Radiative Corrections

We are now (finally) ready to compute the first-order radiative correction to the magnetic moment of the electron

$$a_e^{(1)} = F_2^{(1)}(0)$$

$$= \frac{1}{12m^2} \operatorname{tr}\Big[\Big(m^2\gamma_\mu - P_\mu \not{P} - \frac{3}{2} m P_\mu \Big) V_1^\mu + \frac{m}{4} \Big(\frac{\not{P}}{2} + m \Big) [\gamma_\mu, \gamma_\nu] \Big(\frac{\not{P}}{2} + m \Big) \delta V_1^{\nu\mu} \Big]$$

where

$$V_1^\mu(P) = \Lambda_1^\mu(P, 0)$$

$$\delta V_1^{\nu\mu}(P) = \frac{\partial \Lambda_1^\mu(P, q)}{\partial q_\nu}\Big|_{q=0}$$

where $\Lambda_1^\mu(P, q) \equiv \Lambda_1^\mu(p', p)$ is the correction to the vertex function

$$\Lambda_1^\mu(p', p) = -e^2 \int \frac{d^4k}{(2\pi)^4} \frac{-i^3 N^\mu(p', p, k)}{k^2[(p'-k)^2 - m^2][(p-k)^2 - m^2]}$$

with $N^\mu = \gamma_\nu [(\not{p}' - \not{k}) + m] \gamma^\mu [(\not{p} - \not{k}) + m] \gamma^\nu$

Let us now replace the kinematics

$$p' = \frac{1}{2}(P + q), \quad p = \frac{1}{2}(P - q)$$

So,

$$\Lambda_1^\mu(P,q) = -ie^2\int\frac{d^4k}{(2\pi)^4}\,\frac{N^\mu(P,q,k)}{k^2[((P+q)/2-k)^2-m^2][((P-q)/2-k)^2-m^2]}$$

with $N^\mu(P,q,k) = \gamma_\alpha[(\not{P}+\not{q})/2 - \not{k} + m]\gamma^\mu[(\not{P}-\not{q})/2 - \not{k} + m]\gamma^\alpha$

At $q=0$,

$$\Lambda_1^\mu(P,0) = -ie^2\int\frac{d^4k}{(2\pi)^4}\,\frac{N^\mu(P,0,k)}{k^2[(P/2-k)^2-m^2]^2}$$

with $N^\mu(P,0,k) = \gamma_\alpha[\not{P}/2 - \not{k} + m]\gamma^\mu[\not{P}/2 - \not{k} + m]\gamma^\alpha$

the derivative is given by

$$\frac{\partial\Lambda_1^\mu}{\partial q_\nu}\Big|_{q=0} = -ie^2\int\frac{d^4k}{(2\pi)^4}\Big\{\frac{1}{k^2[(P/2-k)^2-m^2]^2}\frac{\partial N^\mu}{\partial q_\nu}\Big|_{q=0}$$

$$+\frac{N^\mu(P,0,q)}{k^2}\times\frac{\partial}{\partial q_\nu}\left(\frac{1}{[((P+q)/2-k)^2-m^2][((P-q)/2-k)^2-m^2]}\right)\Big|_{q=0}\Big\}$$

Now, $\displaystyle\frac{\partial N^\mu}{\partial q_\nu}\Big|_{q=0} = \gamma_\alpha\frac{\partial}{\partial q_\nu}[(\not{P}+\not{q})/2 - \not{k} + m]\gamma^\mu[\not{P}/2 - \not{k} + m]\gamma^\alpha$

$$+\gamma_\alpha[\not{P}/2 - \not{k} + m]\gamma^\mu\frac{\partial}{\partial q_\nu}[(\not{P}-\not{q})/2 - \not{k} + m]\gamma^\alpha$$

with $\displaystyle\frac{\partial}{\partial q_\nu}[(\not{P}+\not{q})/2 - \not{k} + m] = \frac{\gamma^\rho}{2}\frac{\partial}{\partial q_\nu}q_\rho = \frac{\gamma^\nu}{2}$

$$\Rightarrow \frac{\partial N^\mu}{\partial q_\nu}\Big|_{q=0} = \gamma_\alpha\frac{\gamma^\nu}{2}\gamma^\mu[\not{P}/2 - \not{k} + m]\gamma^\alpha - \gamma_\alpha[\not{P}/2 - \not{k} + m]\gamma^\mu\frac{\gamma^\nu}{2}\gamma^\alpha$$

