Symmetry

Symmetry is key to organizing many complicated phenomena. In the last ~100 years, notion of "symmetry" has been interpreted mathematically.

$\boxed{\text{Symmetry} \equiv \text{Invariance under a group of transformations}}$

There are two basic types,

Discrete - Physical quantities transform by finite amounts, e.g., C, P, & T

Continuous - Physical quantities transform by any amount, including infinitesimal, e.g., rotational

Focus on this first

N.B. Continuous symmetries can be discussed in terms of infinitesimal transformations

→ This is relation between Lie groups ⟷ Lie Algebras

You are already familiar with a number of continuous symmetries, including

- Rotations in 2 and 3 spatial dimensions
- Lorentz transformations in 3+1 spacetime dimensions
- Global phase transformation of Dirac spinors

$$\psi \to e^{i\theta}\psi$$

Each of these classes of symmetry transformations share the mathematical properties of a group

Groups

A group G is a set $\{g_i\}$ with an operation "group multiplication"

$$G \times G \to G$$

such that $\forall\, g_j, g_k, g_\ell \in G$

(1) Closure: $g_j g_k \in G$

(2) Associativity: $g_j(g_k g_\ell) = (g_j g_k) g_\ell$

(3) Identity: $\exists\, g_0 \in G$ such that $g_0 g_j = g_j$

(4) Inverse: $\exists\, g_j^{-1} \in G$ such that $g_j^{-1} g_j = g_0$

Examples

(a) $G = \{\pm 1, \pm i\}$ under ordinary multiplication is a group. Let's check, make group multiplication table

$\times$	$+1$	-1	$+i$	$-i$
$+1$	$+1$	-1	$+i$	$-i$
-1	-1	$+1$	$-i$	$+i$
$+i$	$+i$	$-i$	-1	$+1$
$-i$	$-i$	$+i$	$+1$	-1

identity element

Each row and column has every element of G $\Rightarrow$ closure

Ordinary multiplication is associative, and the inverse elements are $(\pm 1)^{-1} = \pm 1$, $(\pm i)^{-1} = \mp i$, which are elements of G $\Rightarrow$ G is a group!

N.B. this is an example of a Discrete group.

(b) $G = \{ e^{i\alpha} \mid \alpha \in \mathbb{R} \}$ under ordinary multiplication is a group. Let's check,

Let $\beta, \gamma \in \mathbb{R}$, then

- closure: $e^{i\alpha} e^{i\beta} = e^{i(\alpha+\beta)} = e^{i\gamma} \in G$
- Associativity: $e^{i\alpha}(e^{i\beta} e^{i\gamma}) = (e^{i\alpha} e^{i\beta}) e^{i\gamma} \in G$
- Identity: if $\alpha = 0$, $e^{i0} = 1 \in G$
- Inverse: $e^{-i\alpha} \in G$, so $e^{-i\alpha} e^{i\alpha} = 1 \in G$

$\Rightarrow \{ e^{i\alpha} \}$ with $\alpha \in \mathbb{R}$ is a group!

N.B. this is an example of a continuous group.

Commutative (Abelian) Groups

If $g_i g_k = g_k g_i \quad \forall \; g_i, g_k \in G$,
then the group is called commutative or Abelian

For example, $\{e^{i\alpha}\}$ with $\alpha \in \mathbb{R}$ under ordinary multiplication is an Abelian group.

Non-commutative groups are simply called Non-Abelian

For example, rotations in 3 dimensions form a Non-Abelian group.

Continuous groups can also be defined as a smooth manifolds. Such groups are called Lie groups

Lie groups ≡ continuous groups

They have (1) infinite number of elements

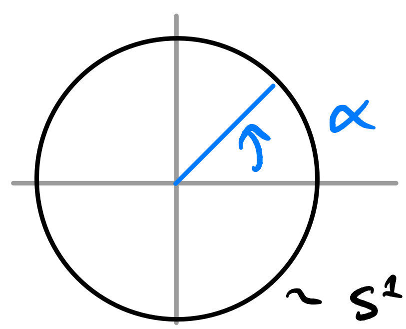
(2) the topological structure of manifold

Manifold = topological space that is locally Euclidean at each point.

Example

The group $G = \{ e^{i\alpha} \mid \alpha \in \mathbb{R} \}$ is a Lie group.

It has an infinite number of elements, labelled by the parameter $\alpha \in \mathbb{R}$, and a topological structure of a circle S^1



There is a useful result to note,

Every Lie group is isomorphic (1-1 correspondence) to a group of square matrices with group multiplication ≡ matrix multiplication.

N.B. Not necessarily compact!

A field F is also a set $\{f_i\}$ of "scalars" with two operations: $\begin{cases} \text{"scalar addition"} \\ \text{"scalar multiplication"} \end{cases}$

Such that

(1) F is Abelian group under addition with identity f_0

(2) F obeys group postulates under multiplication except f_0 has no inverse

(3) distributive: $$f_j(f_k + f_\ell) = f_j f_k + f_j f_\ell$$
$$(f_j + f_k) f_\ell = f_j f_\ell + f_k f_\ell$$

e.g.,

- $\mathbb{R}$ $(f_0 \equiv 0)$ reals numbers under normal $+, \times$
- $\mathbb{C}$ $(f_0 \equiv 0)$ complex numbers under normal $+, \times$
- $\mathbb{Q}$ $(f_0 \equiv 0)$ rational numbers under normal $+, \times$

Some objects not fields, e.g., $\mathbb{Z}$ integers, $\mathbb{N}$ natural numbers

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A <u>vector space</u> V is a set $\{v_i\}$ of vectors and field F with extra operation "vector addition" $V \times V \to V$, and "extended scalar multiplication" $F \times V \to V$ such that

(1) V is Abelian group under vector addition

(2) extended scalar multiplication closes, associative, and has an identity

(3) bilinearity : $f_j(v_u + v_e) = f_j v_u + f_j v_e$

$$(f_j + f_u) v_e = f_j v_e + f_u v_e$$

e.g.,

- $\mathbb{R}^N$ as a vector space
- $\mathbb{C}^N$ as a vector space
- Set of $M \times N$ matrices (over F) under <u>matrix addition</u>

  [The "vectors" here are the matrices]

V is "M dimensional" if it can be spanned by M linearly independent vectors.

