Phenomenology II - QCD

Quantum Chromodynamics (QCD) is a non-abelian gauge theory describing the color interactions of quarks and gluons. The Lagrange density is

$$\mathcal{L} = \frac{1}{2} i \sum_f \bar{q}_f \not{D} q_f + \text{h.c.} - \sum_f m_f \bar{q}_f q_f - \frac{1}{2} \text{tr}\left[G_{\mu\nu} G^{\mu\nu}\right]$$

where $D_\mu = \partial_\mu + i g_s A_\mu$, $A_\mu = A_\mu^a \underbrace{T_a}_{= \frac{\lambda_a}{2}}$

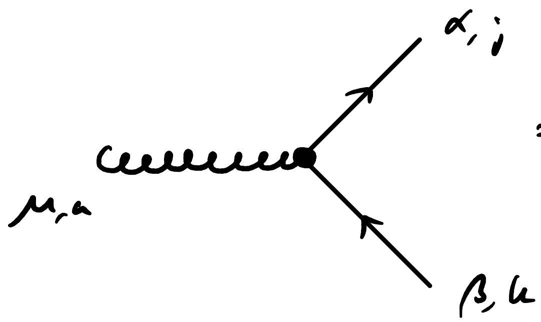
and

$$G_{\mu\nu} = \frac{1}{i g_s}\left[D_\mu, D_\nu\right]$$

There are 6 flavors, $f = \{u, d, s, c, b, \underbrace{t}_{\text{"decays" via weak before hadronization}}\}$

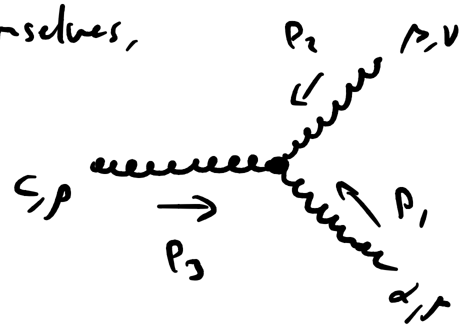
QCD, in principle, describes <u>all</u> hadronic and nuclear phenomena we observe in the universe. It depends on 6 quark masses, and 1 coupling, g_s. Define $\boxed{\alpha_s \equiv \frac{g_s^2}{4\pi}}$

This coupling governs the interactions of massless gluons with quarks, e.g.,

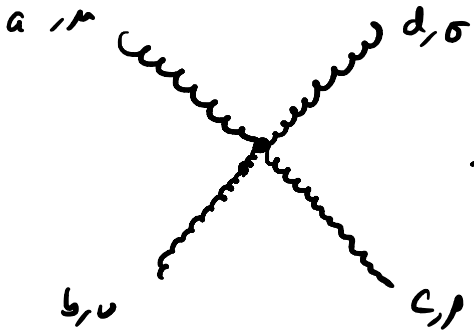


$$= -i g_s \underbrace{(T^a)_{jk}}_{\text{carries 2 color indices}} \gamma^\mu_{\alpha\beta}$$

The same coupling also governs interactions of gluons with themselves,

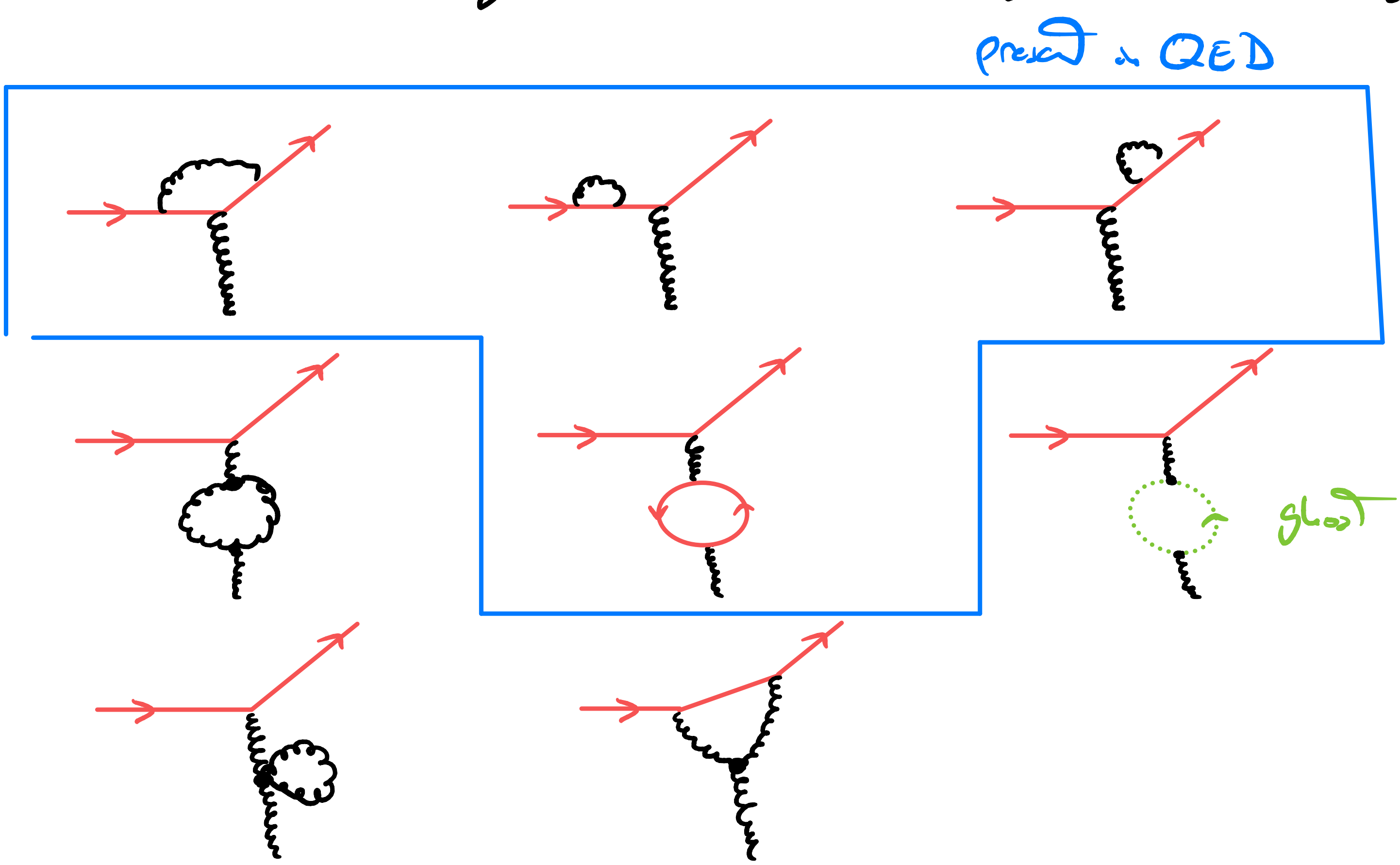


$$= -ig_s f^{abc} \left(p_3^\mu g^{\nu\rho} + \text{perm.} \right)$$



$$= -g_s^2 f^{abe} f^{cde} g^{\mu\rho} g^{\nu\sigma} + \text{perm.}$$

In perturbation theory, higher order corrections augment the coupling.