The derivative of the propagators is

$$\frac{\partial}{\partial q_\nu}\left(\frac{1}{[((P+q)/2-k)^2-m^2][((P-q)/2-k)^2-m^2]}\right)_{q=0}$$

$$= \frac{1}{[(P/2-k)^2-m^2]}\frac{\partial}{\partial q_\nu}\left(\frac{1}{[((P+q)/2-k)^2-m^2]}\right)_{q=0}$$

$$+ \frac{1}{[(P/2-k)^2-m^2]}\frac{\partial}{\partial q_\nu}\left(\frac{1}{[((P-q)/2-k)^2-m^2]}\right)_{q=0}$$

Now,

$$\frac{\partial}{\partial q_\nu}\left(\frac{1}{[((P\pm q)/2-k)^2-m^2]}\right)_{q=0} = \frac{1}{[(P/2-k)^2-m^2]^2}\cdot\frac{\partial}{\partial q_\nu}((P\pm q)/2-k)^2$$

$$= \frac{1}{[(P/2-k)^2-m^2]^2}\cdot 2(P/2-k)^\alpha\cdot\frac{1}{2}(\pm\delta_\alpha{}^\nu)$$

$$= \frac{\pm(P/2-k)^\nu}{[(P/2-k)^2-m^2]^2}$$

Sign means that these terms will sum to zero

$$\Rightarrow \frac{\partial}{\partial q_\nu}\left(\frac{1}{[((P+q)/2-k)^2-m^2][((P-q)/2-k)^2-m^2]}\right)_{q=0} = 0 .$$

So, the derivative of the vertex function is

$$\frac{\partial \Lambda_1^\mu}{\partial q_\nu}\bigg|_{q=0} = -ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2[(P/2-k)^2-m^2]^2} \frac{\partial N^\mu}{\partial q_\nu}\bigg|_{q=0}$$

with,

$$\frac{\partial N^\mu}{\partial q_\nu}\bigg|_{q=0} = \gamma_\alpha \frac{\gamma^\nu}{2} \gamma^\mu [\not{P}/2 - \not{k} + m]\gamma^\alpha - \gamma_\alpha [\not{P}/2 - \not{k} + m]\gamma^\mu \frac{\gamma^\nu}{2} \gamma^\alpha$$

In the projection formula, we need to take traces.

For the first term, we need

$$T_1 = tr\left[\left(m^2\gamma_\mu - P_\mu \not{P} - \frac{3}{2} m P_\mu\right) N^\mu(P, 0, k)\right]$$

with

$$N^\mu(P,0,k) = \gamma_\alpha [\not{P}/2 - \not{k} + m]\gamma^\mu [\not{P}/2 - \not{k} + m]\gamma^\alpha$$

So, the following components are

$\gamma_\alpha \gamma_\mu \gamma^\alpha = -2\gamma_\mu$

- $$tr[\gamma_\mu \gamma_\alpha [\not{P}/2 - \not{k} + m]\gamma^\mu [\not{P}/2 - \not{k} + m]\gamma^\alpha]$$
$$= -2\, tr[\gamma_\mu [\not{P}/2 - \not{k} + m]\gamma^\mu [\not{P}/2 - \not{k} + m]]$$
$$= -2\, tr[\gamma_\mu (\not{P}/2 - \not{k})\gamma^\mu (\not{P}/2 - \not{k})] - 2m^2\, tr[\gamma_\mu \gamma^\mu]$$
$$= +4\, tr[(\not{P}/2 - \not{k})(\not{P}/2 - \not{k})] - 8m^2\, tr[\mathbb{1}]$$
$$= 16(P/2 - k)^2 - 32m^2$$
$$= 16\left(\frac{P^2}{4} + k^2 - P.k\right) - 32m^2$$
$$= 16(k^2 - P.k - m^2)$$