Any such set is called a "<u>basis</u>" for V

Convention: denote basis elements by $\{x_i\}$

An algebra A is a vector space V over a field F with extra operation "vector multiplication" $A \times A \to A$ such that

(1) Closure

(2) "bilinearity" $\begin{cases} (v_j + v_k) v_\ell = v_j v_\ell + v_k v_\ell \\ v_j (v_k + v_\ell) = v_j v_k + v_j v_\ell \\ (f_j v_k)(f_\ell v_m) = (f_j f_\ell)(v_k v_m) \end{cases}$

Other possible combinations for special cases

- Commutative algebra : $v_j v_k = v_k v_j$
- associative algebra : $(v_j v_k) v_\ell = v_j (v_k v_\ell)$
- A with antisymmetry : $v_j v_k = - v_k v_j$
- A with identity : $v_0 v_j = v_j = v_j v_0$
  "unital"

But, many algebras don't have these properties

Example

$N \times N$ matrices under usual scalar multiplication

matrix addition $\quad V$

matrix multiplication $\quad A$

This is a non-commutative, associative, unital algebra

The "vectors" are the matrices

It is N^2 dimensional if $\mathbb{F} = \mathbb{R}$

Example

$\mathbb{C}$ is a 2D algebra over $\mathbb{R}$

"vector over $\mathbb{R}$"

check: For $z \in \mathbb{C}$, can write $z = a + ib = (a, b)$

vector space:

$V \times V \to V \quad : \quad (a,b) + (c,d) \longmapsto (a+c, b+d)$

$\mathbb{F} \times V \to V \quad : \quad r(a,b) \longmapsto (ra, rb)$, $r \in \mathbb{R}$

$A \times A \to A \quad : \quad (a,b) \times (c,d) \longmapsto (ac - bd, ad + bc)$

Required properties satisfied

This is a commutative, associative, unital algebra

Example (exercise)

$\mathbb{R}^3$ as vector space with vector multiplication = cross product

Check properties, find anticommutative algebra

A Lie algebra A is an algebra such that vector multiplication is anticommutative and obeys an identity "Jacobi Identity"

Convention: vector multiplication is denoted by $[\,,\,]$

$$A \times A \to A \ : \ v_j, v_k \longmapsto [v_j, v_k]$$

Lie Bracket

$\Rightarrow$ Lie algebra satisfies

(1) $[v_j, v_k] = -[v_k, v_j]$

(2) Jacobi : $\sum_{(jkl)} [[v_j, v_k], v_l] = 0$

cyclic sum

Useful Result (Ado theorem)

Every Lie algebra is isomorphic to algebra of square matrices with vector multiplication = commutator of matrix multiplication

matrix multiplication

i.e., $[v_j, v_k] \xrightarrow[\text{isomorphism}]{} v_j v_k - v_k v_j$

Given a basis $\{X_j\}$ for Lie algebra, can write

$$\boxed{[X_j, X_k] = C_{jk}{}^{\ell} X_{\ell}}$$

structure constants

The structure constants obey

$$\boxed{\sum_{(jk\ell)} C_{jk}{}^{m} C_{m\ell}{}^{n} = 0}$$

Proof

Recall the Jacobi identity,

$$\sum_{(jk\ell)} [[X_j, X_k], X_{\ell}] = 0$$

From Lie bracket $[X_j, X_k] = C_{jk}{}^{\ell} X_{\ell}$,

Find

$$\sum_{(jk\ell)} [[X_j, X_k], X_{\ell}] = \sum_{(jk\ell)} C_{jk}{}^{m} [X_m, X_{\ell}]$$

$$= \sum_{(jk\ell)} C_{jk}{}^{m} C_{m\ell}{}^{n} X_n$$

$$= 0$$

This is true for any basis set $\{X_j\}$, so

$$\sum_{(jk\ell)} C_{jk}{}^{m} C_{m\ell}{}^{n} = 0 \qquad \blacksquare$$

If $C_{jk}{}^{l} = 0$, Lie algebra is called "Abelian"

By a careful choice of canonical bases,
Lie algebra can be classified and partially enumerated.

Terminology

A mapping of abstract Lie algebra A
into $\begin{cases} \text{definite math structure} \equiv \text{"realization" of A} \\ N\times N \text{ matrices} \equiv \text{"N-dim representation" of A} \end{cases}$

└ Same terminology for group.

Warning

Don't confuse dim(A) with dim(rep)

Example :

dim of algebra

$\dim(su(2)) = 3$ "e.g., 3 pauli matrices"

$\dim(\sigma_j) = 2$ "2×2 matrix = 2-dim rep"

dim of rep

Connection between Lie groups and Lie algebras

Consider Lie group element $g(\alpha^i)$, identity at $\alpha^i = 0$

think of as matrix

Expand in Taylor series about $\alpha^i = 0$

$$g(\alpha^i) = g(0) + \alpha^i X_i + \mathcal{O}(\alpha^2)$$

where $\boxed{X_j = \left.\frac{\partial g}{\partial \alpha_j}\right|_{\alpha_j = 0}}$ "infinitesimal group generator"

Inverse has the form $\left(g(\alpha^i)^{-1} g(\alpha^i) = \mathbb{1} \right)$

$$g(\alpha^i)^{-1} = g(0) - \alpha^i X_i + \mathcal{O}(\alpha^2)$$

Now, consider the group "commutator" of 2 elements

$$g(\beta^i)^{-1} g(\gamma^i)^{-1} g(\beta^i) g(\gamma^i) = g(\alpha^i)$$

true from group axioms

No sum on i, indices for parameters

Expand in Taylor series

$$(g(0) - \beta^j X_j)(g(0) - \gamma^k X_k)(g(0) + \beta^m X_m)(g(0) + \gamma^n X_n)$$

$$= g(0) + \alpha^l X_l$$

Keep $\mathcal{O}(\alpha)$, $\mathcal{O}(\beta)$, and $\mathcal{O}(\gamma)$ terms

