Fermions

All matter in the SM are fermions. Thus, we will review aspects of spinor field theory, which is the field theory describing spin-½ particles.

Dirac spinor field

A "classical" Dirac spinor $\psi(x)$ is a four-component object, ψ_α, $\alpha = 1, 2, 3, 4$, which satisfies the Dirac eqn.

$$\boxed{(i\partial\!\!\!/ - m)\psi = 0}$$

where we have defined for any operator A_μ

$$A\!\!\!/ \equiv \gamma^\mu A_\mu$$

"Feynman slash notation"

with γ^μ being the Dirac matrices. The Dirac matrices are 4×4 matrices satisfying the anticommutation relation

$$\boxed{\{\gamma^\mu, \gamma^\nu\} = 2 g^{\mu\nu}\, \mathbb{1}}$$

$\{A, B\} = AB + BC$

4x4 identity

We also define $\boxed{\gamma_5 \equiv +i\gamma^0\gamma^1\gamma^2\gamma^3}$

which satisfies

$$(\gamma_5)^2 = \mathbb{1} \quad , \quad \{\gamma_5, \gamma^\mu\} = 0$$

Sometimes, it is convenient to work with a particular basis / representation of the γ-matrices.

We will generally use the Chiral (or Weyl) representation

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \quad , \quad \gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}$$

$$\gamma_5 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}$$

where $\mathbb{1} = 2\times 2$ identity matrix

σ^j = Pauli matrices

Recall Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

with

$$\{\sigma^j, \sigma^k\} = 2\delta_{jk}\mathbb{1}$$

$$[\sigma^j, \sigma^k] = 2i\epsilon^{jkl}\sigma^l$$

Chirality

γ_5 is an interesting matrix

$$(\gamma_5)^2 = \mathbb{1} \Rightarrow \gamma_5 \text{ is diagonalizable}$$
$$\Rightarrow \text{eigenvalues are } \pm 1$$

Chirality

A Dirac fermion ψ has a definite chirality if it is either a right-handed or left-handed fermion.

- A fermion is right-handed if $\gamma_5 \psi = +\psi$
- A fermion is left-handed if $\gamma_5 \psi = -\psi$

Any Dirac fermion can be written as

$$\psi = \psi_L + \psi_R$$

We can project any Dirac fermion into a definite chirality using projection operators

$$P_R = \frac{1}{2}(\mathbb{1} + \gamma_5)$$

$$P_L = \frac{1}{2}(\mathbb{1} - \gamma_5)$$

One can check the following properties (exercise)

$$\gamma_5 P_L = -P_L \quad , \quad \gamma_5 P_R = P_R$$

$$(P_{L,R})^2 = P_{L,R} \qquad \text{(idempotent)}$$

$$P_L P_R = P_R P_L = 0$$

$$P_L + P_R = \mathbb{1}$$

Therefore,
$$\psi = \mathbb{1}\psi = (P_L + P_R)\psi$$
$$= P_L\psi + P_R\psi$$
$$= \psi_L + \psi_R$$

we define

$$\boxed{\psi_L = P_L \psi \quad , \quad \psi_R = P_R \psi}$$

In the chiral representation, the projectors are very simple,

$$P_L = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix} \quad , \quad P_R = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix}$$

If ψ and φ are Dirac fermions, then

$$\bar{\psi}_L \varphi_L = \bar{\psi}_R \varphi_R = 0 \qquad (1)$$

$$\bar{\psi}_L \gamma^\mu \varphi_R = \bar{\psi}_R \gamma^\mu \varphi_L = 0 \qquad (2)$$

Proof (1)

Recall the Dirac conjugate

$$\bar{\psi} = \psi^\dagger \gamma^0$$

Then, focusing on the left-hand case

$$\begin{aligned}
\bar{\psi}_L \varphi_L &= \psi_L^\dagger \gamma^0 \varphi_L \\
&= (P_L \psi)^\dagger \gamma^0 (P_L \varphi) \\
&= \psi^\dagger P_L \gamma^0 P_L \varphi \\
&= \psi^\dagger P_L P_R \gamma^0 \varphi \\
&= 0
\end{aligned}$$

$\gamma_5^\dagger = \gamma_5$

$\{\gamma_5, \gamma^0\} = 0$

$\blacksquare$

Proof (2)

$$\begin{aligned}
\bar{\psi}_L \gamma^\mu \varphi_R &= \psi_L^\dagger \gamma^0 \gamma^\mu \varphi_R \\
&= (P_L \psi)^\dagger \gamma^0 \gamma^\mu (P_R \varphi) \\
&= \psi^\dagger P_L \gamma^0 \gamma^\mu P_R \varphi \\
&= \psi^\dagger P_L \gamma^0 P_L \gamma^\mu \varphi \\
&= \psi^\dagger P_L P_R \gamma^0 \gamma^\mu \varphi \\
&= 0
\end{aligned}$$

$\{\gamma_5, \gamma^\mu\} = 0$

$\blacksquare$

Are chiral fermions Dirac spinors?