- $\mathrm{tr}[\not{P}\,\gamma_\alpha\,[\not{P}/2 - \not{k} + m]\,\not{P}\,[\not{P}/2 - \not{k} + m]\,\gamma^\alpha]$

$$= -2\,\mathrm{tr}[\not{P}\,[\not{P}/2 - \not{k} + m]\,\not{P}\,[\not{P}/2 - \not{k} + m]]$$

$$= -2\,\mathrm{tr}[\not{P}(\not{P}/2 - \not{k})\not{P}(\not{P}/2 - \not{k})] - 2m^2\,\mathrm{tr}[\not{P}\,\not{P}]$$

$$= -8\left[2P\cdot\left(\frac{P}{2} - k\right)P\cdot\left(\frac{P}{2} - k\right) - P^2\left(\frac{P}{2} - k\right)^2\right] - 8m^2P^2$$

$$= -8\left[2\left(\frac{P^2}{2} - k\cdot P\right)^2 - 4m^2\left(\frac{P^2}{4} - k\cdot P + k^2\right)\right] - 32m^4$$

$$= -16\left[(2m^2 - k\cdot P)^2 - 2m^2(m^2 - k\cdot P + k^2)\right] - 32m^4$$

$$= -16[4m^4 + (k\cdot P)^2 - 4m^2\,k\cdot P - 2m^4 + 2m^2\,k\cdot P - 2m^2k^2 + 2m^4]$$

$$= -16[4m^4 + (k\cdot P)^2 - 2m^2\,k\cdot P - 2m^2k^2]$$

- $\mathrm{tr}[\gamma_\alpha\,[\not{P}/2 - \not{k} + m]\,\not{P}\,[\not{P}/2 - \not{k} + m]\,\gamma^\alpha]$

$$= 4\,\mathrm{tr}[(\not{P}/2 - \not{k} + m)\not{P}(\not{P}/2 - \not{k} + m)]$$

$$= 4m\,\mathrm{tr}[(\not{P}/2 - \not{k})\not{P}] + 4m\,\mathrm{tr}[\not{P}(\not{P}/2 - \not{k})]$$

$$= 32m\,(P^2/2 - k\cdot P)$$

$$= 32m\,(2m^2 - k\cdot P)$$

So, the first trace is

$$\tau_1 = m^2 \cdot 16(k^2 - P\cdot k - m^2) + 16(4m^4 - 2m^2 P\cdot k - 2m^2 k^2 + (P\cdot k)^2)$$
$$- \frac{3}{2} m \cdot 32 m (2m^2 - k\cdot P)$$
$$= -16 m^2 k^2 + 16 (P\cdot k)^2 - 48 m^4$$
$$= -16 [3m^4 + m^2 k^2 - (k\cdot P)^2]$$

The second trace Needed is

$$\tau_2 = tr\left[\left(\frac{\not{P}}{2} + m\right)[\gamma_\mu, \gamma_\nu]\left(\frac{\not{P}}{2} + m\right) \frac{\partial N^\mu}{\partial q_\nu}\bigg|_{q=0} \right]$$

with

$$\frac{\partial N^\mu}{\partial q_\nu}\bigg|_{q=0} = \frac{1}{2}\Big\{ \gamma_\alpha \gamma^\nu \gamma^\mu (\not{P}/2 - \not{k} + m)\gamma^\alpha$$
$$- \gamma_\alpha (\not{P}/2 - \not{k} + m) \gamma^\mu \gamma^\nu \gamma^\alpha \Big\}$$
$$= \frac{1}{2}\Big\{ -2 \left(\frac{\not{P}}{2} - \not{k}\right) \gamma^\mu \gamma^\nu + 4m g^{\mu\nu}$$
$$+ 2 \gamma^\nu \gamma^\mu \left(\frac{\not{P}}{2} - \not{k}\right) + 4 m g^{\mu\nu} \Big\}$$