So,

$$(g(0) - \beta^j X_j)(g(0) - \gamma^k X_k)(g(0) + \beta^m X_m)(g(0) + \gamma^n X_n)$$

$$= (g(0) - \beta^j X_j - \gamma^k X_k + \beta^j \gamma^k X_j X_k)$$

$$\times (g(0) + \beta^m X_m + \gamma^n X_n + \beta^m \gamma^n X_m X_n)$$

$$= g(0) - \cancel{\beta^j X_j} - \cancel{\gamma^k X_k} + \beta^j \gamma^k X_j X_k$$

$$+ \cancel{\beta^m X_m} + \cancel{\gamma^n X_n} + \beta^m \gamma^n X_m X_n$$

$$+ (- \beta^j X_j - \gamma^k X_k + \beta^j \gamma^k X_j X_k)$$

$$\times (\beta^m X_m + \gamma^n X_n + \beta^m \gamma^n X_m X_n)$$

$$= g(0) + \cancel{\beta^j \gamma^k X_j X_k} + \beta^m \gamma^n X_m X_n$$

$$- \cancel{\beta^j \gamma^n X_j X_n} - \beta^m \gamma^k X_k X_m + \mathcal{O}(\beta^2, \gamma^2)$$

$$= g(0) + \beta^j \gamma^k (X_j X_k - X_k X_j) + \mathcal{O}(\beta^2, \gamma^2)$$

$$= g(0) + \beta^j \gamma^k [X_j, X_k] + \mathcal{O}(\beta^2, \gamma^2)$$

$$= g(0) + \alpha^l X_l + \mathcal{O}(\alpha^2)$$

So, conclude

$$[X_j, X_k] = C_{jk}{}^l X_l \qquad \text{where } \alpha^l = C_{jk}{}^l \beta^j \gamma^k$$

Commutator of matrix multiplication

Therefore,

$$\boxed{\begin{gathered}\{X_j\} \text{ is a basis for Lie algebra} \\ \text{with structure constants } C_{jk}{}^{\ell}.\end{gathered}}$$

Convention: write $\quad g(\alpha^j) = \mathrm{Exp}(\alpha^j X_j)$

generalized exponential
or exponential map

Result: For matrix representations of X_j

$$\mathrm{Exp}(\alpha^j X_j) = \exp(\alpha^j X_j)$$
$$= \mathbb{1} + \alpha^j X_j + \frac{1}{2}\alpha^j\alpha^k X_j X_k + \mathcal{O}(\alpha^3)$$

Suggestive argument:

$$g(\epsilon^j) \approx g(0) + \epsilon^j X_j = g(0) + \frac{\alpha^j}{N} X_j \quad \text{for large } N$$

then,
$$g(\alpha^j) = \left[\mathbb{1} + \frac{\alpha^j}{N} X_j\right]^N \longrightarrow \exp(\alpha^j X_j)$$

can multiply like this
because it is a group

Example: 2D rep of SO(2)

Consider $g(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$, $\alpha \in \mathbb{R}$

These matrices form 2D rep of SO(2) group

special: $\det O = 1$

orthogonal: $O^T O = \mathbb{1}$

2×2

This is an Abelian group. The associated algebra is called $\mathfrak{so}(2)$.

Associated generator X is obtained by Taylor expanding

$$g(\alpha) = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{g(0)} + \underbrace{\alpha}_{\text{parameter}} \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{\text{generator}} + \mathcal{O}(\alpha^2)$$

$$\stackrel{?}{=} \exp\left[\alpha \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right]$$

$$\exp\left[\alpha\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right] = \exp(-i\alpha\sigma^2)$$
$$= \cos\alpha - i\sigma^2 \sin\alpha$$
$$= \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \checkmark$$

Algebra is Abelian: $[X, X] = 0$

Notice: There is one X and it is 2-Dimensional

algebra dim = 1

rep. dim = 2

Evidently, SO(2) useful for situations involving rotations. Generally, physical use of symmetry involves both group and a space on which it acts.

e.g., for rotations in plane

$$g(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \text{ acts on 2D vector } \begin{pmatrix} x \\ y \end{pmatrix}$$

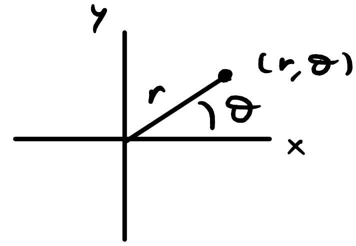
This space is called

"Basis for representation"

or "representation"

Example of realization of SO(2)

Suppose physical space is represented by $f(r,\theta)$. What is realization of SO(2)?



Consider function rotated by α

$$f(r,\theta+\alpha) = f(r,\theta) + \alpha\,\partial_\theta f(r,\theta) + \mathcal{O}(\alpha^2)$$
$$= \exp(\alpha\,\partial_\theta)\, f(r,\theta)$$
$$= g(\alpha)\, f(r,\theta)$$

$$\Rightarrow \quad g(\alpha) = \exp(\alpha\,\partial_\theta) \;\rightarrow\; X = \partial_\theta \;, \quad [X,X]=0$$

realization of SO(2)

Example: 1D rep of U(1)

Suppose physical space is represented by $z = re^{i\theta} \in \mathbb{C}$

To rotate, take $g(\alpha) = e^{i\alpha}$

↳ unitary → U(1)

↳ 1×1

generator is now $X = i$ (or $X = 1$)

$$\Rightarrow [X,X]=0$$

So, $\mathfrak{u}(1) \cong \mathfrak{so}(2) \Rightarrow$ algebras are isomorphic

Some Matrix groups

- General linear groups

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \to \det \neq 0$

  - $GL(N,\mathbb{C})$ = group of invertible $N\times N$ matrices with complex entries

    Has $2N^2$ real parameters,

    generators are $2N^2$ matrices which are $N\times N$ with 1 or i as one non-zero entry.

