Symmetries II - SU(3)

Recall: A Lie group is a continuous group generated by Lie algebra with elements $\{X_j\}$ such that

$$[X_j, X_k] = C_{jk}{}^{l} X_l .$$

The group elements are given by the exponential map

$$g(\alpha^j) = \exp(\alpha^j X_j), \quad \alpha^j \in \mathbb{R}$$

For Quantum systems, take (conventionally) $\boxed{X_j = -i T_j}$, so that

convention

Hermitian for su(N)

$$[T_j, T_k] = i \underbrace{C_{jk}{}^{l}}_{\text{structure constants}} T_l$$

and

$$g(\alpha^j) = \exp(-i\alpha^j T_j) .$$

For SU(N), g is $N \times N$ complex unitary matrix with $\det = +1$

$\Rightarrow$ Number of generators $= N^2 - 1$

Example su(2) algebra

$$[J_j, J_k] = i \epsilon_{jkl} J_l \quad ; \; j,k,l = 1,2,3$$

fundamental rep $\Rightarrow$ $J_j = \frac{1}{2}\sigma_j$ — Pauli matrices

so, $g(\alpha^j) = \exp\left(-\frac{i}{2}\alpha^j \sigma_j\right)$

All reps:  ...

$\underset{\sim}{1}$ $\underset{\sim}{2}$ $\underset{\sim}{3}$ $\underset{\sim}{4}$ $\underset{\sim}{5}$

We continue discussing aspects of SU(N) groups and su(N) algebras, focusing particularly on SU(3) / su(3)

SU(3) = group of 3×3 unitary matrices with det = +1, ∃ 8 generators.

In the fundamental rep: 3×3 matrices acting on 3D column vector ⇒ $\underset{\sim}{3}$

group element is given by
$$\boxed{U(\alpha^a) = \exp\left(-\tfrac{1}{2} i \alpha^a \lambda_a\right) \quad \text{with } a = 1, \dots, 8}$$

The su(3) generators $\frac{1}{2}\lambda_a$ are 8, 3×3 matrices called the "Gell-Mann" matrices.

By convention, they are

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Note that λ_a is hermitian, $\lambda_a^\dagger = \lambda_a$

and satisfies

$$\mathrm{tr}(\lambda_a \lambda_b) = 2\,\delta_{ab}$$

$$\lambda_a \lambda_a = \frac{16}{3}\, \underbrace{\mathbb{1}_3}_{3\times 3 \text{ identity}}$$

Compare to $su(2)$: $\mathrm{tr}(\sigma_j \sigma_k) = 2\delta_{jk}$ and $\sigma_j \sigma_j = 3\mathbb{1}_2$

Notation: $j, k = 1, 2, 3$

$a, b = 1, \dots, 8$

$$(\lambda_a)_{jk}$$

8 of them

3×3 matrix

Lie algebra $su(3)$

$$\boxed{\left[\tfrac{1}{2}\lambda_a, \tfrac{1}{2}\lambda_b\right] = i f_{abc} \tfrac{1}{2}\lambda_c}$$

Compare to $su(2)$: $\left[\tfrac{1}{2}\sigma_j, \tfrac{1}{2}\sigma_k\right] = i\epsilon_{jkl}\sigma_l$

Nonvanishing structure constants (exercise)

$$f_{123} = 1$$

$$f_{147} = f_{165} = f_{246} = f_{257} = f_{345} = f_{376} = \frac{1}{2}$$

$$f_{458} = f_{678} = \frac{\sqrt{3}}{2}$$

Others are zero unless obtained by interchange.

f_{abc} are antisymmetric under interchange of any two indices (exercise)

The λ_a matrices are the $\underset{\sim}{3}$ of SU(3)

It is also true that

$$\{\lambda_a, \lambda_b\} = \frac{4}{3}\delta_{ab}\mathbb{1} + 2 d_{abc}\lambda_c$$

where d_{abc} are symmetric under interchange of any two indices

Note: this is for λ_a, not $\frac{1}{2}\lambda_a$.

$\boxed{\text{Compare to } su(2): \quad \{\sigma_j, \sigma_k\} = 2\delta_{jk}}$

There exists another inequivalent 3×3 representation of su(3) denoted $\underset{\sim}{3}^*$ (or $\bar{\underset{\sim}{3}}$)

To get it, take group element $U \in SU(3)$ for $\underset{\sim}{3}$ and complex conjugate it: U^*

- U^* still obeys $(U^*)^\dagger (U^*) = \mathbb{1}$
  and also $\det U^* = +1$

Issue: Is U^* different from U?

