(Q^2)$$

$$= \frac{-m^2}{q^2(q^2-4m^2)} \int \frac{d\Omega_q}{4\pi} \, tr\Big[\Big\{ \Big(\underline{\tfrac{1}{2} q^2} - \big(\tfrac{\not P}{2} + m\big)\cancel{\not q}\Big)\gamma_\mu + P_\mu\Big(\tfrac{1}{2}(\underline{\not P} - \cancel{\not q}) + \underline{m}\Big) + q_\mu\Big(\tfrac{1}{2}(\cancel{\not P} - \underline{\not q}) + \cancel{m}\Big) + \Big(\frac{q^2+2m^2}{q^2-4m^2}\Big)\frac{P_\mu}{m}\Big[\underline{2m^2 - \tfrac{q^2}{2}} + \not P\Big(\underline{m} + \cancel{\tfrac{\not q}{2}}\Big)\Big]\Big\} V^\mu\Big]$$

$$+ \frac{-m^2}{q^2(q^2-4m^2)} \int \frac{d\Omega_q}{4\pi} \, tr\Big[\Big\{ \Big(\tfrac{1}{2} \cancel{q^2} - \underline{\big(\tfrac{\not P}{2} + m\big)\not q}\Big)\gamma_\mu + P_\mu\Big(\tfrac{1}{2}(\cancel{\not P} - \underline{\not q}) + \cancel{m}\Big) + q_\mu\Big(\tfrac{1}{2}(\underline{\not P} - \cancel{\not q}) + \underline{m}\Big) + \Big(\frac{q^2+2m^2}{q^2-4m^2}\Big)\frac{P_\mu}{m}\Big[\cancel{2m^2} - \cancel{\tfrac{q^2}{2}} + \not P\Big(\cancel{m} + \underline{\tfrac{\not q}{2}}\Big)\Big]\Big\} q^\nu \delta V_\nu^\mu\Big]$$

$$\int \frac{d\Omega_q}{4\pi} = 1, \quad \int \frac{d\Omega_q}{4\pi} q_\mu = 0, \quad \int \frac{d\Omega_q}{4\pi} q_\mu q_\nu = \frac{1}{3} q^2 \Big(g_{\mu\nu} - \frac{P_\mu P_\nu}{P^2}\Big)$$

$$\int \frac{d\Omega_q}{4\pi} q_\mu q_\nu q_\rho = 0$$

$$F_2(Q^2) = \int \frac{d\Omega_q}{4\pi} F_2(Q^2)$$

$$= \frac{-m^2}{q^2(q^2-4m^2)} \int \frac{d\Omega_q}{4\pi} \, tr\Big[\Big\{ \frac{1}{2}q^2\gamma_\mu + P_\mu\left(\frac{1}{2}\not{P}+m\right) - \frac{1}{2}q_\mu q_\alpha \gamma^\alpha$$

$$+ \left(\frac{q^2+2m^2}{q^2-4m^2}\right)\frac{P_\mu}{m}\left[2m^2 - \frac{q^2}{2} + \not{P}m\right]\Big\} V^\mu\Big]$$

$$+ \frac{-m^2}{q^2(q^2-4m^2)} \int \frac{d\Omega_q}{4\pi} \, tr\Big[\Big\{ -\left(\frac{\not{P}}{2}+m\right)q_\alpha \gamma^\alpha \gamma_\mu - \frac{1}{2}q_\alpha P_\mu \gamma^\alpha$$

$$+ q_\mu\left(\frac{1}{2}\not{P}+m\right) + \left(\frac{q^2+2m^2}{q^2-4m^2}\right)\frac{P_\mu}{m}\left[\frac{\not{P}}{2}\gamma^\alpha q_\alpha\right]\Big\} q^\nu \delta V_\nu^\mu\Big]$$

$$F_2(Q^2) = \frac{-m^2}{q^2(q^2-4m^2)}$$

$$\times \, tr\Big[\Big\{\frac{1}{2}q^2\gamma_\mu + P_\mu\left(\frac{\not{P}}{2}+m\right) - \frac{1}{2}\cdot\frac{1}{3}q^2\left(\gamma_\mu - \frac{P_\mu \not{P}}{P^2}\right)$$

$$+ \left(\frac{q^2+2m^2}{q^2-4m^2}\right)\frac{P_\mu}{m}\left(2m^2 - \frac{q^2}{2} + \not{P}m\right)\Big\} V^\mu\Big]$$

$$+ \frac{-m^2}{q^2(q^2-4m^2)} tr\Big[\Big\{ -\frac{1}{3}q^2\left(\frac{\not{P}}{2}+m\right)\left(\gamma^\nu - \frac{P^\nu \not{P}}{P^2}\right)\gamma_\mu$$

$$- \frac{1}{2}P_\mu \frac{1}{3}q^2\left(\gamma^\nu - \frac{P^\nu \not{P}}{P^2}\right) + \left(\frac{1}{2}\not{P}+m\right)\frac{1}{3}q^2\left(g_\mu^{\ \nu} - \frac{P_\mu P^\nu}{P^2}\right)$$

$$+ \left(\frac{q^2+2m^2}{q^2-4m^2}\right)\frac{P_\mu}{m}\left(\frac{\not{P}}{2}\frac{1}{3}q^2\left(\gamma^\nu - \frac{P^\nu \not{P}}{P^2}\right)\right)\Big\}\delta V_\nu^\mu\Big]$$