  - $GL(N,\mathbb{R}) = GL(N,\mathbb{C})$ restricted to $\mathbb{F}=\mathbb{R}$

    N^2 parameters, N^2 generators

    Notice: $GL(N,\mathbb{C}) \supset GL(N,\mathbb{R})$

- Special Linear groups

  - $SL(N,\mathbb{C}) = GL(N,\mathbb{C})$ with $\det = +1$

    $\Rightarrow$ $2(N^2-1)$ real parameters, generators (traceless)

  - $SL(N,\mathbb{R}) = SL(N,\mathbb{C})$ restricted to $\mathbb{F}=\mathbb{R}$

e.g., $SL(2,\mathbb{C})$ is group of quantum Lorentz transformations

- Orthogonal groups

– $O(N,\mathbb{C})$ = group of $N\times N$ complex orthogonal matrices
$\hookrightarrow O^T O = 1$

Notice: $\det O = \det O^T$

$\Rightarrow (\det O)^2 = 1 \Rightarrow \det O = \pm 1$

if $\det O = +1$ $\Rightarrow SO(N,\mathbb{C})$ "special"

so, group is in (at least) two pieces.

$N(N-1)$ real parameters,

generators are $N\times N$ antisymmetric matrices.

entries are ± 1, or $\pm i$.

– $O(N,\mathbb{R}) = O(N,\mathbb{C})$ restricted to $F = \mathbb{R}$

$\frac{1}{2}N(N-1)$ parameters, generators.

Notice: if vector $x = \begin{pmatrix} x^1 \\ \vdots \\ x^N \end{pmatrix}$

then $O(N,\mathbb{R})$ leaves invariant the quadratic form

$$x^T x = \sum_\alpha (x^\alpha)^2$$

$x \to O x$, $O \in O(N,\mathbb{R})$

then, $x^T \to x^T O^T$

and so $x^T x \to x^T \underbrace{O^T O}_{=\mathbb{1}} x = x^T x$

- $O(N, M, \mathbb{R})$ = group of pseudo orthogonal $(N+M)\times(N+M)$ real matrices

  satisfy $O^T \eta O = \eta$

  with $$\eta = \begin{pmatrix} \underbrace{\begin{matrix} + & & \\ & + & \\ & & \ddots \end{matrix}}_{M} & \\ & \underbrace{\begin{matrix} - & & \\ & - & \\ & & \ddots \end{matrix}}_{N} \end{pmatrix}$$

  Leaves invariant

  $$x^T \eta x = \sum_{\alpha, \beta} x^\alpha \eta_{\alpha\beta} x^\beta$$

  $\frac{1}{2}(M+N)\times(M+N-1)$ parameters, generators

  e.g., $SO(3,1)$ is group of classical Lorentz trans.

- Unitary groups

  - $U(N)$ = group of $N\times N$ complex unitary matrices

    $\hookrightarrow U^\dagger U = \mathbb{1}$

    $U^\dagger = (U^T)^*$

    Leaves invariant quadratic form $z^\dagger z = \sum_\alpha z^*_\alpha z_\alpha$

    N^2 real parameters, generators

    $$z = \begin{pmatrix} z^1 \\ \vdots \\ z^N \end{pmatrix}$$

    generators are $N\times N$ antihermitian matrices

- $U(N,M) = U$ obeying $U^\dagger \eta U = \eta$

  Leaves invariant $z^\dagger \eta z$

- $SU(N,M) = U(N,M)$ restricted to $\det = +1$

  In particular, $SU(N,0) \equiv SU(N)$

  has $N^2 - 1$ real parameters

Note

$\exp(\alpha X)$ vs. $\exp(\alpha X)$

if real — antihermitian

if complex ($\alpha \to i\alpha$) — hermitian

choice between the two!

In Quantum theory, conserved quantity leads to symmetry.

Generators are observables associated with quantity

$\Rightarrow$ Observables must be Hermitian

So, for real parameter α^i

$\Rightarrow \quad X_j = i Q_j$

Hermitian

$\Rightarrow U = \exp(i\alpha^i Q_j)$

Note: For SU(N) groups, $\det U = 1$

so,

$$U = \exp(i\alpha^j Q_j)$$

has a constraint.

In general, $\det(\exp(A)) = \exp(\mathrm{tr}(A))$

for A an $N \times N$ matrix.

So,

$$\det U = \det(\exp(i\alpha^j Q_j))$$

$$= \exp(\mathrm{tr}(i\alpha^j Q_j))$$

$$= \exp(i\alpha^j \,\mathrm{tr}\, Q_j)$$

but, $\det U = 1$

$$\Rightarrow \exp(i\alpha^j \,\mathrm{tr}\, Q_j) = 1$$

or

$$\boxed{\mathrm{tr}\, Q_j = 0} \quad \text{for } Q_j \in su(N)$$

Proof that X_j are antihermitian for $U(N)$ groups.

if $g(\alpha^j) = \mathbb{1} + \alpha^j X_j$, $g^+(\alpha^j) = \mathbb{1} + \alpha^j X_j^+$

and $g \in U(N)$, then $g^+ g = \mathbb{1}$

$$\Rightarrow (\mathbb{1} + \alpha^j X_j^+)(\mathbb{1} + \alpha^j X_j)$$

$$= \mathbb{1} + \alpha^j (X_j^+ + X_j) + O(\alpha^2) = \mathbb{1}$$

so, at $O(\alpha)$, we have

$$X_j^+ + X_j = 0$$

or $X_j^+ = - X_j$

So, X_j is antihermitian $\blacksquare$

U(1), SO(3) and SU(2)

Let us consider some specific groups important for the SM. The simplest is U(1)

- $U(1)$ = 1×1 unitary matrix (complex number)

so, $\boxed{U(1) = e^{i\alpha}\,, \quad \alpha \in \mathbb{R}}$

This is a simple phase rotation.