- Not necessarily

Notice that the anticommutation relations give

$$\not\partial \gamma_5 \psi = - \gamma_5 \not\partial \psi$$

Therefore, γ_5 does not commute with the Dirac operator $(i\not\partial - m)$

$$[i\not\partial - m, \gamma_5] = i\not\partial \gamma_5 - i\gamma_5 \not\partial$$
$$= - i\gamma_5 \not\partial \neq 0$$

$\Rightarrow$ ψ_L and ψ_R do not in general satisfy the Dirac equation.

However, if a spinor is <u>massless</u>, $m=0$, then the Dirac eqn. is

$$\boxed{i\not\partial \psi = 0}$$

Massless Dirac eqn.

If $m=0$, then we also have

$$\not\partial \gamma_5 \psi = -\gamma_5 \not\partial \psi = 0$$

$\Rightarrow$ $\gamma_5 \psi$ is a Dirac spinor

$\boxed{\text{If } \psi \text{ is a massless Dirac spinor, then so are } \psi_L,\ \psi_R.}$

To consider this further, consider the Lagrangian density for $m \neq 0$ Dirac spinor

$$\mathcal{L} = \frac{1}{2} i \bar{\psi} \overleftrightarrow{\not{\partial}} \psi - m \bar{\psi}\psi$$

$$= \underbrace{\frac{1}{2} i \bar{\psi} \not{\partial} \psi}_{\text{kinetic term}} + \underbrace{h.c.}_{\text{hermitian conjugate}} - \underbrace{m\bar{\psi}\psi}_{\text{mass term}}$$

where $A \overleftrightarrow{\partial_\mu} B = A \partial_\mu B - (\partial_\mu A) B$

Note - kinetic term

$$\frac{1}{2} i \bar{\psi} \not{\partial} \psi + h.c. = \frac{1}{2} i \bar{\psi} \not{\partial} \psi - \frac{1}{2} i (\bar{\psi} \not{\partial} \psi)^\dagger$$

Now, $(\bar{\psi} \not{\partial} \psi)^\dagger = (\bar{\psi} \gamma^\mu \partial_\mu \psi)^\dagger = \partial_\mu \psi^\dagger \gamma^{\mu\dagger} \bar{\psi}^\dagger$

Recall that

$$\gamma^{\mu\dagger} = \gamma^0 \gamma^\mu \gamma^0$$

and $\bar{\psi}^\dagger = (\psi^\dagger \gamma^0)^\dagger = \gamma^{0\dagger} \psi = \gamma^0 \psi$ since $\gamma^{0\dagger} = \gamma^0$

$$\Rightarrow (\bar{\psi} \not{\partial} \psi)^\dagger = \underbrace{\partial_\mu \psi^\dagger \gamma^0}_{= \partial_\mu \bar{\psi}} \gamma^\mu \underbrace{\gamma^0 \gamma^0}_{= \mathbb{1}} \psi$$

so, $(\bar{\psi}\not{\partial}\psi)^{+} = \partial_\mu \bar{\psi}\gamma^\mu \psi$

$= \bar{\psi}\overleftarrow{\not{\partial}}\psi$

$\Rightarrow \quad \frac{1}{2} i \bar{\psi}\not{\partial}\psi + \text{h.c.} = \frac{1}{2} i \bar{\psi}\overleftrightarrow{\not{\partial}}\psi$

Why this Lagrange density v.s.

$$\mathcal{L}' = i\bar{\psi}\not{\partial}\psi - m\bar{\psi}\psi$$

i.e., Derivative only on ψ ?

Note that

$$\int d^4x \, (i\bar{\psi}\not{\partial}\psi) = \int d^4x \left(\frac{1}{2} i \bar{\psi}\not{\partial}\psi - \frac{1}{2} i \bar{\psi}\overleftarrow{\not{\partial}}\psi \right)$$

+ surface terms

so, Eqns. of motion for $\mathcal{L}$ and $\mathcal{L}'$ are identical.