Note, $[\gamma_\mu, \gamma_\nu] g^{\mu\nu} = [\gamma_\mu, \gamma^\mu] = 0$

So, $$\tau_2 = -tr\left[\left(\frac{\not{P}}{2} + m\right)\underline{[\gamma_\mu, \gamma_\nu]}\left(\frac{\not{P}}{2} + m\right)\left(\frac{\not{P}}{2} - \not{k}\right)\gamma^\mu \gamma^\nu\right]$$
$$+ tr\left[\left(\frac{\not{P}}{2} + m\right)\underline{[\gamma_\mu, \gamma_\nu]}\left(\frac{\not{P}}{2} + m\right)\gamma^\nu \gamma^\mu \left(\frac{\not{P}}{2} - \not{k}\right)\right]$$

- $\mathrm{tr}\left[\left(\frac{\not P}{2}+m\right)\left(\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu\right)\left(\frac{\not P}{2}+m\right)\left(\frac{\not P}{2}-\not k\right)\gamma^\mu\gamma^\nu\right]$

$$= m\,\mathrm{tr}\left[\frac{\not P}{2}\left(\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu\right)\left(\frac{\not P}{2}-\not k\right)\gamma^\mu\gamma^\nu\right]$$

$$+ m\,\mathrm{tr}\left[\left(\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu\right)\frac{\not P}{2}\left(\frac{\not P}{2}-\not k\right)\gamma^\mu\gamma^\nu\right]$$

$$= m\,\mathrm{tr}\left[\frac{\not P}{2}\gamma_\mu\gamma_\nu\left(\frac{\not P}{2}-\not k\right)\gamma^\mu\gamma^\nu\right] - m\,\mathrm{tr}\left[\frac{\not P}{2}\gamma_\nu\gamma_\mu\left(\frac{\not P}{2}-\not k\right)\gamma^\mu\gamma^\nu\right]$$

$$+ m\,\mathrm{tr}\left[\gamma_\mu\gamma_\nu\frac{\not P}{2}\left(\frac{\not P}{2}-\not k\right)\gamma^\mu\gamma^\nu\right] - m\,\mathrm{tr}\left[\gamma_\nu\gamma_\mu\frac{\not P}{2}\left(\frac{\not P}{2}-\not k\right)\gamma^\mu\gamma^\nu\right]$$

$$= 4m\,\mathrm{tr}\left[\frac{\not P}{2}\left(\frac{\not P}{2}-\not k\right)\right] + 2m\,\mathrm{tr}\left[\frac{\not P}{2}\gamma_\nu\left(\frac{\not P}{2}-\not k\right)\gamma^\nu\right]$$

$$- 8m\,\mathrm{tr}\left[\frac{\not P}{2}\left(\frac{\not P}{2}-\not k\right)\right] - 16m\,\mathrm{tr}\left[\frac{\not P}{2}\left(\frac{\not P}{2}-\not k\right)\right]$$

$$= -20m\,\mathrm{tr}\left[\frac{\not P}{2}\left(\frac{\not P}{2}-\not k\right)\right] - 4m\,\mathrm{tr}\left[\frac{\not P}{2}\left(\frac{\not P}{2}-\not k\right)\right]$$

$$= -24\cdot 4m\,\frac{P}{2}\cdot\left(\frac{P}{2}-k\right)$$

$$= -96m\left(\frac{P^2}{4}-\frac{k\cdot P}{2}\right)$$

$$= -48m\left(2m^2-k\cdot P\right)$$

- $\mathrm{tr}\left[\left(\frac{\not{P}}{2}+m\right)(\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu)\left(\frac{\not{P}}{2}+m\right)\gamma^\nu\gamma^\mu\left(\frac{\not{P}}{2}-\not{k}\right)\right]$

$$= m\,\mathrm{tr}\left[\frac{\not{P}}{2}(\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu)\gamma^\nu\gamma^\mu\left(\frac{\not{P}}{2}-\not{k}\right)\right]$$

$$+ m\,\mathrm{tr}\left[(\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu)\frac{\not{P}}{2}\gamma^\nu\gamma^\mu\left(\frac{\not{P}}{2}-\not{k}\right)\right]$$

$$= m\,\mathrm{tr}\left[\frac{\not{P}}{2}\gamma_\mu\gamma_\nu\gamma^\nu\gamma^\mu\left(\frac{\not{P}}{2}-\not{k}\right)\right] - m\,\mathrm{tr}\left[\frac{\not{P}}{2}\gamma_\nu\gamma_\mu\gamma^\nu\gamma^\mu\left(\frac{\not{P}}{2}-\not{k}\right)\right]$$