Can show (cours QFT II), that the coupling g_s runs with the momentum transfer Q^2, and is described by a differential equation

"beta function"

$$\frac{dg_s}{d\ln Q} = \beta(g_s)$$

# of quarks with $m_q < Q$

$$= -\frac{g_s^3}{16\pi^2}\underbrace{\left[11 - \frac{2}{3}N_f\right]}_{\text{LO } \beta_0 \text{ (1-loop)}} + \mathcal{O}(g_s^5)$$

Notice that for $N_f < 16$, $\beta(g_s) < 0$

$\Rightarrow$ g_s decreases as Q increases. This is known as $\boxed{\text{Asymptotic freedom}}$

Conversely, as Q decreases, g_s increases, and eventually perturbation theory breaks down. This indicates that low-energy physics must be computed with non-perturbative techniques. This is the $\boxed{\text{confinement}}$ region. Gluons are essential for this, cf. QED with electrons and photons

$$\frac{de}{d\ln Q} = \frac{e^3}{16\pi^2}\cdot\frac{4}{3} + \mathcal{O}(e^5) > 0$$

Can write a general for α_s,

$$\frac{d\alpha_s}{d\ln Q^2} = -\alpha_s\left[\left(\frac{\alpha_s}{4\pi}\right)\underbrace{\beta_0}_{11-2N_f/3} + \left(\frac{\alpha_s}{4\pi}\right)^2\beta_1 + \cdots\right]$$

Solving the 1-loop equation, let $x = 1/\alpha_s$, $L = \ln Q^2$

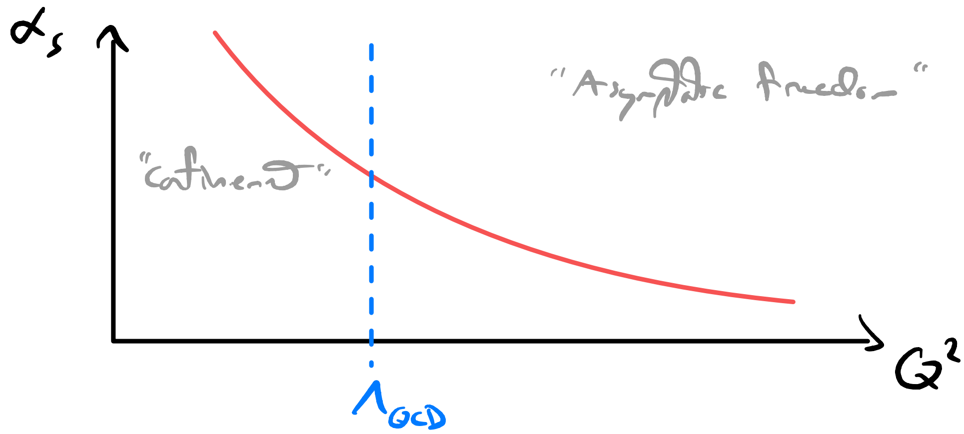
$$\Rightarrow \frac{dx}{dL} = \frac{-1}{\alpha_s^2}\frac{d\alpha_s}{dL} = \frac{\beta_0}{4\pi} \Rightarrow x = \frac{\beta_0}{4\pi}L + \text{const.}$$

$$\Rightarrow \boxed{\alpha_s(Q^2) = \frac{4\pi}{\beta_0}\frac{1}{\ln(Q^2/\Lambda_{QCD}^2)}}$$

integration const.

This is the running coupling.

Λ_{QCD} is value of Q at which $\alpha_s = 1$. Consider massless QCD, with only the dimensionless coupling (1 parameter). But, the running introduces a dimensionful scale. This is known as dimensional transmutation. In effect, Λ_{QCD} sets the scale for strong interactions involving light quarks.



To study low-energy hadrons, we need a non-perturbative technique to access observables. There is another issue we need to deal with, and that is we do not observe quarks and gluons, but observe them only as confined inside hadrons.

What are the allowed quark & gluon combinations that can give a valid hadron? We have seen from the Quark Model

$q\bar{q}$ – Mesons

qqq – Baryons, $\bar{q}\bar{q}\bar{q}$ – Antibaryons

In QCD, we need color singlet objects made from q_j, $\bar{q}_j$, and A^{μ}_{jk} (Field operators)

color indices

For hadrons to exist, there must be a color singlet operator to create it from the QCD vacuum

e.g.,

- $\bar{q}_j \gamma_5 q'_j$ for $J^{PC} = 0^{-+}$ mesons
- $\epsilon_{jkl}\, q_j q'_k q''_l$ for baryons

But, also can have

- $G^{\mu\nu}_{jk} G_{\nu\mu, kj}$ for scalar glueball
- $\bar{q}_j [\gamma^\mu, \gamma^\nu] G_{\mu\nu, jk} q_k$ for $q\bar{q}g$ hybrid
- $\bar{q}_i \gamma^\mu q'_j \, \bar{q}''_k \gamma_\mu q'''_k$ for "tetraquark"
- $\epsilon_{jkl} q_j q'_k q''_l \, \bar{q}'''_m q''''_m$ for "pentaquark"

... any many (∞) more.

The problem is that there is no 1-to-1 correspondence between operators and states in QFT. Operators with the same quantum numbers can create the same state

e.g.,

$$\mathcal{O}_1 = \bar{u}\gamma_5 u + \bar{d}\gamma_5 d + \bar{s}\gamma_5 s$$ looks like $\bar{q}q$

$$\mathcal{O}_2 = \epsilon_{\mu\nu\rho\sigma} G^{\mu\nu}_{jk} G^{\rho\sigma}_{jk}$$ looks like pseudoscalar glueball

Both can create η & η', $\langle 0 | \mathcal{O}_{1,2} | \eta^{(\prime)} \rangle \neq 0$

From a QFT pov, asking whether η' is a $\bar{q}q$ state or a glueball is almost meaningless.

Moreover, states created by a given operator can consist of multiple hadrons, e.g.,

$$\bar{u}\gamma_5 d\, \bar{u}\gamma_5 d\, |0\rangle \sim |\pi^+\pi^+\rangle$$

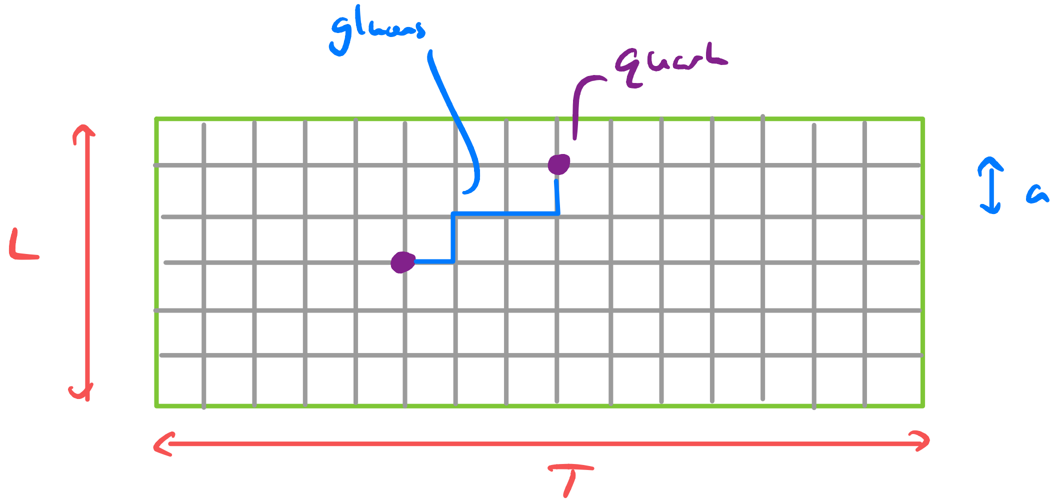
$\Rightarrow$ Existence of an operator Does Not imply existence of a new hadron.

How can we access such complicated non-perturbative objects from QCD? Lattice QCD is a numerical technique to stochastically compute the non-perturbative QCD path integral. Can now calculate light hadron masses w/ $\lesssim 1\%$ uncertainties and reaction amplitudes involving a couple of hadrons. This is the only known method for making quantitative predictions from QCD in the low-energy region.

To compute with Lattice QCD, we rotate the action S_{QCD} to Euclidean time, $t \to -i\tau_E$

$$\Rightarrow e^{-iEt} \to e^{-E\tau_E}$$

and discretize spacetime (w/ spacing a) & use a finite size box $L^3 \times T$.