But, $\mathcal{L}'^{*} \neq \mathcal{L}'$ $\Rightarrow$ Cannot create Hermitian Hamiltonian from $\mathcal{L}'$!

So,

$$\mathcal{L} = \frac{1}{2} i \bar{\psi} \overleftrightarrow{\partial\!\!\!/} \psi - m \bar{\psi} \psi$$

with $\psi = \psi_L + \psi_R$, then (Recall $\bar{\psi}_L \gamma^\mu \psi_R = \bar{\psi}_R \gamma^\mu \psi_L = 0$)

$$\mathcal{L} = \frac{1}{2} i \bar{\psi}_L \overleftrightarrow{\partial\!\!\!/} \psi_L + \frac{1}{2} i \bar{\psi}_R \overleftrightarrow{\partial\!\!\!/} \psi_R$$

$$- m (\bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L)$$

Kinetic term Diagonal in Chiral fields

Mass term mixes Chiral fields

So, if spinor field is massless, ψ_L, ψ_R are separate objects that do not "talk" to each other.

This is Important for the SM!

We have observed that the weak interaction only couples to left-handed fermions. What we will show is that this forces the fermion fields to be massless!

However, fermions such as the electron are NOT massless. Therefore, the fermion masses in the SM cannot come from a direct mass term in the Lagrangian, it must come from some other mechanism — the Higgs mechanism

We can further illustrate how the mass term makes a huge difference. Notice that <u>if</u> $m=0$, then the Lagrange density is invariant under a set of transformations

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \longrightarrow \begin{pmatrix} e^{i\alpha_L}\psi_L \\ e^{i\alpha_R}\psi_R \end{pmatrix}$$

and

$$\bar{\psi} = \begin{pmatrix} \bar{\psi}_L \\ \bar{\psi}_R \end{pmatrix} \longrightarrow \begin{pmatrix} e^{-i\alpha_L}\bar{\psi}_L \\ e^{-i\alpha_R}\bar{\psi}_R \end{pmatrix}$$

where $\alpha_L, \alpha_R \in \mathbb{R}$ constants

Note

> This transformation is a global symmetry called
> $$U(1)_L \times U(1)_R$$
> where $e^{i\alpha_L} \in U(1)_L$, $e^{i\alpha_R} \in U(1)_R$

If $m=0$,

$$\mathcal{L} = \frac{1}{2} i \bar{\psi}_L \overleftrightarrow{\not{\partial}} \psi_L + \frac{1}{2} i \bar{\psi}_R \overleftrightarrow{\not{\partial}} \psi_R$$

Then, under $\psi_L \to e^{i\alpha_L}\psi_L$, $\psi_R \to e^{i\alpha_R}\psi_R$

we find

$$\mathcal{L} = \frac{1}{2} i \bar{\psi}_L \overleftrightarrow{\partial\!\!\!/} \psi_L + \frac{1}{2} i \bar{\psi}_R \overleftrightarrow{\partial\!\!\!/} \psi_R$$

$$\longrightarrow \frac{1}{2} i e^{-i\alpha_L} \bar{\psi}_L \overleftrightarrow{\partial\!\!\!/} \psi_L e^{i\alpha_L}$$

$$+ \frac{1}{2} i e^{-i\alpha_R} \bar{\psi}_R \overleftrightarrow{\partial\!\!\!/} \psi_R e^{i\alpha_R}$$

$$= \frac{1}{2} i \bar{\psi}_L \overleftrightarrow{\partial\!\!\!/} \psi_L + \frac{1}{2} i \bar{\psi}_R \overleftrightarrow{\partial\!\!\!/} \psi_R$$

$$= \mathcal{L}$$

However, if $m \neq 0$, notice the mass term

$$\bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L \to e^{-i\alpha_L} e^{i\alpha_R} \bar{\psi}_L \psi_R$$

$$+ e^{-i\alpha_R} e^{i\alpha_L} \bar{\psi}_R \psi_L$$

$$= e^{i(\alpha_R - \alpha_L)} \bar{\psi}_L \psi_R + e^{-i(\alpha_R - \alpha_L)} \bar{\psi}_R \psi_L$$

Not Invariant!

$$\neq \bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L$$

The only way for $\mathcal{L}$ to remain invariant is if $\alpha_L = \alpha_R \equiv \alpha$.

$\Rightarrow$ The symmetry reduces to a single $U(1)$ symmetry!

