Standard Model Phenomenology

Let us look at a couple consequences of the MSM concerning the weak interactions of Hadrons.

Pion Decay

Let's consider the leptonic decay $\pi^- \to \ell^- \bar{\nu}_\ell$.
The π^- is a QCD bound state of $\bar{u}d$. We cannot perturbatively compute the QCD effects, but using chiral effective theory, we know the matrix element

$$\boxed{\langle 0 | \underset{\text{Axial current}}{A^\mu} | \pi^-(p) \rangle = i\sqrt{2}\, \underset{\text{pion decay constant}}{F_\pi}\, p^\mu}$$

We also know a vector current does not contribute,

$$\langle 0 | V^\mu | \pi^-(p) \rangle = 0 .$$

Let's assume the low-energy effective four-fermi interaction,

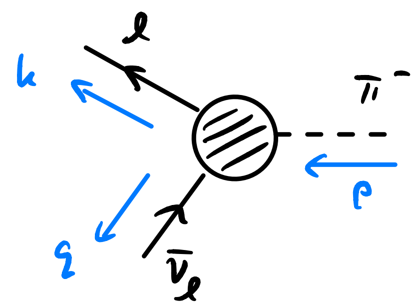
$$\mathcal{L}_{eff} = -\frac{G}{\sqrt{2}} J^{\mu\,\dagger}_{lepton} J_{\mu,\,hadron}$$

where
$$J^\mu_{\text{lepton}} = \bar{\nu}_\ell \gamma^\mu (1-\gamma_5)\,\ell$$

$$J^\mu_{\text{hadron}} = \bar{u}\gamma^\mu (1-\gamma_5)\left(V_{ud}\, d + V_{us}\, s + V_{ub}\, b\right)$$
$$\equiv V^\mu_{\text{had}} - A^\mu_{\text{had}}$$

So, the decay amplitude is

$$i\mathcal{M} =$$



$$= -\frac{G_F}{\sqrt{2}}\, \bar{u}_\ell(k)\gamma_\mu(1-\gamma_5)v_{\nu_\ell}(q)\,\langle 0|\, V^\mu_{\text{had}} - A^\mu_{\text{had}}\,|\pi^-(p)\rangle$$

$$= \frac{G_F}{\sqrt{2}}\, \bar{u}_\ell(k)\gamma_\mu(1-\gamma_5)v_{\nu_\ell}(q)\; i\sqrt{2}\, F_\pi\, p^\mu\, V_{ud}$$

$$= i G_F F_\pi V_{ud}\, \bar{u}_\ell(k)\, \not{p}\, (1-\gamma_5) v_{\nu_e}(q)$$

By momentum conservation, $p = k + q$, & from the Dirac equation, $\bar{u}_\ell \not{k} = \bar{u}_\ell m_\ell$, $\not{q}\, v_{\nu_\ell} = 0$

$$\Rightarrow \mathcal{M} = G_F F_\pi V_{ud}\, m_\ell\, \bar{u}_\ell(k)(1-\gamma_5)v_{\nu_\ell}(q)$$

Let's compute the spin-averaged matrix element

$$\langle |\mathcal{M}|^2 \rangle = \sum_{s,s'} |\mathcal{M}|^2$$

$$= |G_F F_\pi V_{ud} m_\ell|^2 \, tr[\bar{u}_\ell (1-\gamma_5) v_{\nu_\ell} \bar{v}_{\nu_\ell} (1+\gamma_5) u_\ell]$$

$$= G_F^2 F_\pi^2 m_\ell^2 |V_{ud}|^2 \, tr[(\not{k} + m_\ell)(1-\gamma_5) \not{q} (1+\gamma_5)]$$