Notice that the δV_ν^μ term is proportional to q^2. Let us manipulate two expressions* in the first term

Manipulating the two expressions

$$P_\mu\left(\frac{\not{P}}{2}+m\right)+\left(\frac{q^2+2m^2}{q^2-4m^2}\right)\frac{P_\mu}{m}\left(2m^2+\not{P}m\right)$$

$$=\frac{1}{q^2-4m^2}P_\mu\left[q^2-4m^2+2(q^2+2m^2)\right]\left(\frac{\not{P}}{2}+m\right)$$

$$=\frac{1}{q^2-4m^2}P_\mu\cdot 3\left(\frac{\not{P}}{2}+m\right)$$

Thus, every term in the trace is proportional to q^2, which cancels the $1/q^2$ pre-factor. Canceling this factor allows us to take the limit $q\to 0$.

$$F_2(Q^2)=\frac{-m^2}{(q^2-4m^2)}$$

$$\times\, tr\left[\left\{\frac{1}{2}\gamma_\mu-\frac{1}{2}\cdot\frac{1}{3}\left(\gamma_\mu-\frac{P_\mu\not{P}}{P^2}\right)\right.\right.$$

$$\left.\left.+\frac{1}{q^2-4m^2}P_\mu\cdot 3\left(\frac{\not{P}}{2}+m\right)-\frac{1}{2}\left(\left(\frac{q^2+2m^2}{q^2-4m^2}\right)\frac{P_\mu}{m}\right)\right\}V^\mu\right]$$

$$+\frac{-m^2}{(q^2-4m^2)}tr\left[\left\{-\frac{1}{3}\left(\frac{\not{P}}{2}+m\right)\left(\gamma^\nu-\frac{P^\nu\not{P}}{P^2}\right)\gamma_\mu\right.\right.$$

$$-\frac{1}{2}P_\mu\frac{1}{3}\left(\gamma^\nu-\frac{P^\nu\not{P}}{P^2}\right)+\left(\frac{1}{2}\not{P}+m\right)\frac{1}{3}\left(g_\mu{}^\nu-\frac{P_\mu P^\nu}{P^2}\right)$$

$$\left.\left.+\left(\frac{q^2+2m^2}{q^2-4m^2}\right)\frac{P_\mu}{m}\left(\frac{\not{P}}{2}\frac{1}{3}\left(\gamma^\nu-\frac{P^\nu\not{P}}{P^2}\right)\right)\right\}\delta V_\nu{}^\mu\right]$$

So, as $q \to 0$

recall $P^2 = 4m^2$

$$
\begin{aligned}
F_2(0) = +\frac{1}{4}\,\mathrm{tr}\Big[\Big\{ & \frac{1}{2}\gamma_\mu - \frac{1}{6}\Big(\gamma_\mu - \frac{P_\mu \not P}{P^2}\Big) \\
& - \frac{3}{4m^2} P_\mu \Big(\frac{\not P}{2} + m\Big) + \frac{1}{4m} P_\mu \Big\} V^\mu \\
& + \Big\{ -\frac{1}{3}\Big(\frac{\not P}{2} + m\Big)\Big(\gamma^\nu - \frac{P^\nu \not P}{P^2}\Big)\gamma_\mu + m\,\frac{1}{3}\Big(g_\mu{}^\nu - \frac{P_\mu P^\nu}{P^2}\Big) \\
& - \frac{1}{2} P_\mu \frac{1}{3}\Big(\gamma^\nu - \cancel{\frac{P^\nu \not P}{P^2}}\Big) + \frac{1}{2}\not P \frac{1}{3}\Big(g_\mu{}^\nu - \cancel{\frac{P_\mu P^\nu}{P^2}}\Big) \\
& - \frac{1}{2}\frac{P_\mu}{m}\Big(\frac{1}{3}\frac{\not P}{2}\Big(\gamma^\nu - \frac{P^\nu \not P}{P^2}\Big)\Big)\Big\}\delta V_\nu{}^\mu \Big]
\end{aligned}
$$

$$
\begin{aligned}
= +\frac{1}{4}\,\mathrm{tr}\Big[\Big\{ & \frac{1}{3}\gamma_\mu + \frac{1}{6} \frac{P_\mu \not P}{4m^2} - \frac{3}{8m^2} P_\mu \not P - \frac{3P_\mu}{4m} + \frac{1}{4m} P_\mu \Big\} V^\mu \\
& + \Big\{ -\frac{1}{3}\Big(\frac{\not P}{2} + m\Big)\gamma^\nu\gamma_\mu + \frac{1}{3}\Big(\frac{\not P}{2} + m\Big) P^\nu \frac{\not P}{P^2}\gamma_\mu + \frac{m}{3}\Big(g_\mu{}^\nu - \frac{P_\mu P^\nu}{P^2}\Big) \\
& - \frac{1}{6} P_\mu \gamma^\nu + \frac{1}{6}\not P g_\mu{}^\nu - \frac{1}{12m} P_\mu \not P \gamma^\nu + \frac{1}{12m} P_\mu P^\nu \overset{1}{\cancel{\frac{\not P \not P}{P^2}}} \Big\}\delta V_\nu{}^\mu \Big]
\end{aligned}
$$