The QCD path integral is formulated as

$$Z_{QCD} = \underbrace{\int \mathcal{D}q\, \mathcal{D}\bar{q}\, \mathcal{D}U}_{\text{integrate over all field "configurations"}} \; e^{-S^{(E)}_{QCD}[q,\bar{q},U]}$$

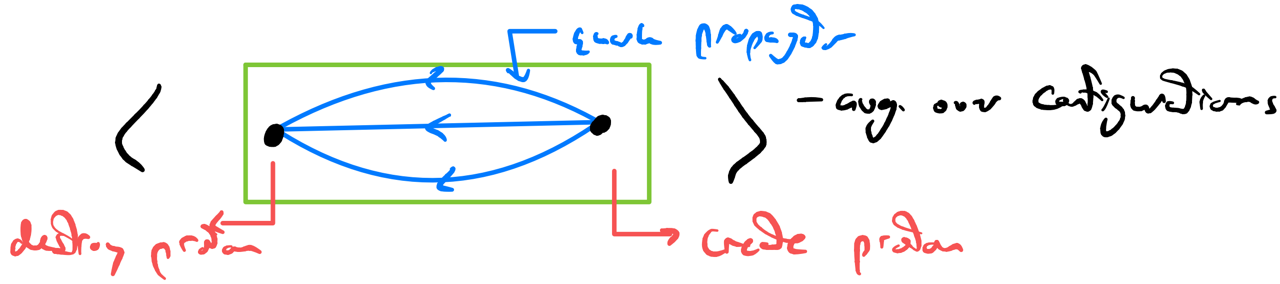
$U \rightarrow e^{i g_s a A_\mu \cdot n_\mu}$

On a lattice, this is millions of integrals. We use Monte Carlo methods to stochastically estimate correlation functions. For example, to get proton mass, want

$$\langle O_p(\tau_E)\, O_p^\dagger(0) \rangle \qquad \text{where } O_p \propto \epsilon_{jkl}\, u_j u_k d_l$$

create proton at $\tau_E = 0$, and destroy @ τ_E.

$$= Z_p\, e^{-\underbrace{m_p}_{\text{ground state proton mass}} T_E} + Z_{p'}\, e^{-\underbrace{m_{p'}}_{\text{excited proton}} T_E} + \cdots$$

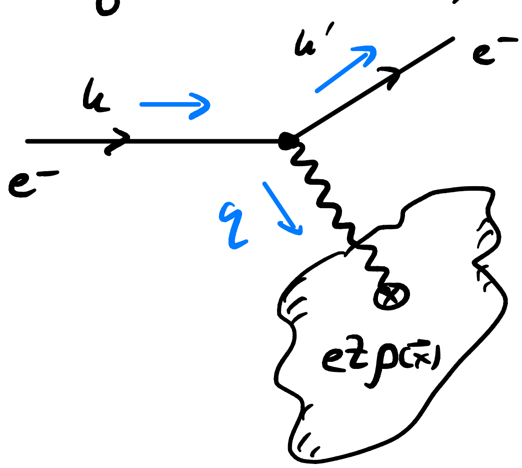


Deep Inelastic Scattering

If QCD is so complicated, how did we accept the theory as an explanation for hadrons? Consider the high-energy limit, $Q^2 \to \infty$, then the coupling is small, and QCD exhibits asymptotic freedom. That is, at high-energies quarks and gluons "behave" as "free" particles, before they hadronize.

There are many types experiments that show this, we will focus on a particular process called Deep Inelastic Scattering (DIS).

Consider electron scattering off a static classical charge distribution,



The electrostatic potential is

$$A^\mu(x) = (\varphi(q\vec{x}), \vec{0})$$

$$\vec{\nabla}^2 \varphi = -eZ\rho$$

$$\int d^3\vec{x}\, \rho(\vec{x}) = 1$$

Fourier transform, $\tilde{A}^\mu(q)$

$$\tilde{A}^0(q) = 2\pi\,\delta(q^0)\int d^3\vec{x}\; e^{i\vec{q}\cdot\vec{x}}\,\varphi(\vec{x})$$

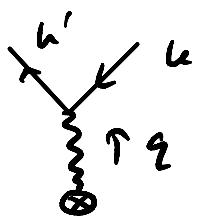
$$= 2\pi\,\delta(q^0)\left(\frac{-1}{|\vec{q}|^2}\int d^3\vec{x}\; e^{i\vec{q}\cdot\vec{x}}\,\vec{\nabla}^2\varphi(\vec{x})\right)$$

$$= 2\pi\,\delta(q^0)\underbrace{\left[\frac{eZ}{|\vec{q}|}\int d^3\vec{x}\; e^{i\vec{q}\cdot\vec{x}}\,\rho(\vec{x})\right]}_{\equiv F(\vec{q})}$$

"form-factor"

$Q^2 = -q^2$

Scattering amplitude is

$i\mathcal{M} =$ 

$$= ie\,\bar{u}(k')\gamma^\mu u(k)\,\tilde{A}_\mu(q)$$

$$= 2\pi i e\;\delta(E-E')\,\frac{Ze}{|\vec{q}|^2}\,F(\vec{q})$$

So cross-section is

$$\frac{d\sigma}{d\Omega} = \frac{Z\alpha^2}{4E^2\sin^4\frac{\theta}{2}}\,|F(\vec{q})|^2 \quad ; \quad |\vec{q}|^2 = 2E^2(1-\cos\theta) = 4E^2\sin^2\frac{\theta}{2}$$

$$= \frac{Z\alpha^2}{Q^4}\,|F(\vec{q})|^2$$

can learn about charge distribution

experimentally measure

Let's Taylor expand the form-factor,

$$F(\vec{q}) = \int d^3\vec{x}\left[1 - i\vec{q}\cdot\vec{x} - \frac{(\vec{q}\cdot\vec{x})^2}{2} + \mathcal{O}(q^3)\right]\rho(\vec{x})$$

assume spherically symmetric

$$= 1 - \frac{|\vec{q}|^2}{6}\int d^3\vec{x}\,|\vec{x}|^2\rho(\vec{x}) + \mathcal{O}(q^4)$$

$$\equiv 1 - \frac{|\vec{q}|^2}{6}\underbrace{\langle|\vec{x}|^2\rangle}_{\text{mean squared charge radius}} + \mathcal{O}(q^4)$$

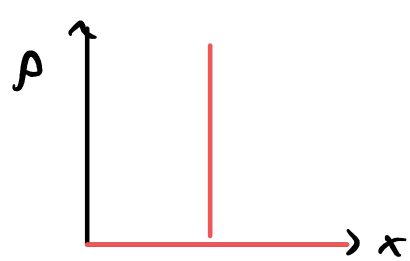
$$\Rightarrow \boxed{\langle r^2\rangle \equiv \langle|\vec{x}|^2\rangle = 6\left.\frac{dF}{dq^2}\right|_{q^2=0}} \qquad q^2 = -|\vec{q}|^2$$

For example, the proton is observed to have (to lowest approximation)

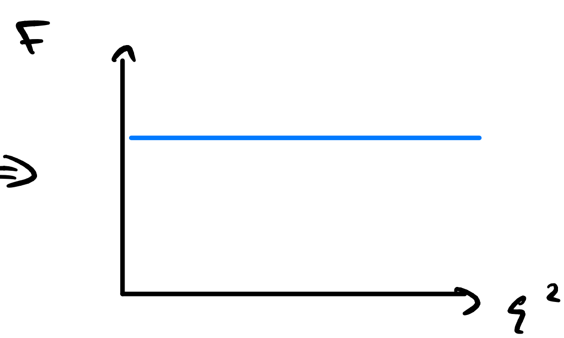
$$F(\vec{q}) = \frac{1}{\left(1-\frac{q^2}{\Lambda^2}\right)^2} \Rightarrow \rho(\vec{x}) \propto e^{-\Lambda|\vec{x}|}$$

where $\Lambda \simeq 1\ \text{fm}^{-1} = 1$ GeV.

e.g., point particle

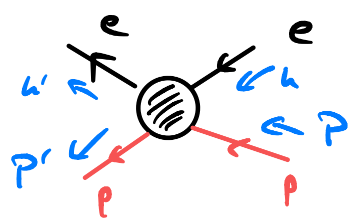


$\Longleftrightarrow$

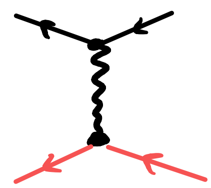


In a realistic experiment, the proton recoils.
Consider elastic $e^- p \to e^- p$ scattering, where the proton is a point particle. To leading order in α, we have