Now, $\gamma^\mu \gamma_5 = -\gamma_5 \gamma^\mu$

$$\Rightarrow (1-\gamma_5)\gamma^\mu (1+\gamma_5) = 2(1-\gamma_5)\gamma^\mu$$

So, $$\langle |\mathcal{M}|^2 \rangle = 2 G_F^2 F_\pi^2 m_\ell^2 |V_{ud}|^2 \, tr[(\not{k} + m_\ell)(1-\gamma_5)\not{q}]$$

$$= 8 G_F^2 F_\pi^2 m_\ell^2 |V_{ud}|^2 (k \cdot q)$$

In the π^- rest frame, $\vec{p} = \vec{0} \Rightarrow \vec{k} = -\vec{q}$

So, $k \cdot q = E E' - \vec{k} \cdot \vec{q} = E E' + \vec{q}^{\,2}$

also, $E' = |\vec{q}| \Rightarrow k \cdot q = |\vec{q}| (E + |\vec{q}|)$

From momentum conservation

$$|\vec{q}| = \frac{1}{2m_\pi} \lambda^{\frac{1}{2}}(m_\pi^2, m_\ell^2, 0)$$

$$= \frac{1}{2m_\pi} (m_\pi^2 - m_\ell^2)$$

The decay rate is

$$d\Gamma = \frac{1}{32\pi^2} \langle |\mathcal{M}|^2 \rangle \frac{|\vec{q}|}{m_\pi^2} \, d\Omega$$

So,

$$\Gamma = \frac{1}{32\pi^2} \frac{|\vec{q}|}{m_\pi^2} \int d\Omega \; 8 G_F^2 F_\pi^2 m_\ell^2 |V_{ud}|^2 |\vec{q}| (E + |\vec{q}|)$$

Now, $E = \dfrac{m_\pi^2 + m_\ell^2}{2m_\pi}$

$$\Rightarrow \quad E + |\vec{q}| = \frac{1}{2m_\pi}\left(m_\pi^2 + m_\ell^2 + m_\pi^2 - m_\ell^2\right) = m_\pi$$

So, $$\Gamma = \frac{1}{32\pi^2} \frac{|\vec{q}|^2}{m_\pi} \cdot 4\pi \cdot 8 G_F^2 F_\pi^2 m_\ell^2 |V_{ud}|^2$$

$$\Rightarrow \boxed{\Gamma = \frac{1}{4\pi} G_F^2 F_\pi^2 |V_{ud}| \, m_\ell^2 m_\pi \left(1 - \frac{m_\ell^2}{m_\pi^2}\right)^2}$$

So, measuring τ_π gives access to F_π, $|V_{ud}|$.

Let's compare ratio of $\Gamma(\pi^- \to e^- \bar{\nu}_e) / \Gamma(\pi^- \to \mu^- \bar{\nu}_\mu)$

$$\Rightarrow \quad \frac{\Gamma(\pi^- \to e^- \bar{\nu}_e)}{\Gamma(\pi^- \to \mu^- \bar{\nu}_\mu)} = \frac{m_e^2}{m_\mu^2} \left(\frac{m_\pi^2 - m_e^2}{m_\pi^2 - m_\mu^2}\right)$$

$$\simeq 1.28 \times 10^{-4} \qquad \blacksquare$$

Experimentally, ratio is $1.230(4) \times 10^{-4}$

$\Rightarrow$ good agreement.

Note that $m_e \ll m_\mu$, so more phase space for $\pi^- \to e^- \bar{\nu}_e$ compared with $\pi^- \to \mu^- \bar{\nu}_\mu$.

Why is it then that $\dfrac{\Gamma(\pi^- \to e^- \bar{\nu}_e)}{\Gamma(\pi^- \to \mu^- \bar{\nu}_\mu)} \ll 1$?

The answer is helicity suppression. The $e^- \bar{\nu}_e$ system is much more relativistic compared with $\mu^- \bar{\nu}_\mu$. Thus, electron chirality ~ helicity. Since π is spin 0, & W^- is spin-1, angular momentum conservation must couple left-right-handed pair $\Rightarrow$ gives Dirac mass term,

$\Rightarrow$ gives helicity suppression.

$$\Gamma = \frac{1}{4\pi} G_F^2 F_\pi^2 |V_{ud}| \boxed{m_\ell^2} m_\pi \left(1 - \frac{m_\ell^2}{m_\pi^2}\right)^2$$

$K^0 - \bar{K}^0$ mixing

As our final example, let's look at neutral kaon oscillations and the effects of CP violation.