- $SO(3)$ = 3×3 real matrices obeying $O^T O = \mathbb{1}$, $\det O = +1$

  Generators are antisymmetric matrices L_1, L_2, L_3

  $N_{\text{Generators}} = \frac{1}{2}\, 3(3-1) = 3$

Can pick

$$L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & +1 & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & 0 & +1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad L_3 = \begin{pmatrix} 0 & -1 & 0 \\ +1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Find Lie algebra $so(3)$ is

$$[L_j, L_k] = \epsilon_{jkl} L_l \quad , \quad \epsilon_{123} = +1 \quad , \quad j,k,l \in \{1,2,3\}$$

The group has 3 parameters α^i

$\Rightarrow$ 3D rep of group is

$$\mathcal{O}(\alpha^i) = \exp(\alpha^i L_j)$$ (Pset 4)

$$= \mathbb{1}_3 + \frac{\alpha^j}{\alpha} L_j \sin\alpha + \left(\frac{\alpha^j}{\alpha} L_j\right)^2 (1 - \cos\alpha)$$

with $\alpha = |\vec{\alpha}|$

- $SU(2)$ = group of 2×2 complex matrices obeying
$$U^\dagger U = \mathbb{1}, \quad \det U = +1$$

$$N_{\text{generators}} = N^2 - 1 = 2^2 - 1 = 3 \text{ generators}$$

must be traceless, Hermitian matrices X_j

Recall: Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are Hermitian

→ to get antihermitian

So, take $X_j = -\frac{1}{2} i \sigma_j$

↳ convenient, normalizes algebra

$\Rightarrow$ algebra $su(2) \cong so(3)$ $\qquad [X_j, X_k] = \epsilon_{jkl} X_l$

SU(2) has 3 parameters, α^j

$\Rightarrow$ 2D rep. is

$$U(\alpha^j) = \exp(\alpha^j X_j) \qquad \text{(PS0 4)}$$
$$= \mathbb{1}_2 \cos \tfrac{1}{2}\alpha - i \frac{\alpha^j}{\alpha} \sigma_j \sin \tfrac{1}{2}\alpha$$

with $\alpha = |\vec{\alpha}|$

- Although $su(2) \cong so(3)$ as algebras, the groups SU(2) and SO(3) are different. In fact,

  $SU(2) \longrightarrow SO(3)$ is $2 \to 1$ map (double cover)

To see this, start at identity $\alpha^j = 0$, pick a direction $\hat{\alpha} = \frac{\vec{\alpha}}{\alpha}$ in group space, move away, and see what happens

Find:
$$O(\alpha) = +O(\alpha + 2\pi) = +O(\alpha + 4\pi)$$
$$U(\alpha) = -U(\alpha + 2\pi) = +U(\alpha + 4\pi)$$

So,
$$SO(3): \ \mathbb{1} \to \mathbb{1} \to \mathbb{1}$$
$$SU(2): \ \mathbb{1} \to -\mathbb{1} \to \mathbb{1}$$

These groups are different!

For low dimensionalities, different groups may have same algebra,

$$\mathfrak{so}(3) \cong \mathfrak{su}(2) \quad - \quad SO(3) \cong SU(2)/\mathbb{Z}_2$$

$$\mathfrak{so}(4) \cong \mathfrak{su}(2) \times \mathfrak{su}(2) \quad - \quad SO(4) \cong (SU(2) \times SU(2))/\mathbb{Z}_2$$

$$\vdots$$

Representations

In the previous examples, we found a 2D and 3D representation (rep.) for algebra $\mathfrak{su}(2) \sim \mathfrak{so}(3)$

$\mathfrak{su}(2)$	$\mathfrak{so}(3)$
$-\frac{1}{2} i \sigma_j$	L_j

Call these $\quad \underset{\sim}{2} \quad \underset{\sim}{3}$

Here, $\underset{\sim}{n}$ denotes the rep. of the algebra.

What about other dimensions? Obviously $X_i = 0$ ($\underset{\sim}{1}$) satisfies the algebra (trivial rep).

Can show $\exists$ reps. for this algebra at every $\underset{\sim}{n} = \underset{\sim}{1}, \underset{\sim}{2}, \underset{\sim}{3}, \underset{\sim}{4}, \ldots$

Can also refer to group rep this way, and to basis for rep.

Terminology for groups: $\underset{\sim}{2}$ $\left\{\begin{matrix}\text{fundamental rep} \\ \text{basic rep} \\ \text{vector rep}\end{matrix}\right\}$ of $SU(2)$

$\underset{\sim}{3}$ vector rep of $SO(3)$

$\underset{\sim}{2}$ is also called the "spinor rep. of $SO(3)$", the point being that

rep. of $SO(3)$ are $\underset{\sim}{1}, \underset{\sim}{3}, \underset{\sim}{5}, \underset{\sim}{7}, \ldots$

rep. of $SU(2)$ are $\underset{\sim}{1}, \underset{\sim}{2}, \underset{\sim}{3}, \underset{\sim}{4}, \ldots$

For $SO(N)$ groups, standard procedure that permits "filling in" the "missing" reps.

$\Rightarrow$ $\exists$ another type of group $Spin(N)$ and it happens that $SU(2) \sim Spin(3)$

Basis for $\underset{\sim}{2}$ is a 2-vector $\begin{pmatrix} x \\ y \end{pmatrix}$ = $SU(2)$ vector

= $SO(3)$ spinor

= $Spin(3)$ vector

The basis $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is used for spin up, down.

Algebra $so(3) \sim su(2)$ is familiar from QM,

$\frac{1}{2}\sigma_j$ generators $\Rightarrow$ $[\frac{1}{2}\sigma_j, \frac{1}{2}\sigma_k] = i\,\epsilon_{jkl}\left(\frac{1}{2}\sigma_l\right)$

or, more generally,

$$[J_j, J_k] = i\,\epsilon_{jkl}\, J_l$$

Introduce notion of "Casimir operator" for algebra

$\equiv$ nonlinear function of generators that commutes with $\underline{\text{all}}$ generators

e.g., $su(2)$: $J^2 \equiv J_1^2 + J_2^2 + J_3^2$

satisfies $[J^2, J_j] = 0$

Casimirs are important because they can be used to label reps.

e.g. $su(2)$ can be labeled by eigenstates (= basis for rep) using J^2, J_3 eigenvalues of $|j,m\rangle$.

$$J^2 |j,m\rangle = j(j+1)\,|j,m\rangle$$

$$J_3 |j,m\rangle = m\,|j,m\rangle$$

with $m \in \{-j, \dots, +j\}$, $2j+1$ values.

multiplets: $(2j+1)$	$\underset{\sim}{1}$	$\underset{\sim}{2}$	$\underset{\sim}{3}$	$\underset{\sim}{4}$	$\underset{\sim}{5}$	$\cdots$
"Spin"	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\cdots$
"Casimir" $j(j+1)$	0	$\frac{3}{4}$	2	$\frac{15}{4}$	6	$\cdots$

In general, eigenvalues of Casimirs fix representation, eigenvalues of other operators span the space within the rep.