$i\mathcal{M} =$

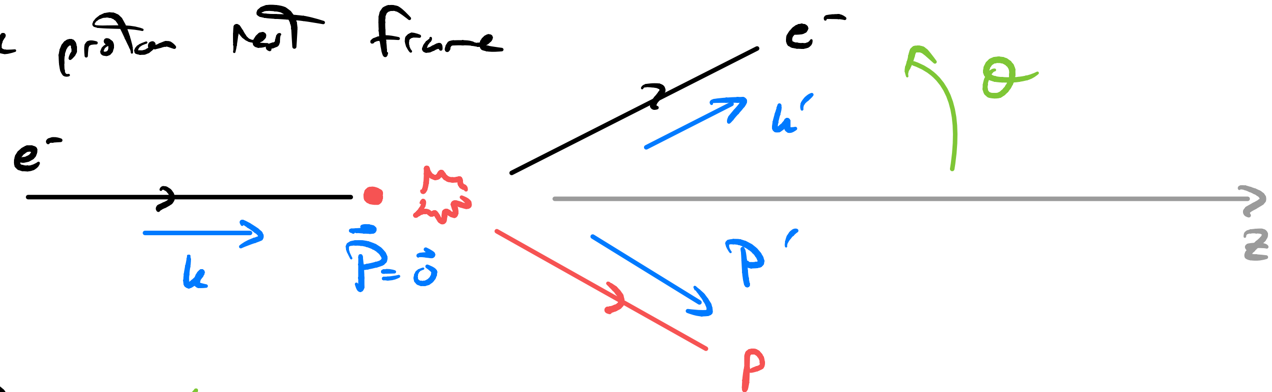


$=$



$+ \quad \mathcal{O}(\alpha^2)$

In the proton rest frame



Cross-section is (exercise)

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E^2 \sin^4\frac{\theta}{2}} \underbrace{\frac{E'}{E}}_{\text{recoil factor}} \left(\cos^2\frac{\theta}{2} + \frac{Q^2}{2m_p^2} \sin^2\frac{\theta}{2} \right)}$$

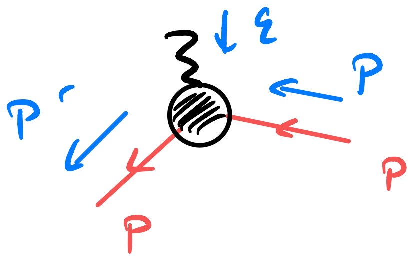
kinematics: $k = (E, 0, 0, E)$ where $E \gg m_e$

$k' = (E', E'\sin\theta, 0, E'\cos\theta)$ $\quad E' \gg m_e$

$P = (m_p, 0, 0, 0)$

and $q^2 = (k-k')^2 = -2EE'(1-\cos\theta) = -Q^2$

For a realistic proton, we need to account for it being a composite particle. As in QED, can parametrize in terms of form-factors,



$$\Rightarrow \Gamma^\mu = F_1(Q^2)\gamma^\mu + \frac{i\sigma^{\mu\nu}}{2m_p} q_\nu F_2(Q^2)$$

In e-μ scattering of QED, $F_1(Q_1^2) - F_1(Q_2^2)$

$$\simeq -\frac{\alpha}{4\pi}\log\left(\frac{Q_1^2}{Q_2^2}\right)$$

a relatively mild change

For ep scattering, it is observed that

$$F_1(Q^2) \simeq \frac{1}{\left(1+\frac{Q^2}{\Lambda^2}\right)^2} \qquad \text{with } \Lambda^2 \simeq 0.71\ \mathrm{GeV}^2$$

This includes non-perturbative QCD effects.

Can show, in proton rest frame

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E^2\sin^4\frac{\theta}{2}}\frac{E'}{E}\left[\left(F_1^2 + \frac{Q^2}{4m_p^2}F_2^2\right)\cos^2\frac{\theta}{2} + \frac{Q^2}{2m_p^2}(F_1+F_2)^2\sin^2\frac{\theta}{2}\right]}$$

Convenient to define electric and magnetic form-factors

$$G_E \equiv F_1 - \tau F_2 \qquad , \quad \tau = \frac{Q^2}{4m_p}$$
$$G_M \equiv F_1 + F_2$$

So,

Rosenbluth formula

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E^2 \sin^4\frac{\theta}{2}} \frac{E'}{E}\left[\frac{G_E^2 + \tau G_M^2}{1+\tau} \cos^2\frac{\theta}{2} + 2\tau G_M^2 \sin^2\frac{\theta}{2} \right]}$$

Experimentally, for proton $G_E(0) = 1$, $G_M(0) = 2.79$

neutron $G_E(0) = 0$, $G_M(0) = -1.91$

Recall for point particle, Dirac showed

$$g = 2G_M(0) = 2(1 + F_2(0)) = 2(1 + \mathcal{O}(\alpha))$$

For the proton $F_2(0) = G_M(0) - 1 = 1.79 \neq \mathcal{O}(\alpha)$

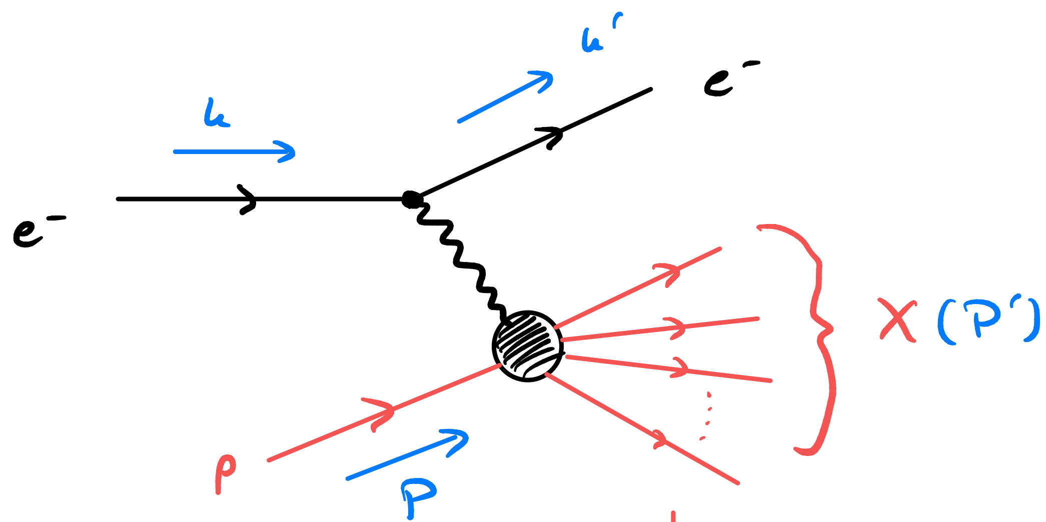
neutron $F_2(0) = G_M(0) - 1 = -2.91 \neq \mathcal{O}(\alpha)$

proton & neutron are not elementary fermions!

At higher virtualities $Q^2 \gg m_p^2$, then we can blow the proton apart in an inelastic collision

$e^- p \to e^- X$

↳ A bunch of hadrons



↳ A number of hadrons, $p, n, \pi, K, \ldots$

In this process, we do not measure X (inclusive process).

Can think of this as a high-energy photon probing the structure of the proton.

Let's define a few more useful kinematic variables,

$$\nu = E - E' = \frac{P \cdot q}{m_p} \qquad \text{energy loss}$$

For DIS, $Q^2 \gg m_p^2 \Rightarrow \nu \gg m_p$

↳ $\Rightarrow \lambda \ll \frac{1}{m_p} \ll r_p$

"photon probes internal structure"

"Bjorken x"

$$\boxed{x = \frac{Q^2}{2m_p\nu} = \frac{Q^2}{2p\cdot q}}$$

Since $p'^2 = M_x^2 \geq m_p^2$ $\longrightarrow$ = for elastic scattering

$$= (q+p)^2$$

$$= -Q^2 + 2p\cdot q + m_p^2$$

So, $$x = \frac{Q^2}{Q^2 + M_x^2 - m_p^2} \Rightarrow \boxed{0 \leq x \leq 1}$$

$x = 0$ if $Q^2 = 0$
$x = 1$ if $M_x^2 = m_p^2$

It is convenient to parametrize DIS by x & Q^2.