Kaons are pseudoscalar ($J^P = 0^-$) QCD bound states

$$K^0 (\bar{s} d) \qquad K^+ (\bar{s} u)$$
$$\bar{K}^0 (\bar{d} s) \qquad K^- (\bar{u} s)$$

Let's understand its properties under CP. Recall that $P|K^0\rangle = -|K^0\rangle$, $P|\bar{K}^0\rangle = -|\bar{K}^0\rangle$, and $C|K^0\rangle = |\bar{K}^0\rangle$, $C|\bar{K}^0\rangle = |K^0\rangle$.

So, we find

$$CP|K^0\rangle = -|\bar{K}^0\rangle$$
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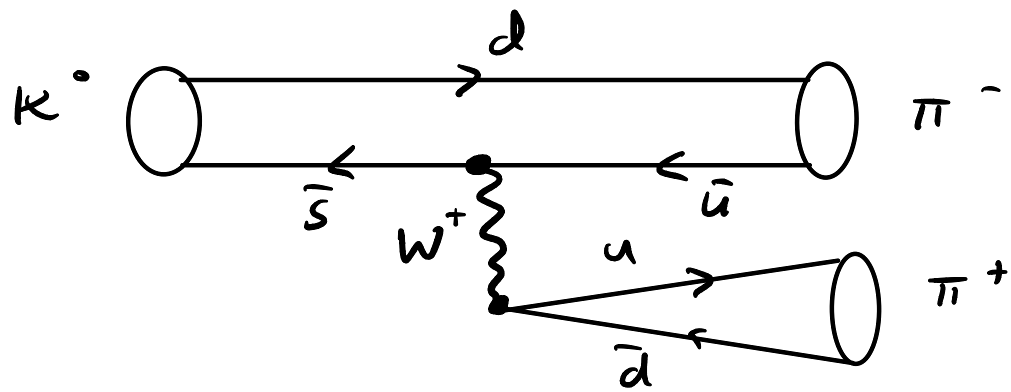
Let's construct CP eigenstates $|K^0_\pm\rangle$

$$|K^0_\pm\rangle = \frac{1}{\sqrt{2}}\left(|K^0\rangle \mp |\bar{K}^0\rangle\right)$$

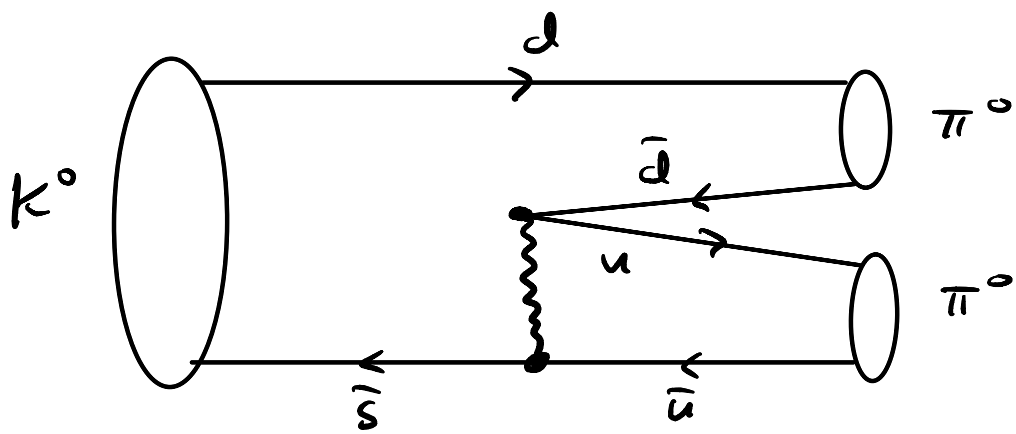
so that $\boxed{CP|K^0_\pm\rangle = \pm|K^0_\pm\rangle}$

Consider two possible 2π decay modes,

$K^0 \to \pi^+ \pi^-$



$K^0 \to \pi^0 \pi^0$



Let's use conservation of angular momentum

$$|\pi\pi\rangle = \sum_{\ell} Y_{\ell m_\ell}(\Omega)\, |\pi\pi, \ell m_\ell\rangle$$

