Aspects of QFT

We begin by discussing some familiar concepts from QFT I. In particular, chiral and gauge symmetries of fermion fields, and the theoretical formulation of Quantum Electrodynamics (QED). This will be a bit of a review, which will serve to set our signs and notational conventions.

Natural Units

We will work almost exclusively with natural units, where

$$\boxed{\hbar = c = 1}$$

In this system,

$$[\text{Length}] = [\text{time}] = [\text{energy}]^{-1} = [\text{mass}]^{-1}$$

The mass, m, of a particle is equal to its rest energy, mc^2, and also its inverse Compton wavelength, $mc/\hbar$.

For example,

$$m_{\text{electron}} = 9.109 \times 10^{-28}\ g$$
$$= 0.511\ \text{MeV}$$
$$= (3.862 \times 10^{-11}\ \text{cm})^{-1}$$

useful conversion

$$\boxed{\begin{aligned} 1 = \hbar c &= 197.3\ \text{MeV} \cdot \text{fm} \\ &\simeq 200\ \text{MeV} \cdot \text{fm} \end{aligned}}$$

Relativity

We follow the conventions of Jackson and Peskin & Schroeder.

The metric tensor $g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} +1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$

Where Greek indices

$$\mu, \nu = 0, 1, 2, 3$$
$$= t, x, y, z$$

and Roman indices

$$i, j = 1, 2, 3$$

Four vectors are $x^\mu = (x^0, \vec{x})$ contravariant

$x_\mu = g_{\mu\nu} x^\nu = (x^0, -\vec{x})$ covariant

Einstein summation convention $x_\mu = \sum_{\nu=0}^{3} g_{\mu\nu} x^\nu$

Scalar products

$$p \cdot x = g_{\mu\nu} p^\mu x^\nu$$
$$= p^0 x^0 - \vec{p} \cdot \vec{x}$$

The relativistic dispersion relation is

$$\boxed{p^2 = p^\mu p_\mu = E^2 - \vec{p}^{\,2} = m^2 \geq 0}$$

Derivatives are covariant, $\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial x^0}, \vec{\nabla}\right)$

notice

The Levi-Civita tensor, $\epsilon^{\mu\nu\rho\sigma}$ totally antisymmetric

$$\epsilon^{0123} = +1 \Rightarrow \begin{cases} \epsilon_{0123} = -1 \\ \epsilon^{1230} = -1 \end{cases}$$

Fourier transforms

Recall the n-dimensional Dirac delta function: $\delta^{(n)}(x)$

which satisfies, $\int d^n x \, \delta^{(n)}(x) = 1$

The Fourier transform pairs are

$k \cdot x = k^0 x^0 - \vec{k} \cdot \vec{x}$

$$\begin{cases} f(x) = \int \frac{d^4 k}{(2\pi)^4} \, e^{-ik\cdot x} \, \tilde{f}(k) \\ \tilde{f}(k) = \int d^4 x \, e^{ik\cdot x} \, f(x) \end{cases}$$

so that $\int d^4 x \, e^{ik\cdot x} = (2\pi)^4 \delta^{(4)}(k)$

Scalar Fields

Consider a real scalar field $\varphi(x)$ in 4D spacetime

Lagrange density

$$\boxed{\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 - V(\varphi)}$$

kinetic term   mass term   interactions (e.g., $\frac{1}{4}\lambda\varphi^4, \ldots$)

The equations of motion for a real scalar field can be given by the Euler-Lagrange eqns.

$$\boxed{\partial_\mu\left(\frac{\delta\mathcal{L}}{\delta\partial_\mu\varphi}\right) - \frac{\delta\mathcal{L}}{\delta\varphi} = 0}$$

which gives

$$\underbrace{(\Box + m^2)}_{\text{Klein-Gordon operator}}\varphi = -\frac{\partial V}{\partial\varphi}$$

Note: $\Box \equiv \partial_\mu\partial^\mu$

The action is given by $\boxed{S = \int d^4x\, \mathcal{L}}$

Functional Quantization of Scalar Field

There are two primary formalisms to construct quantum theories of fields. One is canonical quantization, which was discussed in QFT I. The other is known as functional quantization or Path Integral quantization. The Path Integral (PI) has many advantages over canonical quantization:

(1) Simple covariant way to quantize gauge theories

(2) Suitable for non-perturbative methods

(3) Based on commuting numbers (c-numbers) instead of operators.