$$+ m\,\mathrm{tr}\left[\gamma_\mu\gamma_\nu\frac{\not{P}}{2}\gamma^\nu\gamma^\mu\left(\frac{\not{P}}{2}-\not{k}\right)\right] - m\,\mathrm{tr}\left[\gamma_\nu\gamma_\mu\frac{\not{P}}{2}\gamma^\nu\gamma^\mu\left(\frac{\not{P}}{2}-\not{k}\right)\right]$$

$$= 16m\,\mathrm{tr}\left[\frac{\not{P}}{2}\left(\frac{\not{P}}{2}-\not{k}\right)\right] + 8m\,\mathrm{tr}\left[\frac{\not{P}}{2}\left(\frac{\not{P}}{2}-\not{k}\right)\right]$$

$$+ 4m\,\mathrm{tr}\left[\frac{\not{P}}{2}\left(\frac{\not{P}}{2}-\not{k}\right)\right] - 4m\,\mathrm{tr}\left[\frac{\not{P}}{2}\left(\frac{\not{P}}{2}-\not{k}\right)\right]$$

$$= 24\cdot 4m\,\frac{P}{2}\cdot\left(\frac{P}{2}-k\right)$$

$$= 96m\left(\frac{P^2}{4}-\frac{P\cdot k}{2}\right)$$

$$= 48m\,(2m^2-k\cdot P)$$

So, $\tau_2 = 96m(2m^2 - k\cdot P)$

Therefore, we find for the anomaly

$$a_e^{(1)} = \frac{1}{12m^2} \operatorname{tr}\Big[\Big(m^2\gamma_\mu - P_\mu \not{P} - \frac{3}{2} m P_\mu \Big) V_1^\mu + \frac{m}{4}\Big(\frac{\not{P}}{2}+m\Big)[\gamma_\mu,\gamma_\nu]\Big(\frac{\not{P}}{2}+m\Big)\,\delta V_1^{\nu\mu} \Big]$$

$$= -\frac{ie^2}{12m^2}\int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2[(P/2-k)^2-m^2]^2}\cdot\Big[\tau_1 + \frac{m}{4}\tau_2\Big]$$

Now,

$$\tau_1 + \frac{m}{4}\tau_2 = -16[3m^4 + m^2k^2 - (k\cdot P)^2] + 24m^2(2m^2 - k\cdot P)$$

$$= -16m^2k^2 + 16(k\cdot P)^2 - 24m^2\, k\cdot P$$

Notice that the integrand is finite as $k \to 0$, since the numerator is proportional to k. Naively, we see that the integral could be UV divergent, however we will see that this integral is finite.

So,

$$a_e^{(1)} = -ie^2\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2[(P/2-k)^2-m^2]^2}\Big(-\frac{4}{3}k^2 + \frac{4}{3m^2}(k\cdot P)^2 - 2k\cdot P\Big)$$

We now introduce Feynman parameters

$$\frac{1}{A^2 B} = 2\int_0^1 dx_1 \int_0^1 dx_2 \; \delta(1-x_1-x_2) \frac{x_1}{[x_1 A + x_2 B]^3}$$

$$= 2\int_0^1 dx \frac{x}{[xA + (1-x)B]^3}$$

$$= 2\int_0^1 dx \frac{x}{[x(A-B)+B]^3}$$

So, with

$$A = (P/2 - k)^2 - m^2 = \frac{P^2}{4} + k^2 - k\cdot P - m^2$$

$$= k^2 - k\cdot P$$

and

$$B = k^2$$

so,

$$\frac{1}{k^2[(P/2-k)^2 - m^2]^2} = 2\int_0^1 dx \frac{x}{[k^2 - k\cdot P\, x]^3}$$

So,

$$a_e^{(1)} = -2ie^2 \int_0^1 dx \; x \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - k\cdot P x)^3}\left(-\frac{4}{3}k^2 + \frac{4}{3m^2}(k\cdot P)^2 - 2k\cdot P\right)$$

To perform the momentum integral, let us perform this in d-dimensions to see the cancellation of the UV divergence.