Since $K^0 (J^P = 0^-) \to \pi\pi \Rightarrow \underline{\ell = J = 0}$

Applying CP, noting that relative phases of π^+, π^- cancel, we have

$$CP|\pi^+\pi^-\rangle = (-1)^{\ell}|\pi^+\pi^-\rangle = |\pi^+\pi^-\rangle$$

$\hookrightarrow$ parity from angular momentum
$\ell = 0$

Similarly,

$$CP|\pi^0\pi^0\rangle = |\pi^0\pi^0\rangle$$

Therefore, $\pi\pi$ is always an eigenstate of CP w/ $+1$.
We know that CP is conserved in EM & strong interactions, but $K^0 \to \pi\pi$ is a weak process ($\Delta S \neq 0$).
Weak interactions violate P, but if CP was conserved, then expect

$$K_+^0 \to \pi\pi \qquad \text{allowed}$$

$$K_-^0 \not\to \pi\pi \qquad \text{Not allowed}$$

K_-^0 can of course still decay, but to more elaborate chanels, e.g., $K_-^0 \to \pi\pi\pi$.

We call K_+^0 "short-lived" & K_-^0 "long-lived" since the easier 2π decay mode is supressed.

Experimentally, we find two neutral Kaons K_S^0, K_L^0.
Here, K_S^0 has a "short" lifetime of $\tau_S \simeq 9 \times 10^{-11}$ s,
while K_L^0 has a "long" lifetime of $\tau_L \simeq 5 \times 10^{-8}$ s.

If CP preserved in weak interactions then

$$|K_S^0\rangle = |K_+^0\rangle$$

$$|K_L^0\rangle = |K_-^0\rangle$$

$$\Rightarrow \qquad K_L^0 \not\to \pi\pi .$$

But, it is observed that $K_L^0 \to \pi\pi$ with

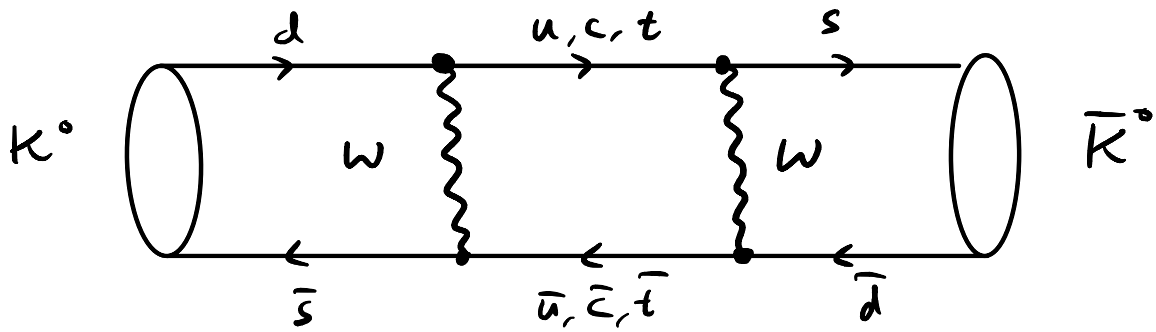
$$BR(K_L^0 \to \pi^+\pi^-) \simeq BR(K_L^0 \to \pi^0\pi^0) = 2 \times 10^{-3}$$

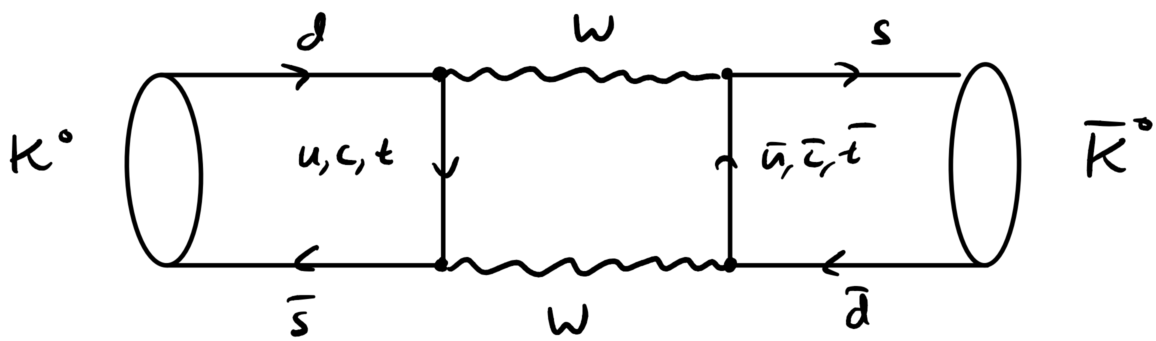
$\Rightarrow$ CP is violated in weak interactions!