More than one Casimir is typical

$su(N)$ $N-1$ Casimirs

$so(2N)$, $so(2N+1)$ N Casimirs

There is always a quadratic Casimir, but others may differ.

e.g., $su(3)$ has quadratic and cubic.

Combining Representations in su(2)

Consider 2 systems with spins j_1, j_2.

One basis is the tensor product basis

$$|j_1 m_1\rangle \otimes |j_2 m_2\rangle \equiv |m_1 m_2\rangle$$

Another possible basis uses $\vec{J} = \vec{J}_1 + \vec{J}_2$, which satisfies su(2) algebra as well. Eigenvalues are $J(J+1)$, label eigenstates with M, $2J+1$ values.

Denote this state by $|j_1 j_2 J M\rangle \equiv |J M\rangle$.

Relationship

$$|J M\rangle = \sum_{m_1, m_2} C(m_1 m_2, J M) \, |m_1, m_2\rangle$$

Clebsch-Gordan coefficients

e.g.,

$j_1 = j_2 = \frac{1}{2}$

singlet, antisymmetric state

$$J=0, \; M=0 \qquad |00\rangle = \frac{1}{\sqrt{2}} |\tfrac{1}{2}, -\tfrac{1}{2}\rangle - \frac{1}{\sqrt{2}} |-\tfrac{1}{2}, \tfrac{1}{2}\rangle$$

$$J=1, \; M = \begin{cases} +1 & |11\rangle = |\tfrac{1}{2}, \tfrac{1}{2}\rangle \\ 0 & |10\rangle = \frac{1}{\sqrt{2}} |\tfrac{1}{2}, -\tfrac{1}{2}\rangle + \frac{1}{\sqrt{2}} |-\tfrac{1}{2}, \tfrac{1}{2}\rangle \\ -1 & |1 -1\rangle = |-\tfrac{1}{2}, -\tfrac{1}{2}\rangle \end{cases}$$

triplet, symmetric states

From the SU(2) group, find

$$\underbrace{\tfrac{1}{2}}_{\text{spin}-\frac{1}{2}} \times \tfrac{1}{2} = \underbrace{0}_{\text{spin}-0} + \underbrace{1}_{\text{spin}-1}$$

In group rep. language : $\underset{\sim}{2} \times \underset{\sim}{2} = \underset{\sim}{1} + \underset{\sim}{3}$

Other examples

spin : $1 \times 1 = 0 + 1 + 2$

rep : $\underset{\sim}{3} \times \underset{\sim}{3} = \underset{\sim}{1} + \underset{\sim}{3} + \underset{\sim}{5}$

spin : $\frac{1}{2} \times 1 = \frac{1}{2} + \frac{3}{2}$

reps : $\underset{\sim}{2} \times \underset{\sim}{3} = \underset{\sim}{2} + \underset{\sim}{4}$

spin : $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \left(\frac{1}{2} \times \frac{1}{2}\right) \times \frac{1}{2}$

$$= (0 + 1) \times \tfrac{1}{2}$$

$$= \tfrac{1}{2} + \left(\tfrac{1}{2} + \tfrac{3}{2}\right)$$

$$= \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{3}{2}$$

reps : $\underset{\sim}{2} \times \underset{\sim}{2} \times \underset{\sim}{2} = (\underset{\sim}{2} \times \underset{\sim}{2}) \times \underset{\sim}{2}$

$$= (\underset{\sim}{1} + \underset{\sim}{3}) \times \underset{\sim}{2}$$

$$= \underset{\sim}{2} + (\underset{\sim}{2} + \underset{\sim}{4})$$

$$= \underset{\sim}{2} + \underset{\sim}{2} + \underset{\sim}{4}$$

This idea generalizes to other groups. There are numerous methods to do these calculations. The three common ones: "Cartan matrix", "Dynkin diagrams" and "Young Tableaux"

Introduction to Young Tableaux for SU(2)

Basis for $\underset{\sim}{2}$ of SU(2)

$$u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad d = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow \quad |\tfrac{1}{2}, +\tfrac{1}{2}\rangle, \quad |\tfrac{1}{2}, -\tfrac{1}{2}\rangle$$

Notation:

$$\square \;, \quad u = \boxed{1} \;, \quad d = \boxed{2} \quad \Rightarrow \quad \underset{\underset{\sim}{2}}{\square}$$

represents doublet

tableaux

diagrams

Two particle states

- convention - boxes
  - horizontally: side-by-side = symmetric combo
  - vertically: top & bottom = antisymmetric combo

$$\left.\begin{array}{ll}
\begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array} & uu \\[6pt]
\begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array} & \frac{1}{\sqrt{2}}(ud+du) \\[6pt]
\begin{array}{|c|c|}\hline 2 & 2 \\ \hline\end{array} & dd
\end{array}\right\}\ \text{triplet symmetric} \Rightarrow \underset{\underset{\sim}{3}}{\begin{array}{|c|c|}\hline & \\ \hline\end{array}}$$

$$\begin{array}{|c|}\hline 1 \\ \hline 1 \\ \hline\end{array} = \begin{array}{|c|}\hline 2 \\ \hline 2 \\ \hline\end{array} \equiv 0 \Rightarrow \text{can't antisymmetrize same object}$$