Details on PI will be given in QFT II. Here, we will summarize main results. We will assume you are familiar with a cursory intro to PI in NRQM.

The vacuum-to-vacuum matrix element is defined as

$$\langle 0 | 0 \rangle = \lim_{T \to \infty(1-i\epsilon)} \int \mathcal{D}\varphi \, e^{iS[\varphi]}$$

where,

- $S[\varphi]$ is the action → weight for integral
- The $T \to \infty(1-i\epsilon)$ ensures convergence

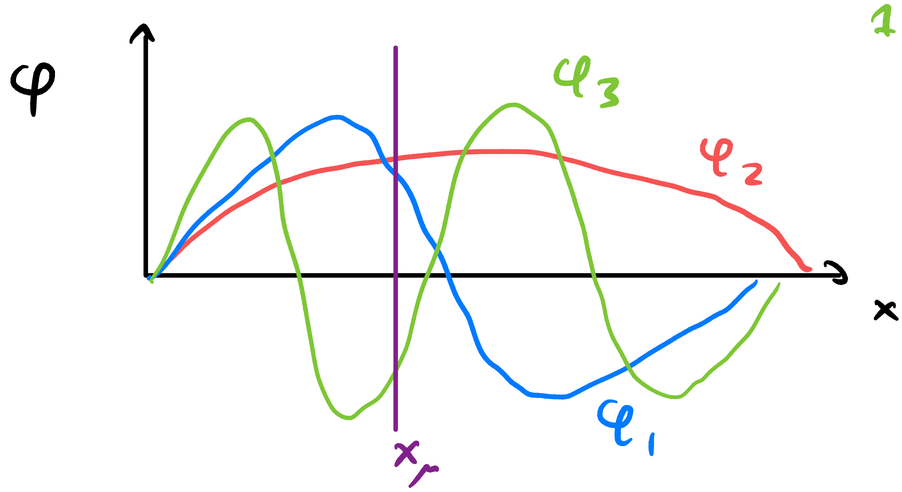
  (often we will drop the limit)

- $\mathcal{D}\varphi$ is defined on a lattice

$$\mathcal{D}\varphi \equiv \lim_{N \to \infty} \prod_{x_\mu} d\varphi(x_\mu)$$

Essentially, we are "summing" over field configurations at some spacetime point x_μ

1-D example



$$\int d\varphi(x) \simeq \varphi_1(x) + \varphi_2(x) + \varphi_3(x) + \cdots$$

Generating Functions

The central objects of all QFTs are time-ordered correlation functions, or Green's functions. The P.I. allows us to easily compute Green's functions using generating functions.

Define the generating function $Z[J]$ as

$$\boxed{Z[J] = \frac{1}{N}\int \mathcal{D}\varphi\, e^{i\int d^4x\,(\mathcal{L} + J\varphi)}}$$

Normalization such that $Z[0] = 1$ $\Rightarrow \langle 0|0\rangle \to 1$

Source term

Correlation functions are then given by

$$\langle 0|T\{\mathcal{O}[\hat{\varphi}]\}|0\rangle = \frac{1}{N}\int \mathcal{D}\varphi\, \mathcal{O}[\varphi]\, e^{iS[\varphi]}$$

operators

c-numbers

$\hookrightarrow$ $\mathcal{O}[\hat{\varphi}]$ generic operator

e.g., $\mathcal{O}[\hat{\varphi}] = \varphi(x_1)\cdots\varphi(x_n)$

So, the generating functional gives

$$\boxed{\begin{aligned} G_n(x_1, \dots, x_n) &= \frac{1}{\mathcal{N}} \int \mathcal{D}\varphi \, \varphi(x_1) \cdots \varphi(x_n) \, e^{iS[\varphi]} \\ &= \frac{(-i)^n \delta^n}{\delta J(x_1) \cdots \delta J(x_n)} Z[J] \Big|_{J=0} \end{aligned}}$$