CP violation is ultimately due to the δ phase in the CKM matrix. Since the $K_{L/S}^0$ are neutral, quantum mechanical oscillations between K^0 & $\bar{K}^0$ can occur, which involve weak interactions.

$$K^0 \leftrightarrow \bar{K}^0$$

Consider the dominant $\Delta S = 2$ "box diagrams",





Since K_L°, K_S° aren't exactly K_+° & K_-°, we have some corrections

$|\varepsilon_1| \ll 1$

$$|K_S^\circ\rangle = \frac{1}{\sqrt{1+|\varepsilon_1|^2}}\left(|K_+^\circ\rangle + \varepsilon_1 |K_-^\circ\rangle\right) \simeq |K_+^\circ\rangle$$

$$|K_L^\circ\rangle = \frac{1}{\sqrt{1+|\varepsilon_2|^2}}\left(|K_-^\circ\rangle + \varepsilon_2 |K_+^\circ\rangle\right) \simeq |K_-^\circ\rangle$$

$|\varepsilon_2| \ll 1$

here, $\varepsilon_1, \varepsilon_2 \in \mathbb{C}$. These arise from CP violation.

Assuming a two-state mixing model,

$$|K_S(t)\rangle = a_S(t)\,|K^0\rangle + b_S(t)\,|\bar{K}^0\rangle$$
$$|K_L(t)\rangle = a_L(t)\,|K^0\rangle + b_L(t)\,|\bar{K}^0\rangle$$

for $a_S, b_S, a_L, b_L \in \mathbb{C}$. Schrödinger's equation gives

$$i\frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle$$

$$\Rightarrow \quad i\frac{d}{dt}\begin{pmatrix} a \\ b \end{pmatrix} = R\begin{pmatrix} a \\ b \end{pmatrix}$$

$R \neq R^\dagger$ since K decays.

where,
$$R = \begin{pmatrix} \langle K^0|H'|K^0\rangle & \langle K^0|H'|\bar{K}^0\rangle \\ \langle \bar{K}^0|H'|K^0\rangle & \langle \bar{K}^0|H'|\bar{K}^0\rangle \end{pmatrix}$$

$\hookrightarrow$ H' is NLO weak Hamiltonian

Can in general decompose $R = M - \frac{i}{2}\Gamma$

mass matrix $M = M^\dagger$

decay matrix $\Gamma = \Gamma^\dagger$

We won't compute R, but use symmetries to constrain it.

Consider action of CPT, $\Theta = CPT$.

The CPT theorem says observables are invariant under CPT, if $A = A^{\dagger} \Rightarrow \Theta A \Theta^{-1} = A$.

Θ is an antiunitary operator,

$$\Rightarrow \quad \Theta i A \Theta^{-1} = -i \, \Theta A \Theta^{-1} = -iA$$

So, we find for H' (Not Hermitian)

$$H' = A + iB$$

$$\Rightarrow \quad \Theta H' \Theta^{-1} = A - iB = H'^{\dagger}$$

For Kaons at rest, $\Theta |\bar{K}^0\rangle = -|K^0\rangle$

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So,
$$R_{11} = \langle K^0 | H' | K^0 \rangle$$
$$= \langle K^0 | \Theta^{-1} \Theta H' \Theta^{-1} \Theta | K^0 \rangle$$
$$= \left(\langle \bar{K}^0 | H'^{\dagger} \right) |\bar{K}^0\rangle^* \quad \longleftarrow \text{Antiunitary}$$
$$= \langle \bar{K}^0 | H' | \bar{K}^0 \rangle = R_{22}$$