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array} \quad \frac{1}{\sqrt{2}}(ud-du) \Big\}\ \text{singlet} \Rightarrow \underset{\underset{\sim}{1}}{\begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}} \equiv \bullet$$

So now,

$$\underset{\sim}{2} \times \underset{\sim}{2} = \underset{\sim}{1} + \underset{\sim}{3}$$

$$\begin{array}{|c|}\hline \\ \hline\end{array} \quad \begin{array}{|c|}\hline \\ \hline\end{array} = \begin{array}{|c|}\hline \\ \hline \\ \hline\end{array} + \begin{array}{|c|c|}\hline & \\ \hline\end{array}$$

$$= \bullet + \begin{array}{|c|c|}\hline & \\ \hline\end{array}$$

What about 3-particle states?

e.g., $\begin{array}{|c|c|c|}\hline & & \\ \hline\end{array}$, $\begin{array}{|c|c|}\hline & \\ \hline \\ \cline{1-1}\end{array}$, $\cancel{\begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline\end{array}}$ can't antisymmetrize three things in 2 ways.

Multiplicities?

$$\left.\begin{array}{l}\ytableaushort{111} \\ \ytableaushort{112} = \ytableaushort{121} \\ \ytableaushort{122} \\ \ytableaushort{222}\end{array}\right\} \Rightarrow \underset{\underset{\sim}{4}}{\ydiagram{3}}$$

$$\left.\begin{array}{l}\ytableaushort{11,2} \\ \ytableaushort{12,2}\end{array}\right\} \Rightarrow \underset{\underset{\sim}{2}}{\ydiagram{2,1}} = \underset{= \bullet}{\ydiagram{2,1}} = \underset{\underset{\sim}{2}}{\ydiagram{1}}$$

so,

$$\underset{\underset{\sim}{3}}{\ydiagram{2}} \times \underset{\underset{\sim}{2}}{\ydiagram{1}} = \underset{\underset{\sim}{4}}{\ydiagram{3}} + \underset{\underset{\underset{\sim}{2}}{\ydiagram{1}}}{\overset{\shortparallel}{\ydiagram{2,1}}}$$

therefore, 3 particles are

$$\begin{aligned}\ydiagram{1} \times \ydiagram{1} \times \ydiagram{1} &= \left(\ydiagram{2} + \ydiagram{1,1}\right) \times \ydiagram{1} \\ &= \ydiagram{3} + \ydiagram{2,1} + \ydiagram{1} \\ &= \ydiagram{3} + \ydiagram{1} + \ydiagram{1} \\ &= \underset{\sim}{4} + \underset{\sim}{2} + \underset{\sim}{2}\end{aligned}$$

For SU(2), get simple dimension formula

$$\underbrace{\ydiagram{3}\cdots\ydiagram{2}}_{n \text{ boxes}} = \underset{\sim}{n+1}$$

Comments about physical implications of Symmetry

General result: symmetry $\rightarrow$ conservation law

Very useful in particle physics, because have many empirical conservation laws

$$Q,\ L_e,\ L_\mu,\ L_\tau,\ B,\ S, \ldots$$
$$E,\ \vec{P},\ \vec{J}, \ldots$$

$\Rightarrow$ Can hope to understand some aspects of complexity in the SM as consequences of symmetries of interactions. For a continuous symmetry, result is "Noether's theorem"

In the SM, need quantum regime.

Suppose a state $|\psi\rangle$ of some QM system transforms under action of Lie group G as

$$|\psi\rangle \rightarrow |\psi'\rangle = U(g)\,|\psi\rangle$$

rep. of g acting on basis $|\psi\rangle$

$$\Rightarrow \langle\psi| \longrightarrow \langle\psi'| = \langle\psi|\,U(g)^\dagger$$

If $H \to H' = H$ symmetry, the physics is invariant if (1) probabilities are unchanged,

(2) Hamiltonian matrix elements are preserved.

$\Rightarrow$ $U(g)$ is a unitary (or antiunitary) rep of G

Wigner's theorem

Discrete Symmetries

In addition to continuous symmetries, there are a few discrete symmetries far important for understanding SM physics. These are C, P, T.

Operational Definitions

Charge conjugation C : particles $\leftrightarrow$ antiparticles ← definition of applying C.

(3 momenta, spin unchanged)

$$\boxed{C\left(X(\vec{p},s)\right) \to \bar{X}(\vec{p},s)}$$

Parity Inversion P : Spatial inversion ≡ mirror reflection + 180° rotation about axis ⊥ to the mirror.

(3 momenta change, spin unchanged)

$$\boxed{P\left(X(\vec{p},s)\right) \to X(-\vec{p},s)}$$

Time reversal T : change sign of time coordinate

(reverses sign of momenta and spin)

$$\boxed{T\left(X(\vec{p},s)\right) \longrightarrow X(-\vec{p},-s)}$$

For a broad class of theories, C, P, T are NOT independent.

CPT theorem

Under mild conditions (locality, flat spacetime, vacuum exists, finite dim. reps.), any QFT invariant under Lorentz transformations is also invariant under CPT.

Various proofs: Bell, Pauli, Lüders

CPT symmetry $\Rightarrow$ particles and antiparticles have same mass, lifetimes, ...