Functional Derivatives

Definition : $$\boxed{\frac{\delta}{\delta J(x)} J(y) = \delta^{(4)}(x-y)}$$

Example

$$\frac{\delta}{\delta J(x)} \int d^4y \, J(y) \varphi(y) = \varphi(x)$$

Example

$$\begin{aligned} &\frac{\delta}{\delta J(x)} \exp\left[i \int d^4y \, J(y) \varphi(y) \right] \\ &= \exp\left[i \int d^4y \, J(y) \varphi(y) \right] \frac{\delta}{\delta J(x)} i \int d^4y \, J(y) \varphi(y) \\ &= i \varphi(x) \exp\left[i \int d^4y \, J(y) \varphi(y) \right] \end{aligned}$$

Example

$$\frac{\delta}{\delta J(x)} \int d^4y \, \partial_\mu J(y) \, V^\mu(y)$$

$$= \frac{\delta}{\delta J(x)} \int d^4y \, J(y) \left(-\partial_\mu V^\mu(y)\right) + \frac{\delta}{\delta J(x)} \left(\text{Boundary terms}\right)$$

$$= -\partial_\mu V^\mu(x)$$

gives zero except for topologically interesting theories.

Free Klein-Gordon theory

The generating function can be evaluated in closed form for the free KG theory.

$$Z_{KG}[J] = \frac{1}{N_{KG}} \int \mathcal{D}\varphi \, e^{i\int d^4x \left(\frac{1}{2} (\partial_\mu \varphi)^2 - \frac{m^2}{2} \varphi^2 + J_x \varphi_x \right)}$$

$N_{KG} = Z[0]$

$(\partial_\mu\varphi)^2 = \cancel{\partial_\mu(\varphi \partial^\mu \varphi)} - \varphi \Box \varphi$

surface term

$$= \frac{1}{N_{KG}} \int \mathcal{D}\varphi \, e^{\frac{i}{2} \int_{x,y} \varphi_x \, iD^{-1}_{y,x} \varphi_y + i \int_x J_x \varphi_x}$$

where $\int_x \equiv \int d^4x$ and $\int_y iD^{-1}_{y,x} \varphi_y \equiv -(\partial_x^2 + m^2) \varphi_x$

Now,

$$\int_y iD^{-1}_{y,x}\,\varphi_y = -(\partial_x^2+m^2)\varphi_x$$

$$\Rightarrow \quad iD^{-1}_{y,x} = -(\partial_x^2+m^2)\,\delta^{(4)}(y-x) \qquad (y-x)_0 \to T(1-i\epsilon)$$

$$= -(\partial_x^2+m^2)\int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot(x-y)}$$

$$= \int \frac{d^4p}{(2\pi)^4}\,(p^2-m^2+i\epsilon)\, e^{-ip\cdot(y-x)}$$

Then, we get Feynman propagator

$$\boxed{D_{y,x} = \int \frac{d^4p}{(2\pi)^4}\,\frac{i}{p^2-m^2+i\epsilon}\, e^{-ip\cdot(y-x)} \equiv i\Delta(y-x)}$$

Such that $\int d^4y\, D_{x,y}\, D^{-1}_{y,z} = \delta^{(4)}(x-z)$

The integrals are now Gaussians

Recall

$$\int \prod_{i=1}^{N} dx_i\, e^{-x_i A_{ij} x_j} = (2\pi)^{N/2}\frac{1}{\sqrt{\det A}}$$