$$
\begin{aligned}
= +\frac{1}{12m^2}\,\mathrm{tr}\Big\{ & \Big[m^2\gamma_\mu - P_\mu \not P - \frac{3}{2} m P_\mu \Big] V^\mu \\
& + \Big[-m^2\Big(\frac{\not P}{2} + m\Big)\gamma_\nu\gamma_\mu + m^2\Big(\frac{\not P}{2} + m\Big)\frac{P_\nu \not P}{4m^2}\gamma_\mu \\
& + m^3\Big(g_{\mu\nu} - \cancel{\frac{P_\mu P_\nu}{4m^2}}\Big) - \frac{m^2}{2} P_\mu \gamma_\nu + \frac{m^2}{2}\not P g_{\mu\nu} \\
& - \frac{m}{4} P_\mu \not P \gamma_\nu + \cancel{\frac{m}{4} P_\mu P_\nu} \Big] \delta V_{\nu,\mu} \Big\}
\end{aligned}
$$

$$F_2(0) = +\frac{1}{12m^2}\,\mathrm{tr}\Big\{\Big[m^2\gamma_\mu - P_\mu \not{P} - \frac{3}{2} m P_\mu \Big] V^\mu$$

$$+\Big[-m^2\Big(\frac{\not{P}}{2}+m\Big)\gamma_\nu\gamma_\mu + m^2\Big(\frac{\not{P}}{2}+m\Big)\frac{\not{P}P_\nu}{4m^2}\gamma_\mu$$

$$+ m^3 g_{\mu\nu} - \frac{m^2}{2}P_\mu\gamma_\nu + \frac{m^2}{2}\not{P} g_{\mu\nu} - \frac{m}{4}P_\mu\not{P}\gamma_\nu \Big]\delta V^{\nu;\mu}\Big\}$$

For the second term, note

$$\{\gamma_\mu,\gamma_\nu\} = 2g_{\mu\nu} \quad , \text{ and } \quad \not{P}^2 = P^2 = 4m^2$$

So,

$$2^{\text{nd}}\text{ term} = m\Big[-\Big(\frac{\not{P}}{2}+m\Big)\gamma_\nu\gamma_\mu m + \Big(\frac{\not{P}}{2}+m\Big)\frac{\not{P}P_\nu}{4m^2}\gamma_\mu m$$

$$+\Big(\frac{\not{P}}{2}+m\Big) g_{\mu\nu} m - \Big(\frac{\not{P}}{2}+m\Big)\gamma_\nu\frac{P_\mu}{2}\Big]$$

$$= m\Big(\frac{\not{P}}{2}+m\Big)\Big[-\gamma_\nu\gamma_\mu m + \frac{\not{P}}{4m^2}P_\nu\gamma_\mu m + \underbrace{g_{\mu\nu}}_{=\frac{1}{2}\gamma_\mu\gamma_\nu+\frac{1}{2}\gamma_\nu\gamma_\mu} m - \gamma_\nu\frac{P_\mu}{2}\Big]$$

$$= \frac{m}{2}\Big(\frac{\not{P}}{2}+m\Big)\Big[[\gamma_\mu,\gamma_\nu]m + \frac{\not{P}}{2m^2}P_\nu\gamma_\mu m - \gamma_\nu P_\mu\Big]$$

One can show, e.g. Mathematica, that the second term can be written as

$$2^{\text{nd}}\text{ term} = \frac{m}{4}\Big(\frac{\not{P}}{2}+m\Big)[\gamma_\mu,\gamma_\nu]\Big(\frac{\not{P}}{2}+m\Big)$$

N.B. I'm stuck at analytically proving this, but Mathematica can do the algebra faster than me...

So, we find that the anomalous magnetic moment can be found directly by

$$\boxed{\begin{aligned} a &= F_2(0) \\ &= \frac{1}{12m^2} \operatorname{tr}\Big[\Big(m^2 \gamma_\mu - P_\mu \not{P} - \frac{3}{2} m P_\mu \Big) V^\mu \\ &\qquad + \frac{m}{4} \Big(\frac{\not{P}}{2} + m \Big) [\gamma_\mu, \gamma_\nu] \Big(\frac{\not{P}}{2} + m \Big) \delta V^{\nu\mu} \Big] \end{aligned}}$$

where we take $P^2 \to 4m^2$ after taking the trace, and where

$$V^\mu(P) = \Gamma^\mu(P, 0)$$

$$\delta V^{\nu\mu}(P) = \frac{\partial \Gamma^\mu(P, q)}{\partial q_\nu}\bigg|_{q=0}$$

Thus, we only need to find the vertex function linear in q^μ to compute a.