If T is good symmetry (CP is good) $\Rightarrow R_{12} = R_{21}$

Can then show that

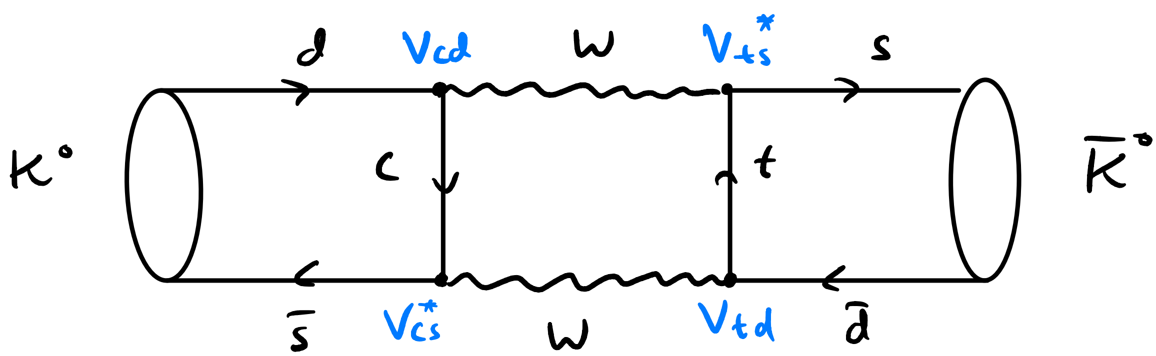
$$\varepsilon_1 = \varepsilon_2 \equiv \varepsilon = \frac{\sqrt{R_{21}} - \sqrt{R_{12}}}{\sqrt{R_{12}} + \sqrt{R_{21}}}$$

If CP conserved, $R_{21} = R_{12} \Rightarrow \varepsilon = 0.$

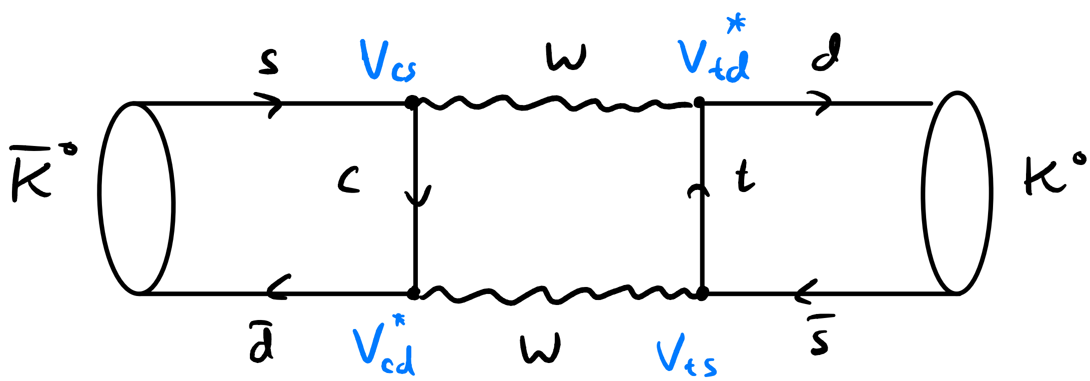
To have mixing, need $\varepsilon \neq 0$. Experimentally, find $\varepsilon \simeq 2 \times 10^{-3}$. Since CP is violated, T violated.

Thus,
$$\Gamma(\bar{K}^0 \to K^0) \neq \Gamma(K^0 \to \bar{K}^0)$$

Compare two box diagrams,



$$\mathcal{M} \propto V_{cd} V_{cs}^{*} V_{td} V_{ts}^{*}$$



$$\mathcal{M}' \propto V_{cd}^{*} V_{cs} V_{td}^{*} V_{ts} = \mathcal{M}^{*}$$

find

$$\Gamma(K^0 \to \bar{K}^0) - \Gamma(\bar{K}^0 \to K^0) \propto \mathcal{M} - \mathcal{M}^*$$

$$= 2\,\mathrm{Im}\,\mathcal{M}$$

So, difference in rates is proportional to $\mathrm{Im}\,\mathcal{M}$. Can show

$$|\varepsilon| \propto \mathrm{Im}\,\mathcal{M} \propto \mathrm{Im}\left(V_{cd} V_{cs}^* V_{td} V_{ts}^*\right)$$