Let us complete the square

$$\frac{i}{2}\int_{y,x} \varphi_y \,(iD^{-1}_{y,x})\,\varphi_x + i\int_x J_x\, \varphi_x$$

$$= \frac{i}{2}\int_{y,x}\left(\varphi_y - \int_z J_z\, iD_{zy}\right)\left(iD^{-1}_{y,x}\right)\overbrace{\left(\varphi_x - \int_{z'} iD_{x,z'}\, J_{z'}\right)}^{\equiv\, \varphi'_x}$$

$$+ \frac{i}{2}\int_{y,x} J_y\, iD_{yx}\, J_x$$

$$= \frac{i}{2}\int_{y,x}\varphi'_y\, iD^{-1}_{yx}\,\varphi'_x + \frac{i}{2}\int_{y,x} J_y\, iD_{y,x}\, J_x$$

So, integrate over φ', find

$$\boxed{Z_{KG}[J] = e^{\frac{i}{2}\int d^4x \int d^4y\, J(x)\, i\Delta(x-y)\, J(y)}}$$

Simple application,

$$G_2(x_1,x_2) = \langle 0|T\{\varphi(x_1)\varphi(x_2)\}|0\rangle$$

$$= \frac{1}{N_{KG}}\int \mathcal{D}\varphi\; \varphi(x_1)\varphi(x_2)\, e^{i\mathcal{L}_{KG}}$$

$$= \frac{(-i)^2\,\delta^2}{\delta J(x_1)\,\delta J(x_2)} Z_{KG}[J]\Big|_{J=0}$$

$$= i\Delta(x-y)$$

Interacting φ^4-theory — Perturbation Theory

Consider $\lambda\varphi^4$ theory,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\varphi)^2 - \frac{1}{2}m^2\varphi^2 - \frac{\overset{\text{coupling}}{\lambda}}{4!}\varphi^4$$

$$= \mathcal{L}_{KG} + \mathcal{L}_{int}$$

where
$$\mathcal{L}_{int} = -\frac{\lambda}{4!}\varphi^4$$

In general, can't integrate $Z[J]$. But, if λ is small, can generate perturbation series

$$Z[J] = \frac{1}{N_{int}}\int \mathcal{D}\varphi\, e^{i\int d^4x\,(\mathcal{L} + J\varphi)}$$

$$= \frac{1}{N_{int}}\int \mathcal{D}\varphi\, \underbrace{e^{i\int d^4x\,\mathcal{L}_{int}}}_{=\sum_{n=0}^{\infty}\frac{1}{n!}\left(i\int d^4x\,\mathcal{L}_{int}\right)^n}\, e^{i\int d^4x(\mathcal{L}_{KG} + J\varphi)}$$

Note:
$$N_{int} = Z[0]$$

$$= N_{KG}\langle 0|T\{e^{i\int d^4x\,\mathcal{L}_{int}}\}|0\rangle$$ - vacuum-to-vacuum diagrams

So,

$$Z[J] = \frac{1}{N_{int}} \int \mathcal{D}\varphi \, e^{i\int d^4x \, \mathcal{L}_{int}} \underbrace{e^{i\int d^4x (\mathcal{L}_{KG} + J\varphi)}}_{= N_{KG} Z_{KG}[J]}$$

$$= N_{KG} \langle 0 | T (e^{i\int \mathcal{L}_{int}}) | 0 \rangle$$

$$= \frac{1}{N_{vac}} e^{i\int d^4x \, \mathcal{L}_{int}[-i\frac{\delta}{\delta J}]} Z_{KG}[J]$$

where
$$N_{vac} = e^{i\int d^4x \, \mathcal{L}_{int}[-i\frac{\delta}{\delta J}]} Z_{KG}[J] \Big|_{J=0}$$

and
$$f(\varphi) \, e^{i\int J\varphi} = f\left(-i\frac{\delta}{\delta J}\right) e^{i\int J\varphi}$$