Helicity

Another important difference the mass term makes is in the notion of helicity. This is a quantum effect, so let's review some aspects of the quantum theory of the Dirac field.

A quantum Dirac field can be expanded into definite momentum modes

$$\psi(x) = \sum_s \int \frac{d^3\vec{p}}{(2\pi)^3 2E_{\vec{p}}} \left[u_s(p)\, b_s(p)\, e^{-ip\cdot x} + v_s(p)\, d_s^\dagger(p)\, e^{ip\cdot x} \right]$$

spin, $s = \pm\frac{1}{2}$

$E_{\vec{p}} = \sqrt{m^2 + \vec{p}^{\,2}}$ is on-shell Energy (positive)

forward propagating

backward propagating

$b^\dagger, d^\dagger$ are creation operators for positive, negative frequency particles, respectively.

A single particle state

$$|p,s\rangle = b_s^\dagger(p)\,|0\rangle \quad \text{Vacuum}$$

with relativistic normalization

$$\langle p', s' | p, s\rangle = \delta_{s's}\, (2\pi)^3\, 2E_{\vec{p}}\, \delta^{(3)}(\vec{p}\,' - \vec{p})$$

The wave function

$$\langle 0|\psi(x)|p,s\rangle = \sum_{s'}\int\frac{d^3\vec{p}'}{(2\pi)^3 2E_{\vec{p}'}}\, u_{s'}(p')\, e^{-ip'\cdot x}\langle 0|b_{s'}(p')|p,s\rangle$$

Now, $\underbrace{\langle 0|b_{s'}(p')}_{=\langle p',s'|}|p,s\rangle = \langle p',s'|p,s\rangle = (2\pi)^3 2E_{\vec{p}}\,\delta^{(3)}(\vec{p}'-\vec{p})\,\delta_{s's}$

$$\Rightarrow \langle 0|\psi(x)|p,s\rangle = u_s(p)\,e^{-ip\cdot x}$$

$u_s(p)$ and $v_s(p)$ form a basis of the solution space to the (classical) Dirac eqn. in momentum space.

i.e.,

$$u_s(p)\,e^{-ip\cdot x}\ ,\quad v_s(p)\,e^{ip\cdot x}$$

are solutions to the Dirac eqn. for any p, s.

e.g.,

$$0 = (i\partial\!\!\!/ - m)\,u_s(p)\,e^{-ip\cdot x}$$
$$= (p\!\!\!/ - m)\,u_s(p)\,e^{-ip\cdot x}$$

$$\Rightarrow (p\!\!\!/ - m)\,u_s(p) = 0$$

Choose a basis for γ^μ to find solution for $u_s(p)$.

In the chiral representation,

$$u_s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi_s \\ \sqrt{p\cdot\bar{\sigma}}\,\xi_s \end{pmatrix}, \quad v_s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\eta_s \\ -\sqrt{p\cdot\bar{\sigma}}\,\eta_s \end{pmatrix}$$

where

$$\sigma^\mu = (\mathbb{1}, \vec{\sigma}), \quad \bar{\sigma}^\mu = (\mathbb{1}, -\vec{\sigma})$$

and $\xi_{\pm\frac{1}{2}}$, $\eta_{\pm\frac{1}{2}}$ are two component spinors.

We can define a quantum operator which corresponds to chirality, known as <u>helicity</u>

Helicity

The helicity is the projection of the angular momentum onto the direction of the momentum

$$h = \vec{J}\cdot\hat{p} = \vec{S}\cdot\hat{p}$$

where

$$\vec{J} = -i\vec{r}\times\vec{\nabla} + \vec{S}$$

is the total angular momentum and $\vec{S}$ is the spin operator,

$$S_i = \frac{i}{4}\epsilon_{ijk}\gamma^k\gamma^l = \frac{1}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}$$

Claim

For a massless spinor, helicity = chirality

i.e., if u is a massless spinor, then

$$\boxed{h\, u(p) = \frac{\gamma_5}{2}\, u(p)}$$

Proof

For a massless particle, $\not{p}\, u_s = 0$.