Another useful variable,

$$y = \frac{E - E'}{E}$$ relative energy loss

$$= \frac{\nu}{E} = \frac{p\cdot q}{p\cdot k}$$

Can show that the cross-section for DIS is of the form

$$\frac{d\sigma}{d\Omega\, dE'} = \frac{\alpha^2}{4\pi m_p Q^4} \frac{E'}{E} L^{\mu\nu} W_{\mu\nu}$$

↳ More d.o.f. since $M_X > m_p$

where $L^{\mu\nu} = \frac{1}{2} \mathrm{tr}[\not{k}' \gamma^\mu \not{k} \gamma^\nu]$ lepton tensor

&

$$W^{\mu\nu} = F_1 \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2} \right) + F_2 \left(p^\mu - \frac{p \cdot q}{q^2} q^\mu \right)\left(p^\nu - \frac{p \cdot q}{q^2} q^\nu \right)$$

↳ Ward identity enforced

Hadronic tensor (parametrize)

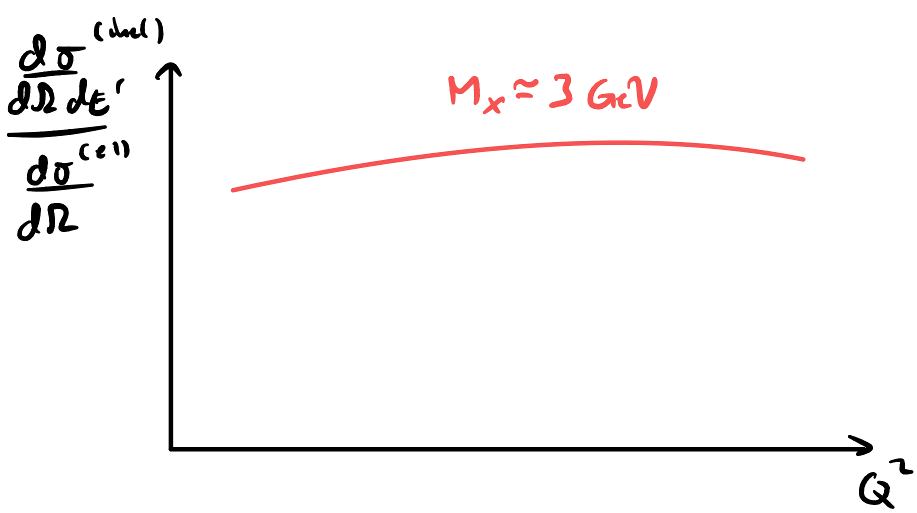
& $F_{1,2} = F_{1,2}(x, Q^2)$ have been measured.

↳ structure functions (not form-factors)

We find,

$$\frac{d\sigma}{d\Omega\, dE'} = \frac{\alpha^2}{4E^2 \sin^4\frac{\theta}{2}} \left[\frac{1}{\nu} F_2(x,Q^2) \cos^2\frac{\theta}{2} + \frac{2}{m_p} F_1(x,Q^2) \sin^2\frac{\theta}{2} \right]$$

It has been experimentally observed that



Result looks approx constant ⇒ looks like scattering off point-like constituents.

Useful to reformulate cross-section as

$$\frac{d^2\sigma}{dx\,dQ^2} = \frac{4\pi\alpha^2}{Q^2}\left[(1-y)\frac{F_2(x,Q^2)}{x} + y^2 F_1(x,Q^2)\right]$$

Can measure separately due to y-dependence

From y-dependence, $0 \leq y \leq 1$, find

as $y \sim 0 \Rightarrow \frac{d^2\sigma}{dx\,dQ^2} \propto \frac{F_2(x,Q^2)}{x}$

as $y \sim 1 \Rightarrow \frac{d^2\sigma}{dx\,dQ^2} \propto F_1(x,Q^2)$

Some features of F's

- For sufficiently large Q^2 (deep probe)

$$F_1(x,Q^2) \simeq F_1(x)$$
$$F_2(x,Q^2) \simeq F_2(x)$$

Q independence!

$\boxed{\text{Bjorken Scaling}}$

- For sufficiently large Q^2

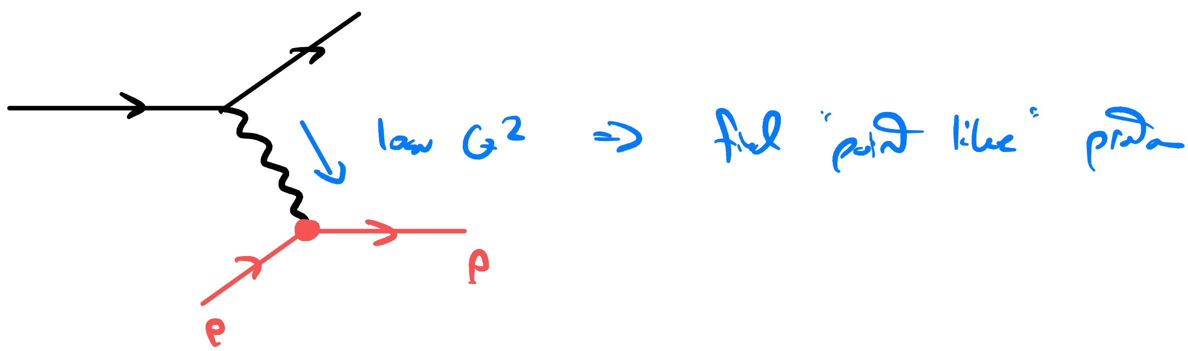
$$\boxed{2F_1(x) = \frac{F_2(x)}{x}}$$

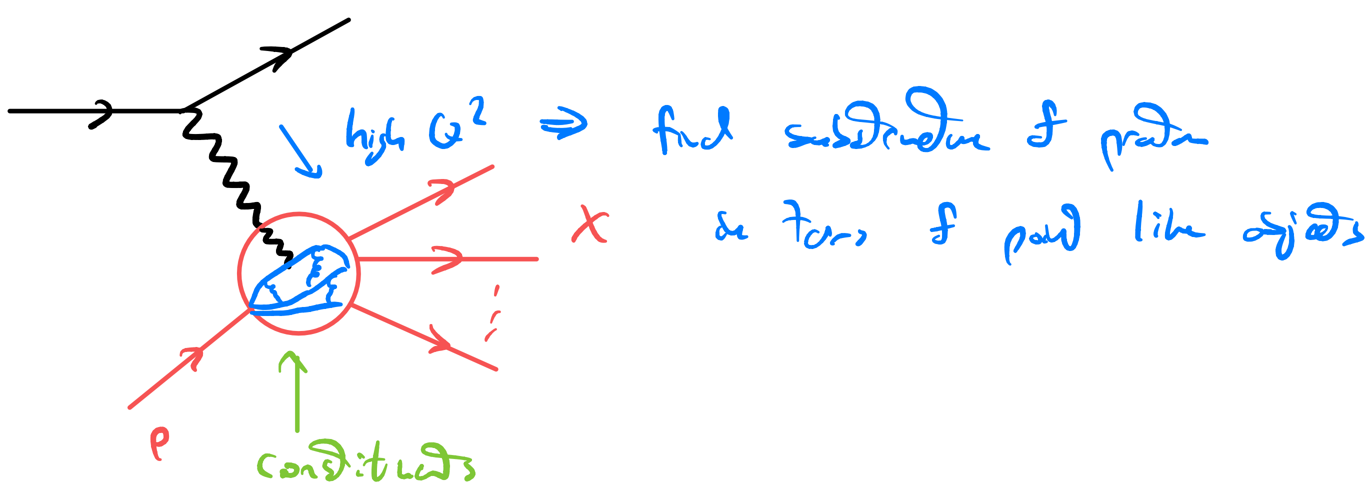
This is the Callan-Gross relation

The Parton Model

From the data, $d^2\sigma/dx\,dQ^2$ looks like a cross-section for a point particle for large Q^2. Since F_1, F_2 are Q^2 independent for DIS, it suggests that the proton is made up of point-like constituents. Feynman called these Partons.