Note the following integrals (No proof)

$$I(n) = \int d^d k \frac{1}{(k^2 + 2\hat{P}\cdot k - M^2)^n} = i\pi^{d/2} \frac{(-1)^n}{\Gamma(n)} \frac{\Gamma(n-d/2)}{(\hat{P}^2 + M^2)^{n-d/2}}$$

$$I_\mu(n) = \int d^d k \frac{k_\mu}{(k^2 + 2\hat{P}\cdot k - M^2)^n} = -\hat{P}_\mu I(n)$$

$$I_{\mu\nu}(n) = \int d^d k \frac{k_\mu k_\nu}{(k^2 + 2\hat{P}\cdot k - M^2)^n} = \left[\hat{P}_\mu \hat{P}_\nu - \frac{1}{2} g_{\mu\nu} \left(\frac{\hat{P}^2 + M^2}{n-1-d/2} \right) \right] I(n)$$

For us, $\hat{P} = -\frac{P_x}{2}$, $M^2 = 0$, $n = 3$, and let $d = 4 - 2\varepsilon$ ← take $\varepsilon \to 0$ at end

So

$$I(3) = -\frac{i\pi^2}{2\hat{P}^2} = -\frac{2i\pi^2}{P^2 x^2} = -\frac{i\pi^2}{2m^2 x^2}$$

and

$$I_\mu(3) = +\frac{x}{2} P_\mu I(3)$$

$$I_{\mu\nu}(3) = \left[\frac{x^2}{4} P_\mu P_\nu - \frac{1}{2} g_{\mu\nu} \frac{(P^2 x^2/4)}{2 - d/2} \right] I(3)$$

$$= \frac{x^2}{4} \left[P_\mu P_\nu - \frac{2m^2}{\varepsilon} g_{\mu\nu} \right] I(3)$$

Therefore,

$$\int d^4 k \frac{1}{(k^2 - k\cdot P_x)^3} \left(-\frac{4}{3} k^2 + \frac{4}{3m^2} (k\cdot P)^2 - 2k\cdot P \right)$$

$$= -\frac{4}{3} g^{\mu\nu} I_{\mu\nu}(3) + \frac{4}{3m^2} P^\mu P^\nu I_{\mu\nu}(3) - 2P^\mu I_\mu(3)$$

$$\int d^4k \, \frac{1}{(k^2 - k\cdot P_x)^3}\left(-\frac{4}{3}k^2 + \frac{4}{3m^2}(k\cdot P)^2 - 2k\cdot P\right)$$

$$= -\frac{4}{3} g^{\mu\nu} I_{\mu\nu}(3) + \frac{4}{3m^2} P^\mu P^\nu I_{\mu\nu}(3) - 2P^\mu I_\mu(3)$$

$$= -\frac{4}{3} g^{\mu\nu} \frac{x^2}{4}\left[P_\mu P_\nu - \frac{2m^2}{\varepsilon} g_{\mu\nu}\right] I(3)$$

$$+ \frac{4}{3m^2} P^\mu P^\nu \frac{x^2}{4}\left[P_\mu P_\nu - \frac{2m^2}{\varepsilon} g_{\mu\nu}\right] I(3)$$

$$- 2P^\mu \frac{x}{2} P_\mu I(3)$$

$$= \left\{-\frac{4}{3}\frac{x^2}{4}\left[P^2 - \frac{2m^2}{\varepsilon}\cdot 4\right] + \frac{4}{3m^2}\frac{x^2}{4}\left[P^2P^2 - \frac{2m^2}{\varepsilon}P^2\right] - xP^2\right\} I(3)$$

$$= \left\{\frac{x^2}{3}\left[-P^2 + \frac{2m^2}{\varepsilon}\cdot 4 + \frac{P^2P^2}{m^2} - \frac{2m^2}{\varepsilon}\frac{P^2}{m^2}\right] - xP^2\right\} I(3)$$

$$= \left\{\frac{x^2}{3}\left[-4m^2 + 16m^2\right] - x\cdot 4m^2\right\} I(3)$$

Naive UV divergence cancels.