So, $\gamma^\mu p_\mu u_s = (\gamma^0 p^0 - \vec{\gamma}\cdot\vec{p})\, u_s = 0$

Now, multiply by $\gamma_5 \dfrac{\gamma^0}{p^0}$ using $(\gamma^0)^2 = \mathbb{1}$

$$\Rightarrow \gamma_5 u_s = \gamma_5 \gamma^0 \gamma^i \frac{p^i}{p^0}\, u_s$$

Since the particle is massless, $(p^0)^2 - \vec{p}\cdot\vec{p} = 0$

$$\Rightarrow \hat{p} = \frac{\vec{p}}{p^0}$$

Also, by direct computation (in Chiral rep.)

$$\gamma_5\gamma^0\gamma^i = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}\begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} = \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}$$

$$= 2 S^i$$

So, it follows that $\gamma_5 u_s = 2h\, u_s(p)$ $\blacksquare$

In particular,

$$h u_{L/R} = \frac{\gamma_5}{2} u_{L/R} = \mp \frac{1}{2} u_{L/R}$$

where $u_{L/R}$ has helicity $\mp \frac{1}{2}$.

Here we make another crucial observation. Helicity is the spin in the direction of momentum, & spin is what has to be conserved. On the other hand, chirality is what determines whether the particle interacts with the weak force.

For massless particles, these notions coincide. Thus the spin of the particles participating in the weak interactions is constrained by the fact that the weak interaction only couples to the left-handed particles.

Consequently, spin conservation will forbid certain interactions from happening. Why spin conservation? Recall that for a massless particle, there does NOT exist a rest frame.

Therefore, there is <u>NO</u> Lorentz transformation that flips helicity

as $h = \vec{S} \cdot \hat{p}$ and $\hat{p} \neq 0$ for $m = 0$.

However, once particles have mass, helicity is different from spin. Generally, helicity is closely related to spin, especially when mass is small. So, interactions that were previously forbidden, are now possible, albeit suppressed.

Functional Quantization of Spinor fields

To use PIs for fermions, we need an extension of c-numbers to a new class of anticommuting numbers called Grassmann-variables, or G-numbers. If θ_i, θ_j are G-numbers, they obey

$$\theta_i \theta_j = -\theta_j \theta_i$$

It follows that $\theta_i^2 = 0$!

So, all functions of θ are at most linear

$$f(\theta) = a + b\theta$$

Example

$$e^{a\theta} = 1 + a\theta = \frac{1}{1 - a\theta}$$

Integration over G-numbers is defined such that translational invariance of ordinary c-numbers,

$$\int_{-\infty}^{\infty} dx \, f(x) = \int_{-\infty}^{\infty} dx \, f(x+c)$$

carries over

$$\int d\theta \, \varphi(\theta) \equiv \int d\theta \, \varphi(\theta + \xi)$$

Since $\varphi(\theta) = a + b\theta$ in general,

$$a\int d\theta + b\int d\theta\,\theta = \int d\theta\,[a + b\xi] + b\int d\theta\,\theta$$

only 2 indep. integrals

Since this must hold for all φ $(\forall a, b)$

$$\Rightarrow \int d\theta = 0$$

and furthermore defines $\int d\theta\,\theta = 1$ such that

$$\int d\theta\,(A + B\theta) = B.$$

It then follows that G-integral is the same as G-derivatives

$$\int d\theta\,\varphi(\theta) = \frac{\partial}{\partial\theta}\varphi(\theta)$$

Moreover,

anticommute $\Rightarrow -\eta\theta$

$$\int d\theta \int d\eta\;\eta\theta = +1 \quad ; \quad \int d\theta \int d\eta\;\theta\eta = -1$$

and $\theta = \dfrac{\theta_1 + i\theta_2}{\sqrt{2}} \Rightarrow \theta^* = \dfrac{\theta_1 - i\theta_2}{\sqrt{2}}$

so, $\int d\theta\, d\theta^*\, \theta^*\theta = 1 \Rightarrow \int d\theta\, d\theta^*\, e^{\theta^* b\theta} = \int d\theta\, d\theta^*\,(1 + \theta^* b\theta)$

$$= +b\int d\theta\, d\theta^*\, \theta^*\theta = +b \quad \text{, etc.}$$

Multidimensional Gaussian G-Integrals

Can easily extend above to ∞-number of variables,

$$I_F(A) = \int \prod_{i=1}^{N} d\theta_i d\bar{\theta}_i \; e^{\sum_{i,j} \bar{\theta}_i A_{ij} \theta_j}$$

$$= \int \prod_{i=1}^{N} d\theta_i d\bar{\theta}_i \; \frac{1}{N!} \left(\sum_{j,k} \bar{\theta}_j A_{jk} \theta_k \right)^N$$