We now know that these are constituent quarks & gluons in QCD. The idea of DIS is illustrated as



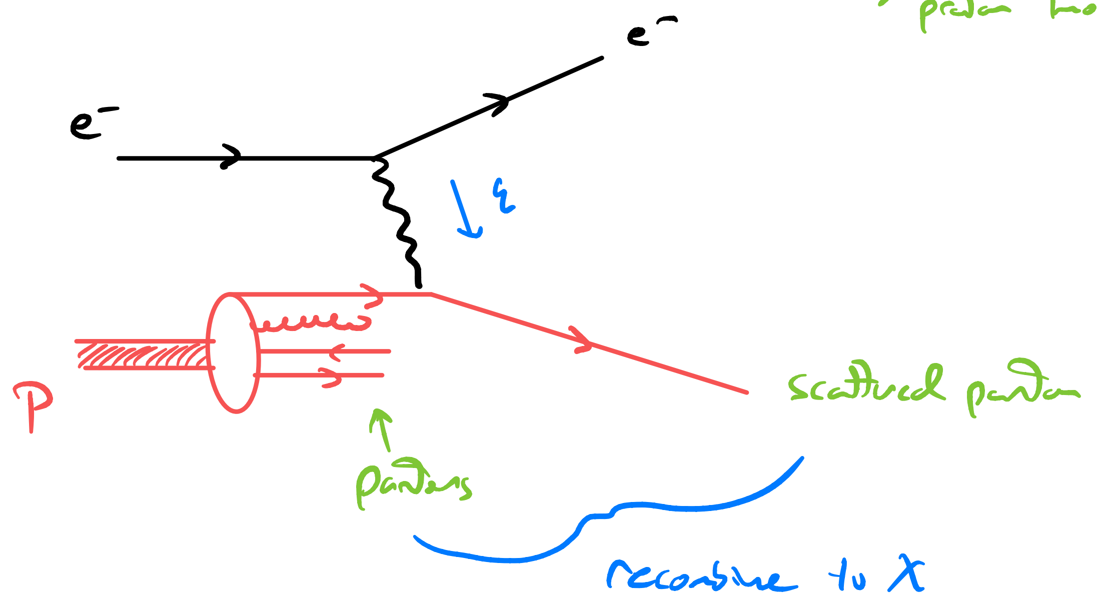


DIS consists of elastic scattering off individual "parts", followed by "hadronization".

Each parton carries a fraction ξ of the proton's momentum,

$$p^\mu_{parton} = \xi P^\mu$$

↳ proton momentum



Recall that $x = \dfrac{Q^2}{2p\cdot q}$, $0 \leq x \leq 1$

for elastic scattering, $M_X = m_p \Rightarrow x = 1 = \dfrac{Q^2}{2p\cdot q}$

For a parton, $p_{parton} = \xi p$

So, x_f (fraction for proton) is

$$x_f = \frac{Q^2}{2p_{parton}\cdot q} = \frac{Q^2}{2\xi p\cdot q} = \frac{1}{\xi}x$$

But, partons scatter elastically, $\Rightarrow x_f = 1$

$\Rightarrow \quad x_f = \frac{1}{\xi} x = 1 \quad \Rightarrow \quad \boxed{x = \frac{Q^2}{2p\cdot q} = \xi}$

So, in the parton model, Bjorken x is the parton's momentum fraction.

Can show that $\quad$ Q_f frac $Q_f e$ charge

$$\left.\frac{d\sigma}{dQ^2}\right|_{\text{parton}} = \frac{4\pi\alpha^2}{Q^4} Q_f^2 \left[(1-y) + \frac{y^2}{2}\right]$$

For a single parton.

Therefore, integrating over all parton distributions,

$$\Rightarrow \boxed{\frac{d^2\sigma}{dx\,dQ^2} = \frac{4\pi\alpha^2}{Q^4}\left[(1-y) + \frac{y^2}{2}\right]\sum_f f(x)\,Q_f^2}$$

$f(x)$ is called a parton distribution function (PDF)

$f(x)\,dx$ = # of f-type quarks in proton
with momentum fractions
between x and $x+dx$

$f(x) = \{ u(x), d(x), s(x), \ldots, \bar{u}(x), \bar{d}(x), \bar{s}(x), \ldots \}$

Can also have gluon distribution $g(x)$.

Compare Parton model with experimental observable,

$$\frac{d^2\sigma}{dx\,dQ^2} = \frac{4\pi\alpha^2}{Q^2}\left[(1-y)\frac{F_2(x,Q^2)}{x} + y^2 F_1(x,Q^2)\right]$$

$$\simeq \frac{4\pi\alpha^2}{Q^2}\left[(1-y) + \frac{y^2}{2}\right] 2F_1(x,Q^2)$$

↳ assumes Callan-Gross

$$2F_1 = \frac{F_2}{x}$$

parton model,

$$\frac{d^2\sigma}{dx\,d\Omega} = \frac{4\pi\alpha^2}{Q^2}\left[(1-y) + \frac{y^2}{2}\right] \sum_f{}' f(x)\, Q_f$$

Find that the Parton model predicts

$$\boxed{\underbrace{2F_1(x,Q^2) = \frac{F_2(x,Q^2)}{x}}_{\text{Callan-Gross}} = \underbrace{\sum_f{}' Q_f^2 f(x)}_{\text{Independent of } Q^2\,!}}$$

For proton,

$$\frac{F_2}{x} = \frac{4}{9}\left[u(x) - \bar{u}(x)\right] + \frac{1}{9}\left[d(x) + \bar{d}(x) + s(x) + \bar{s}(x)\right]$$

So, can measure cross-section for $ep \to eX$,
can measure then $F_1(x,Q^2)$ & $F_2(x,Q^2)$.
Similarly, can measure eD scattering, to
infer neutron structure functions,

$$F_{1,2}^{eD} \simeq \frac{F_{1,2}^{ep} + F_{1,2}^{en}}{2}$$

For neutron, using isospin symmetry, find

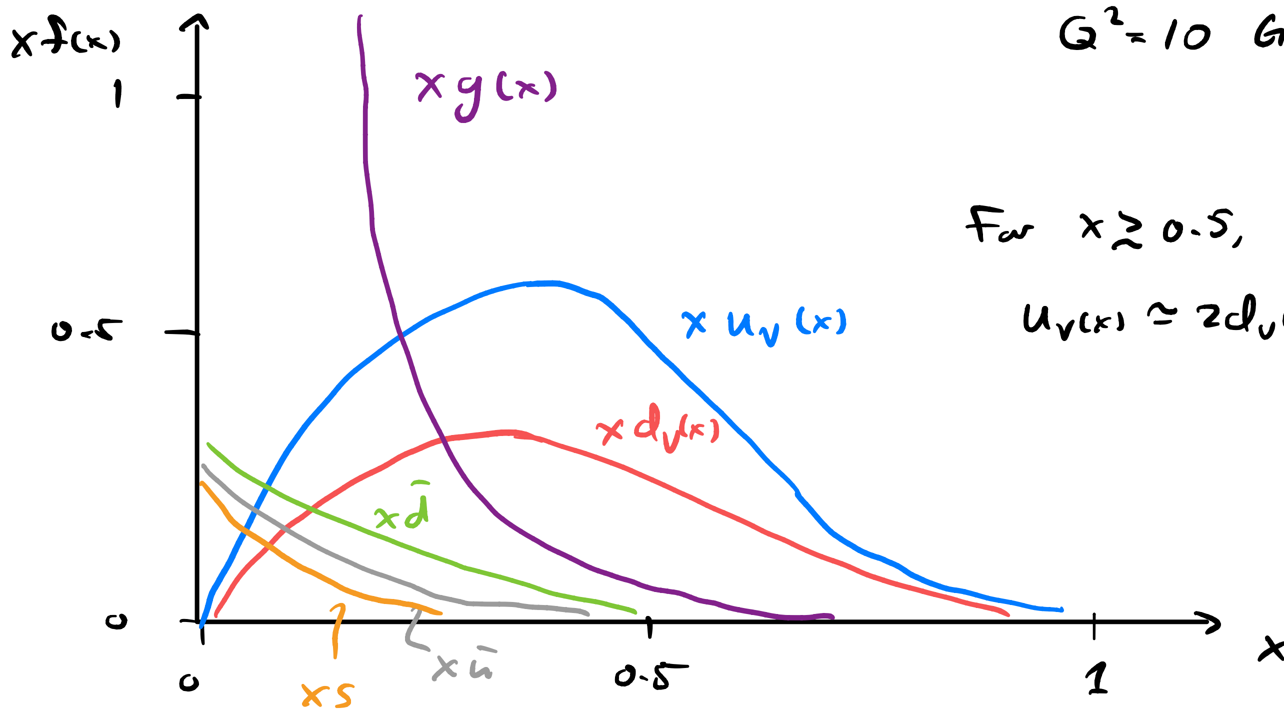
$$\frac{F_2^{en}}{x} \simeq \frac{4}{9}\left[d(x) + \bar{d}(x)\right] + \frac{1}{9}\left[u(x) + \bar{u}(x) + s(x) + \bar{s}(x)\right]$$

isospin : $u \leftrightarrow d$, $\bar{d} \leftrightarrow \bar{u}$, $s \leftrightarrow s$, $\bar{s} \leftrightarrow \bar{s}$