We know $Z_{KG}[J]$, so from $e^{i\int d^4x \, \mathcal{L}_{I}} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(i\int d^4x \, \mathcal{L}_{I} \right)^n$

we can solve for correlation functions

as a series in coupling!

$$\boxed{Z[J] = \frac{1}{N_{vac}} e^{i\int d^4x \, \mathcal{L}_{int}[-i\frac{\delta}{\delta J}]} Z_{KG}[J]}$$

Scattering amplitudes and LSZ theorem

Our primary goal in SM phenomenology is to compute scattering amplitudes of particles, from which we can predict/compare to experiment.

Scattering amplitudes are defined through the S-matrix

$$\boxed{S_{\beta\alpha} = \langle \beta, \text{out} | \alpha, \text{in} \rangle}$$

where

$$|\alpha, \text{in}\rangle = |p_a, p_b, \dots, \text{in}\rangle \quad \text{incoming particles}$$

$$|\beta, \text{out}\rangle = |k_1, k_2, \dots, \text{out}\rangle \quad \text{outgoing particles}$$

We use the standard normalization for a relativistic single particle state

$$\langle p' | p \rangle = (2\pi)^3 \, 2E \, \delta^{(3)}(\vec{p}\,' - \vec{p}\,)$$

$$E = \sqrt{m^2 + \vec{p}^{\,2}}$$

The S-matrix operator converts in-states to out-states,

$$S_{\beta\alpha} = \langle \beta, \text{out} | \alpha, \text{in} \rangle$$
$$= \langle \beta, \text{in} | S | \alpha, \text{in} \rangle$$

The S-matrix is a unitary operator

$$S^\dagger S = S S^\dagger = \mathbb{1}$$

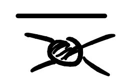
Proof

$$\delta_{\beta\alpha} = \langle \beta, \text{in} | \alpha, \text{in} \rangle$$
$$= \sum_\gamma \underbrace{\langle \beta, \text{in} | \gamma, \text{out} \rangle}_{S^*_{\gamma\beta} = S^\dagger_{\beta\gamma}} \underbrace{\langle \gamma, \text{out} | \alpha, \text{in} \rangle}_{S_{\gamma\alpha}}$$

$$= (S^\dagger S)_{\beta\alpha} = (\mathbb{1})_{\beta\alpha} \qquad \blacksquare$$

The main task now is to connect the S-matrix to the QFT correlation functions. We can do this by the LSZ theorem

$$\boxed{\begin{aligned} S_{\beta\alpha} = \text{D.T.} &+ (iZ)^{-n/2}(iZ)^{-n'/2}\int d^4x_1\cdots\int d^4y_1\cdots \\ &\times \exp\left[-i\left(\sum_{i=1}^{n} k_i\cdot x_i - \sum_{j=1}^{n'} p_j\cdot y_j\right)\right] \\ &\times (\Box_{x_1}+m^2)\cdots\langle 0|T\{\varphi_{x_1}\cdots\varphi_{y_1}\cdots\}|0\rangle(\overleftarrow{\Box}_{y_1}+m^2)\cdots \end{aligned}}$$

Here D.T. are disconnected terms, e.g., ,  , etc. and Z is the wave function renormalization of the scalar field, $Z = \langle 0|\varphi(0)|p\rangle$.

We will focus on tree-level calculations, thus $\boxed{Z=1}$

We define the scattering amplitude $\mathcal{M}$ (or $\mathcal{T}$) as

$$\boxed{S_{\beta\alpha} = \text{D.T.} + (2\pi)^4\,\delta^{(4)}(p'-p)\; i\mathcal{M}_{\beta\alpha}}$$

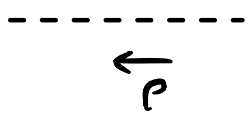
convention

Conserves total momentum

The scattering amplitude is the central observable we want to compute observables. Using the expansion for the correlation functions, we can generate an expansion for the scattering amplitude.