Let $|X\rangle$ be some particle state, $|\bar{X}\rangle$ the antiparticle state, and $|A'\rangle$ state with spins flipped and momentum unchanged, such that

$$\langle Y|X\rangle \xrightarrow{CPT} \langle \bar{X}'|\bar{Y}'\rangle$$

so,

$$m_x = \langle x | H | x \rangle$$

if $H \xrightarrow{CPT} H_{CPT} = H$ (CPT theorem)

then,
$$m_x = \langle x | H | x \rangle \xrightarrow{CPT} \langle \bar{x}' | H_{CPT} | \bar{x}' \rangle$$
$$= \langle \bar{x}' | H | \bar{x}' \rangle$$
$$= \langle \bar{x} | H | \bar{x} \rangle$$
$$= m_{\bar{x}}$$

$$\Rightarrow \quad m_x = m_{\bar{x}} \qquad \blacksquare$$

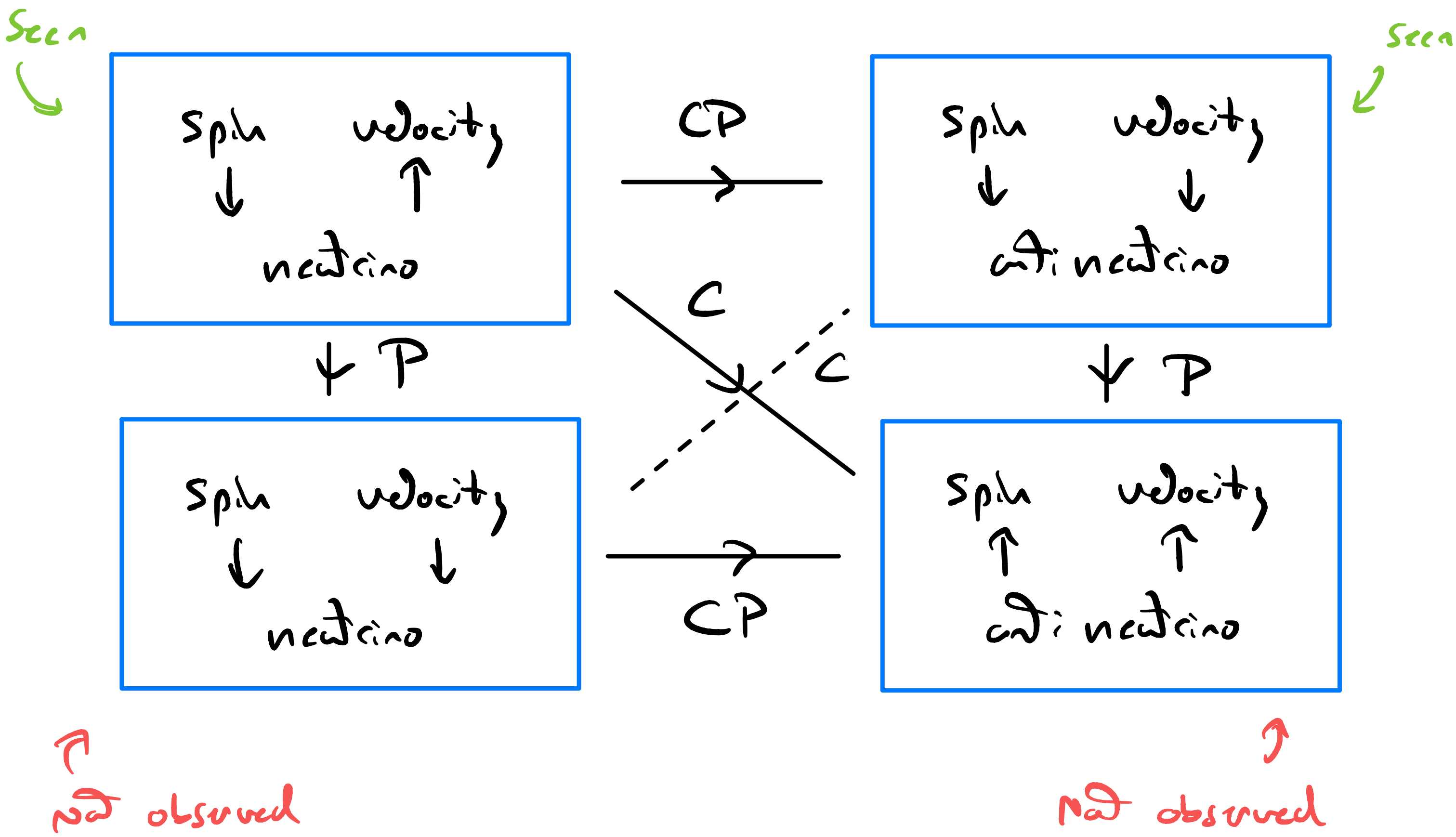
C, P, T properties & fundamental interactions

	Strong	EM	Weak
C	✓	✓	✗
P	✓	✓	✗
T	✓	✓	$\simeq$ (close)
CP	✓	✓	$\simeq$ (close)
CPT	✓	✓	✓

C, P violations in Weak interactions

Consider massless neutrinos (here, only weak ints.)

Experimentally, you can check the helicity of ν & $\bar{\nu}$



At this level, C & P violation, but CP okay...

If CPT is a symmetry, then CP = T invariance

$\Rightarrow$ Enough to discuss CP only because T properties follow.

Both C and P are symmetries that involve one of the simplest finite (discrete) groups.

Group of order 2 (2 elements), e.g.,

$$\{g, e\} \Rightarrow \begin{cases} ee = e & \text{identity} \\ ge = eg = g & \\ gg = e & \end{cases}$$

e.g., real numbers under multiplication: $e = +1$, $g = -1$.

Invariance of physics under action of g

$\Rightarrow$ g represented by unitary operator $U(g)$ such that $[H, U(g)] = 0$

$$\left.\begin{array}{ll} \text{unitary}: & U^{+} = U^{-1} \\ \text{group closure}: & U^{2} = \mathbb{1} \end{array}\right\} \quad U^{+} = U^{-1} = U$$

$\Rightarrow U$ is Hermitian

U observable with conserved eigenvalues,

For eigenstate $|\psi\rangle \Rightarrow U(g)|\psi\rangle = u|\psi\rangle$

then, $U^2|\psi\rangle = u^2|\psi\rangle$ but $U^2|\psi\rangle = |\psi\rangle$

$\Rightarrow u^2 = 1 \Rightarrow u = \pm 1$

Individual particles have intrinsic parity and intrinsic C-parity (If charge neutral).

$$P|X(\vec{p}=\vec{0},s)\rangle = \eta_P |X(\vec{p}=\vec{0},s)\rangle$$
$$C|X^0\rangle = \eta_C |X^0\rangle$$

Note: $\begin{cases}\text{antibosons have same parity as bosons} \\ \text{antifermions have opposite parity as fermions}\end{cases}$