If one ignores $s+\bar{s}$, can use F_2^{ep} and F_2^{en}
together to determine $u(x)+\bar{u}(x)$ & $d(x)+\bar{d}(x)$.

To obtain further PDFs , one uses (anti)-neutrino DIS,
results from $p\overset{(-)}{p}$ scattering (e.g., Drell-Yan $p\overset{(-)}{p} \to \mu^+\mu^- + X$)

Find (roughly) the following distributions,



$Q^2 = 10$ GeV^2

For $x \gtrsim 0.5$,

$u_v(x) \simeq 2d_v(x)$

Here, "valence" PDF are

$$u_v(x) \equiv u(x) - \bar{u}(x)$$
$$d_v(x) \equiv d(x) - \bar{d}(x)$$

Crudely, proton is made up of u_v, u_v, d_v "valence" quarks, with sea quarks & gluons

Note the sum rules (for proton)

$$\int_0^1 dx\, u_v(x) = 2, \quad \int_0^1 dx\, d_v(x) = 1, \quad \int_0^1 dx\, s_v(x) = 0$$

& momentum sum rule

$$\sum_j \int_0^1 dx\, x f_j(x) = 1$$

$\hookrightarrow$ sum over partons

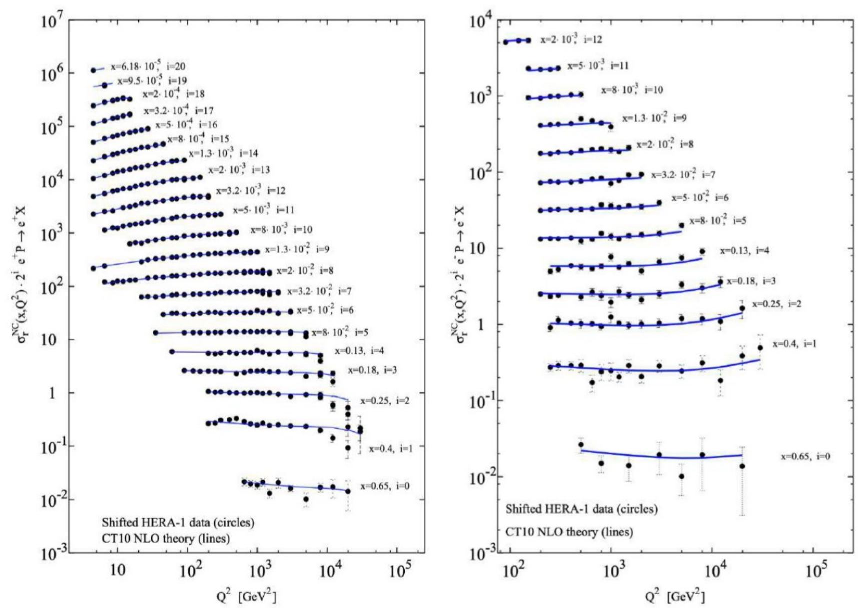
Find that

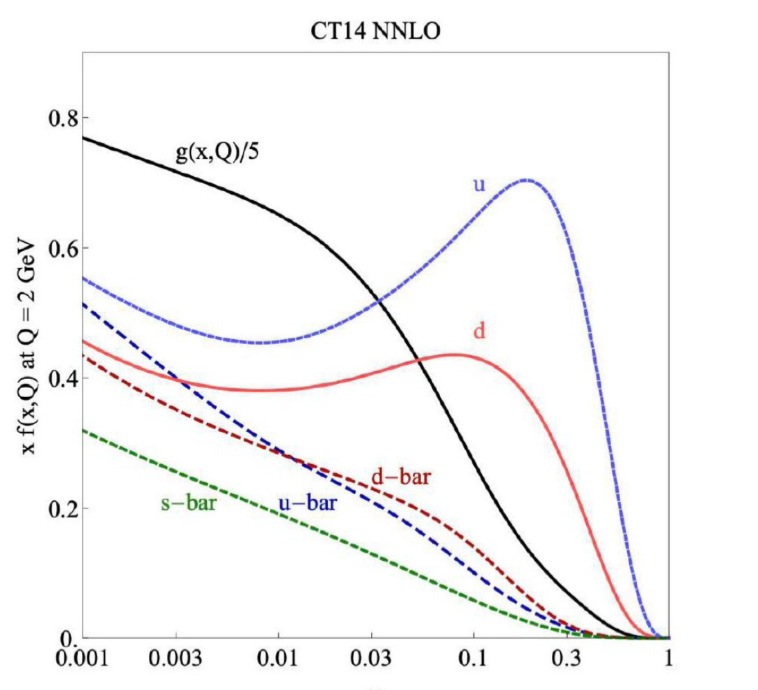
$$\int_0^1 dx\,(u_v + d_v)\,x\,dx \simeq 0.38$$

$$\int_0^1 dx\,g(x)\,x \simeq 0.5$$

rest in sea quarks

$\Rightarrow$ Most momentum in gluons!





Chiral Symmetry Breaking in QCD

In the Parton model, we find that gluons contribute a significant fraction to the momentum distribution of the proton. In general, we have observed that the light quark mass is very small,

$$m_u \simeq 2.2\ \text{MeV}, \quad m_d \simeq 4.7\ \text{MeV}, \quad m_s \simeq 95\ \text{MeV}$$

Compare this to $m_p \simeq 940\ \text{MeV} \gg 2m_u + m_d$. One can show with Lattice QCD that the Nucleon mass goes like

$$m_N \simeq m_N^{(0)} + C_1 m_\pi$$

depends on $m_{\ell = u,d}$

zero quark mass

$m_N^{(0)} \approx 800\ \text{MeV}$

The pion mass is known to behave like

$$m_\pi^2 \propto m_\ell$$

so, as $m_\ell \to 0$, $m_\pi \to 0$

Why is the π special?

The pion, as well as the kaon & eta, play a special role in QCD dynamics.

Consider QCD of only light & strange quarks, $N_f = 3$.

Here, $m_u, m_d, m_s \ll \Lambda_{QCD}$. So, perhaps we treat the quark mass as a perturbative correction.