$$= \int \prod_{i=1}^{N} d\theta_i d\bar{\theta}_i \; \frac{1}{N!} \left(\sum_{j=1}^{N} \bar{\theta}_j \xi_j \right)^N \qquad \xi_j = \sum_{k_j=1}^{N} A_{jk_j} \theta_{k_j}$$

$$= \int \prod_{i=1}^{N} d\theta_i d\bar{\theta}_i \; (\bar{\theta}_1 \xi_1 \cdots \bar{\theta}_N \xi_N)$$

$$= \int \prod_{i=1}^{N} d\theta_i d\bar{\theta}_i \; \prod_{j=1}^{N} \bar{\theta}_j \left(\sum_{k_j=1}^{N} A_{jk_j} \theta_{k_j} \right)$$

$$= \int \prod_{i=1}^{N} d\theta_i d\bar{\theta}_i \; \sum_{\text{perm}} A_{1k_1} A_{2k_2} \cdots A_{Nk_N} \bar{\theta}_1 \theta_{k_1} \underbrace{\bar{\theta}_2 \theta_{k_2}}_{\text{antisymm.}} \cdots \bar{\theta}_N \theta_{k_N}$$

$$= \epsilon_{k_1 \cdots k_N} A_{1k_1} A_{2k_2} \cdots A_{Nk_N} \underbrace{\int \prod_{i=1}^{N} d\theta_i d\bar{\theta}_i \bar{\theta}_i \theta_i}_{=1}$$

$$= \underline{\det A}$$

Compare to bosonic (c-number) Gaussian integral

$$I_B(A) = \int \prod_{i=1}^{N} dx_i \, e^{-x_i A_{ij} x_j} = (2\pi)^{N/2} \frac{1}{\sqrt{\det A}}$$

Generating function for fermions

Using the previous results and taking the continuum limit by $\theta_i \to \psi(x)$, where $\psi(x)$ is a Grassmann valued field.

$$\boxed{Z_D[\eta,\bar\eta] = \frac{1}{N_D}\int \mathcal{D}\psi\,\mathcal{D}\bar\psi\; e^{i\int d^4x\left(\underbrace{\frac{i}{2}\bar\psi\overleftrightarrow{\partial\!\!\!/}\psi - m\bar\psi\psi}_{=\int_{x,y}\bar\psi_x S_F^{-1}(x-y)\psi_y} + \bar\eta\psi + \bar\psi\eta\right)}}$$

Source terms (G-numbers)

This is the generating functional for the free Dirac theory. The action is quadratic, so we can integrate the functional. Shift the fields,

$$\psi_x \to \psi_x - \int_y i S_F(x-y)\,\eta(y) \equiv \psi_x'$$

and integrate over ψ_x' to find

$$\boxed{Z_D[\eta,\bar\eta] = e^{i\iint d^4x\, d^4y\; \bar\eta(x)\, i S_F(x-y)\,\eta(y)}}$$

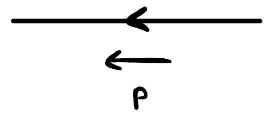
Where $\quad iS_F^{-1}(x-y) \equiv (i\partial\!\!\!/ - m)\,\delta^{(4)}(x-y)$

$$\Rightarrow \quad iS_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot(x-y)}\,\frac{i}{p\!\!\!/ - m + i\epsilon}$$

In momentum space,

$$i S_F(p) = \frac{i}{\not{p} - m + i\epsilon} = \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon}$$

$$=$$



Yukawa theory

We can construct a perturbative theory by considering the generating functions for both scalars and spinors.

if

$$\mathcal{L}_{int.} = -g\,\varphi(x)\,\bar{\psi}(x)\,\psi(x)$$

So,

$$\mathcal{L}_{Yukawa} = \mathcal{L}_{KG} + \mathcal{L}_{D} + \mathcal{L}_{int.}$$

then,

$$Z_{Yukawa}[J, \eta, \bar{\eta}] = \int \mathcal{D}\varphi\, \mathcal{D}\psi\, \mathcal{D}\bar{\psi}\; e^{i\int(\mathcal{L}_{Yukawa} + J\varphi + \bar{\eta}\psi + \bar{\psi}\eta)}$$

$$= e^{-ig\int d^4x \frac{\delta}{\delta J_x}\frac{\delta}{\delta \eta_x}\frac{\delta}{\delta \bar{\eta}_x}}\; Z_{KG}[J]\, Z_D[\eta, \bar{\eta}]$$

Vertex is given by

$$i\Gamma = -ig\,\mathbb{1} \quad =$$