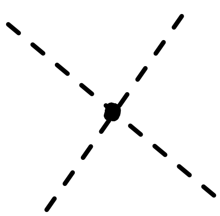
It is convenient to use a diagrammatic representation for the S-matrix called Feynman diagrams

Feynman Rules $-\lambda\varphi^4$



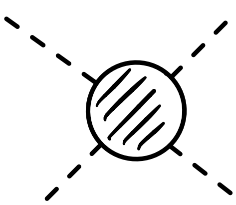
$= i\Delta(p) = \dfrac{i}{p^2 - m^2 + i\epsilon}$

$= \displaystyle\int d^4x \; e^{ip\cdot x} \, i\Delta(x)$



$= i\Gamma = -i\lambda$

Example – $\varphi\varphi \to \varphi\varphi$ amplitude at leading order



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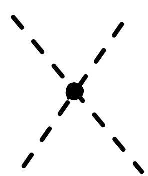

$+ \mathcal{O}(\lambda^2)$

$\Rightarrow \quad i\mathcal{M} = -i\lambda + \mathcal{O}(\lambda^2)$

Observables

Two observables we will concentrate on are cross-sections and decay rates.

Cross-Section (σ) - measures the transition rate of a beam of particles colliding with a target in some region of final states per unit time & unit flux.

$$\sigma = \frac{\#\text{ events}}{\text{unit time} \times \underbrace{\text{unit flux}}_{= \frac{\#\text{ particles}}{\text{cross-sectional area} \times \text{time}}}}$$

For $2 \to n$ processes

$$\boxed{d\sigma_{\alpha\to\beta} = \frac{1}{\mathcal{F}} |\mathcal{M}_{\beta\alpha}|^2 \, d\Phi_n}$$

where $\mathcal{F} = 4\sqrt{(p_a \cdot p_b)^2 - m_a^2 m_b^2}$ is flux factor

and

$$d\Phi_n = (2\pi)^4 \delta(P_\beta - P_\alpha) \prod_{j=1}^{n} \frac{d^3\vec{k}_j}{(2\pi)^3 2E_j}$$

is n-body phase space with $P_\alpha = p_a + p_b$

$$P_\beta = \sum_{j=1}^{n} k_j$$

The total cross-section is then

$$\boxed{\sigma_{\alpha\to\beta} = \frac{1}{F}\int d\Phi_n \, |M_{\beta\alpha}|^2}$$

For particles with spin, can determine unpolarized cross-section by spin-averaging

$$\sigma_{\alpha\to\beta} = \frac{1}{F}\frac{1}{2s_a+1}\frac{1}{2s_b+1}\sum_{\substack{\text{initial}\\ \text{spins}}}\sum_{\substack{\text{final}\\ \text{spins}}}\int d\Phi_n \, |M_{\beta\alpha}|^2$$

Decay Rate (Γ) - measures the disintegration rate for a particle decaying into lighter particles.

$$\Gamma = \frac{\#\ \text{decays}}{\text{unit time}}$$

For $1\to n$, we have

$$\boxed{\Gamma_{\alpha\to\beta} = \frac{1}{2E_\alpha}\int d\Phi_n \, |M_{\beta\alpha}|^2}$$

Here $E_\alpha = \sqrt{M^2 + \vec{p}_\alpha^{\,2}}$ is energy of decaying particle.

Γ is not Lorentz invariant. It is common to quote the rest-frame decay rate, $E_\alpha = M$.

The partial width is the decay rate to some particular decay mode. The total decay rate is the sum of partial widths

$$\Gamma_{\alpha,\,tot} = \sum_\beta \Gamma_{\alpha\to\beta}$$

The branching ratio for some particular channel is

$$BR(\alpha\to\beta) = \frac{\Gamma_{\alpha\to\beta}}{\Gamma_{tot}}$$