Massless QCD is

$$\mathcal{L} = \frac{i}{2} \sum_f \bar{\psi}_f \not{D} \psi_f - \frac{1}{2} \mathrm{tr}\left[G_{\mu\nu} G^{\mu\nu} \right]$$

There is now a local $SU(3)_c$ gauge symmetry, and a global chiral symmetry, $U(3)_{R,L}$.

$$\mathcal{L} = \frac{i}{2} \sum_f \bar{\psi}_{f,L} \not{D} \psi_{f,L} + \frac{i}{2} \sum_f \bar{\psi}_{f,R} \not{D} \psi_{f,R}$$
$$- \frac{1}{2} \mathrm{tr}\left[G_{\mu\nu} G^{\mu\nu} \right]$$

Massless fermions conserve helicity, and interaction with gluons conserve helicity.

$$\psi_{f,R} \to R\, \psi_{f,R} \,, \quad \psi_{f,L} \to L\, \psi_{f,L}$$

with $L, R \in U(3)_{L,R}$

Can rewrite this global symmetry as

$$U(3)_L \times U(3)_R = SU(3)_L \times SU(3)_R \times U(1)_V \times U(1)_A$$

which has 18 generators

$$V_\mu^a = R_\mu^a + L_\mu^a = \bar{\psi}_f \gamma_\mu \frac{\lambda^a}{2} \psi_f$$

$$A_\mu^a = R_\mu^a - L_\mu^a = \bar{\psi}_f \gamma_\mu \gamma_5 \frac{\lambda^a}{2} \psi_f$$

$$V_\mu^0 = R_\mu^0 + L_\mu^0 = \bar{\psi}_f \gamma_\mu \psi_f$$ → Baryon number

$$A_\mu^0 = R_\mu^0 - L_\mu^0 = \bar{\psi}_f \gamma_\mu \gamma_5 \psi_f$$ → broken by Anomaly

Conserved charges $Q = \int d^3\vec{x}\, J^0(x)$ commute with H

$[H, Q] = 0$

So, a state $|\psi_p\rangle$ (e.g., proton), $H|\psi_p\rangle = E_p|\psi_p\rangle$

$$\Rightarrow \quad H e^{iQ} |\psi_p\rangle = e^{iQ} H |\psi_p\rangle$$

$$= E_p \big(\underbrace{e^{iQ} |\psi_p\rangle}_{\text{Another state of same mass}} \big)$$

Another state of same mass
⇒ degenerate multiplets

However, there is a problem with axial generator

$e^{i\hat{Q}_5}|\psi_p\rangle$ is a state of opposite parity w.r.t. $|\psi_p\rangle$

but, observe $J^P = \frac{1}{2}^+$ $m_p \simeq 938$ MeV

and $J^P = \frac{1}{2}^-$ $m_{N(1535)} \simeq 1535$ MeV

i.e, there is no parity doubling observed in the hadron spectrum

To resolve this issue, we note that we have implicitly assumed that the ground state / vacuum is invariant under the symmetry $\Rightarrow$ This need not be true

We shall soon learn details about symmetry breaking mechanisms. For QCD, we will find that it has dynamically broken chiral symmetry breaking.

This means that the quark condensate, $\langle 0|\bar{q}q|0\rangle \neq 0$ under $SU(3)_L \times SU(3)_R$. So, a mass term is induced by interactions, and

$$SU(3)_L \times SU(3)_R \underset{\substack{\uparrow \\ \text{Broken symmetry}}}{\rightarrow} SU(3)_V$$

Some Consequences for QCD

- The remaining $SU(3)_V$ is the flavor symmetry
- We will see that there are 8 massless particles associated with this breaking (Goldstone modes)

However, Chiral symmetry not exact since quarks have mass $\Rightarrow$ can still identify the lightest pseudo scalars as those in the spectrum due to the broken axial generators

$$\pi^{\pm}, \pi^{0}, K^{\pm}, \overset{(-)}{K}{}^{0}, \eta$$

Chiral Effective Theory

Chiral dynamics is interesting and we would like to learn more. To do so, we construct an effective theory for QCD where the symmetries are those of QCD, but the degrees of freedom are the light pseudo scalars. The resulting effective theory is valid for low energies, and can be systematically expanded as a perturbation theory in $\mathcal{O}(p^2)$

Let's go through an overview for SU(2) flavor (isospin).

1. Construct a common field for the $\pi^{\pm}, \pi^0$

$$\underset{\text{unitary}}{U} = \exp\left(i\frac{\varphi}{\underset{\text{Dimensionful constant}}{F}}\right), \quad \varphi = \varphi_i \sigma_i$$
$$= \sqrt{2}\begin{pmatrix} \pi^0/\sqrt{2} & \pi^+ \\ \pi^- & -\pi^0/\sqrt{2} \end{pmatrix}$$

2. Specify transformation under chiral group

$$U \to L U R^{\dagger}$$

3. Construct general $\mathcal{L}$ invariant under $U \to L U R^{\dagger}$
4. Organize according to number of derivatives ($\partial_\mu \leftrightarrow p_\mu$)

$$\mathcal{L} = \mathcal{L}^{(0)} + \mathcal{L}^{(2)} + \mathcal{L}^{(4)} + \cdots$$

- For $\mathcal{L}^{(0)}$, find $\mathrm{tr}[UU^{\dagger}] \xrightarrow[U \to LUR^{\dagger}]{} \mathrm{tr}[UU^{\dagger}]$

  but, $UU^{\dagger} = \mathbb{1} \Rightarrow \mathcal{L}^{(0)}$ is irrelevant constant

- For $\mathcal{L}^{(2)}$, find

$$\boxed{\mathcal{L}^{(2)} = \frac{F^2}{4}\,\mathrm{tr}\left(\partial_\mu U \partial^\mu U^{\dagger}\right)}$$

if $\quad U = \exp\left(i \frac{\varphi}{F} \right) = \mathbb{1} + i \frac{\varphi}{F} - \frac{\varphi^2}{2F^2} + \cdots$

then,

$$\mathcal{L}^{(2)} = \frac{1}{2} \partial_\mu \varphi_j \partial^\mu \varphi_j + \cdots \quad \text{interactions}$$

$$= \frac{1}{2} \partial_\mu \pi^0 \partial^\mu \pi^0 + \partial_\mu \pi^+ \partial^\mu \pi^- + \cdots$$

What is F? Consider vector and Axial vector current

$$V_j^\mu = -i \frac{F^2}{4} \, tr\left(\sigma_j \, [\partial^\mu U, U^\dagger] \right)$$

$$A_j^\mu = i \frac{F^2}{4} \, tr\left(\sigma_j \, \{\partial^\mu U, U^\dagger\} \right)$$

$$\Rightarrow \quad A_j^\mu = -F \partial^\mu \varphi_j + \mathcal{O}(\varphi^3)$$

Take $\boxed{\langle 0 | A_j^\mu | \varphi_k(p) \rangle = i p^\mu \delta_{jk} F}$

So, F is pion decay constant, measured for $\pi^- \to \mu^- \bar{\nu}_\mu$

$$\Rightarrow \boxed{F_\pi \simeq 92.2 \text{ MeV}}$$

Can also include explicit mass breaking term

$$\mathcal{L}^{(2)} = \frac{F^2}{4} \mathrm{tr}(\partial_\mu U \partial^\mu U^\dagger) + \frac{F^2 B}{2} \mathrm{tr}(M U^\dagger + M^\dagger U)$$

where $M = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix}$

B is new const
$\Rightarrow$ related to $\langle 0|\bar{q}q|0\rangle$

Chiral Effective theory is limited, but can produce some useful insight, e.g., mass ratios of pseudoscalars, $\pi\pi$-scattering, ...

Consider $\pi\pi \to \pi\pi$ scattering,

$$\mathcal{M}(\pi^j \pi^k \to \pi^l \pi^m) = A(s,t,u)\, \delta_{jk}\delta_{lm} + A(t,u,s)\, \delta_{jl}\delta_{km} + A(u,s,t)\, \delta_{jm}\delta_{kl}$$

Find from $\mathcal{L}^{(2)}$, $\boxed{A(s,t,u) = \frac{s - m_\pi^2}{F_\pi}}$ parameter free!

In terms of definite isospin amplitudes,

$$\mathcal{M}^{I=0} = 3A(s,t,u) + A(t,u,s) + A(u,s,t)$$

$$\mathcal{M}^{I=1} = A(t,u,s) - A(u,s,t)$$

$$\mathcal{M}^{I=2} = A(t,u,s) + A(u,s,t)$$

The S-wave scattering lengths are

$$a_0^I = \frac{1}{32\pi} \mathcal{M}^I (s = 4m_\pi^2, t=0)$$

expt.

$$\Rightarrow \quad a_0^0 \simeq 0.16\, m_\pi^{-1} \qquad\qquad a_0^0 \simeq 0.26(5)\, m_\pi^{-1}$$

$$a_0^2 \simeq -0.045\, m_\pi^{-1} \qquad\qquad a_0^2 \simeq -0.028(12)\, m_\pi^{-1}$$