$$= 4m^2 x(x-1) I(3)$$

$$= -2\,\frac{(x-1)}{x}\, i\pi^2$$

Therefore, the simplified expression for $a_e^{(1)}$ is

$$a_e^{(1)} = \frac{-2ie^2}{(2\pi)^4}\int_0^1 dx\, x \left\{\frac{2}{x}(1-x)\, i\pi^2\right\}$$

$$= +\frac{(2\pi)^2}{(2\pi)^4} e^2 \int_0^1 dx\,(1-x) = \frac{e^2}{4\pi^2}\left[x - \frac{x^2}{2}\right]_0^1 = \frac{1}{2}\left(\frac{e^2}{4\pi^2}\right)$$

So,

$$a_e^{(1)} = \frac{1}{2}\left(\frac{e^2}{4\pi^2}\right)$$

Recall the fine-structure constant, $\alpha \equiv \frac{e^2}{4\pi^2}$

$$\Rightarrow \boxed{a_e^{(1)} = \frac{\alpha}{2\pi}}$$

Schwinger's triumph!

here, $a_e^{(1)} = 1.16\ldots \times 10^{-3}$

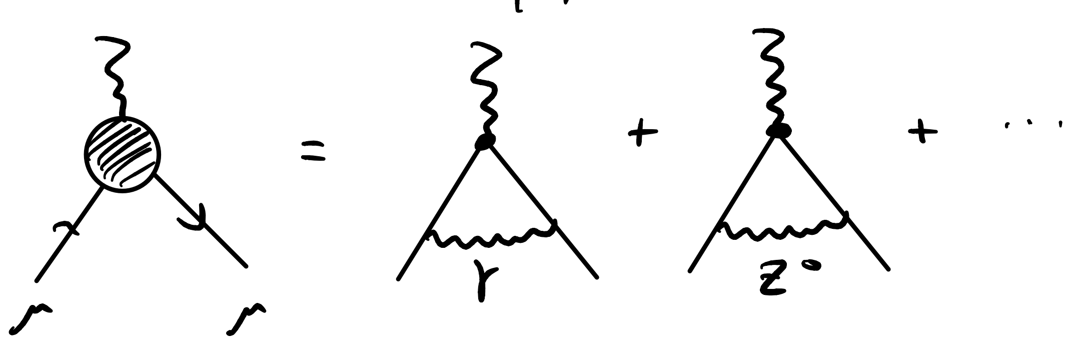
The measured value is $(a_e)^{exp} = 1.15965218073(28) \times 10^{-3}$

while the current theoretical prediction ($\mathcal{O}(\alpha^{10})$!) is

$$(a_e)^{SM} = 1.15965218188(78) \times 10^{-3}$$

A remarkable prediction!

For muons, the leading order is identical, $a_\mu^{(1)} = \frac{\alpha}{2\pi}$, but since $\frac{m_\mu}{m_e} \sim 200$, the value of $\frac{g-2}{2}$ is susceptible to searches for new physics.



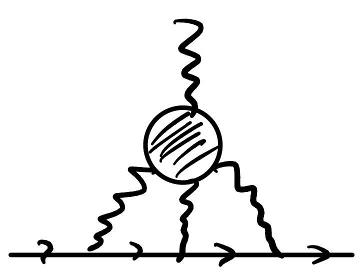
EW coupling.

For the Z^0-boson, $$a_\mu \sim \left(\frac{m_\mu}{m_Z}\right)^2 \cdot \frac{g_e^2}{16\pi} \cdot \log\left(\frac{m_Z^2}{m_\mu^2}\right)$$

There is an intense search for new physics by looking at the muon anomaly. The current (as of Aug. 2023) experimental measurement is

$$(a_\mu)^{exp} = 116592059(22) \times 10^{-11}$$

The SM theory is a little more muddled. There are discrepancies between lattice methods and data-driven analyses. There is an apparent 4.2σ discrepancy with experiment. But, two hadronic processes are the leading systematics



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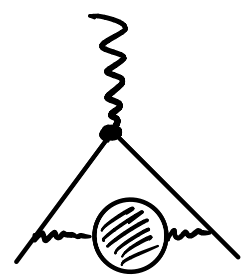

Hadronic Light-by-light

Hadronic vacuum polarization