Suppose $V \to V' = UV$ $\underset{\sim}{3}$

then, $V^* \to V'^* = U^* V^*$ $\underset{\sim}{3}^*$

If $\exists\, S \ni SU^*S^{-1} = U$ "Similarity transformation"

then,
$$(SV^{*\prime}) = (SU^*S^{-1})(SV^*)$$
$$= U(SV^*)$$

$\Rightarrow$ SV^* transforms like V

But SV^* is just linear combo of components of V^*

$\Rightarrow$ Linear combo of V^* behaves like V

$\Rightarrow$ Not different

So, U^* is different from U if cannot find S such that $SU^*S^{-1} = U$

or,

U^* is "equivalent" to U if $SU^*S^{-1} = U$ for some S

Claim: U^* is inequivalent to U

Check: $U = \exp(-\frac{1}{2} i \alpha^a \lambda_a)$

$\Rightarrow$ $U^* = \exp(+\frac{1}{2} i \alpha^a \lambda_a^*)$

Sufficient to show $(-\lambda_a^*)$ cannot be transformed to λ_a by a unitary transformation. (exercise)

In general, the $\underset{\sim}{N}$ representation of $SU(N)$ is inequivalent to the $\underset{\sim}{N}^*$ for all $N \geq 3$.

But, for $N=2$, the $\underset{\sim}{2}^*$ is equivalent to $\underset{\sim}{2}$.

Proof

for $SU(2)$, $U = \exp(-\frac{1}{2} i \alpha^i \sigma_i)$

$\Rightarrow$ $U^* = \exp(\frac{1}{2} i \alpha^i \sigma_i^*)$

So, we seek $S \ni S(-\sigma_i^*)S^{-1} = \sigma_i$

Claim: $S = \pm i\sigma_2$ works, $S^{-1} = \mp i \sigma_2$

Check: for σ_1, find $(\pm i\sigma_2)(-\sigma_1^*)(\mp i\sigma_2)$

$$= \sigma_2(-\sigma_1)\sigma_2$$
$$= -\sigma_2\sigma_1\sigma_2$$
$$= -\sigma_2(i\sigma_3)$$
$$= +\sigma_1 \quad ✓$$

Exercise to check σ_2, σ_3.

So,

$$\underset{\sim}{2} = \begin{pmatrix} u \\ d \end{pmatrix} \rightarrow U \begin{pmatrix} u \\ d \end{pmatrix}$$

transforms the same way as $\underset{\sim}{2}^*$

$$\underset{\sim}{2}^* = \pm i\sigma_2 \begin{pmatrix} u^* \\ d^* \end{pmatrix} = \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} u^* \\ d^* \end{pmatrix}$$

$$= \pm \begin{pmatrix} d^* \\ -u^* \end{pmatrix} \rightarrow \pm U \begin{pmatrix} d^* \\ u^* \end{pmatrix}$$

Not U^* !

Representations of su(3)

So far, see su(3) has reps $\underset{\sim}{1}$, $\underset{\sim}{3}$, $\underset{\sim}{3}^*$

What about other reps?

Don't have "simple" dimensions like su(2) ($\underset{\sim}{1}, \underset{\sim}{2}, \underset{\sim}{3}, \underset{\sim}{4}, \ldots$)

Let us use Young tableaux to find them

$\boxed{}$ $\qquad\qquad$ $\boxed{1}$ $\quad$ $\boxed{2}$ $\quad$ $\boxed{3}$

$\underset{\sim}{3}$

Young Tableaux $\qquad\qquad$ Young Diagrams

Let's look at $\underset{\sim}{3} \times \underset{\sim}{3}$

$$\underset{\underset{\sim}{3}}{\ydiagram{1}} \times \underset{\underset{\sim}{3}}{\ydiagram{1}} = \ydiagram{1,1} + \ydiagram{2}$$

For $\ydiagram{1,1}$, we have 3 Young diagrams

$$\begin{ytableau} 1 \\ 2 \end{ytableau}\,, \quad \begin{ytableau} 1 \\ 3 \end{ytableau}\,, \quad \begin{ytableau} 2 \\ 3 \end{ytableau}$$

So, this is another 3-dim rep, it is the $\underset{\sim}{3}^*$

The symmetric combination $\ydiagram{2}$ is the $\underset{\sim}{6}$

check:

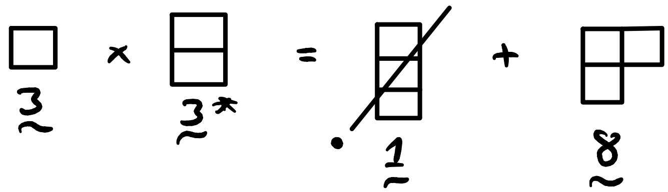
$$\left.\begin{matrix} \begin{ytableau} 1 & 1 \end{ytableau} & \begin{ytableau} 2 & 2 \end{ytableau} \\ \begin{ytableau} 1 & 2 \end{ytableau} & \begin{ytableau} 2 & 3 \end{ytableau} \\ \begin{ytableau} 1 & 3 \end{ytableau} & \begin{ytableau} 3 & 3 \end{ytableau} \end{matrix}\right\} \quad \underset{\underset{\sim}{6}}{\ydiagram{2}}$$

$$\Rightarrow \quad \underset{\underset{\sim}{3}}{\ydiagram{1}} \times \underset{\underset{\sim}{3}}{\ydiagram{1}} = \underset{\underset{\sim}{3}^*}{\ydiagram{1,1}} + \underset{\underset{\sim}{6}}{\ydiagram{2}}$$

What about $\ydiagram{1,1,1}$? $\begin{ytableau} 1 \\ 2 \\ 3 \end{ytableau} = \bullet \quad \underset{\sim}{1}$ singlet of su(3)

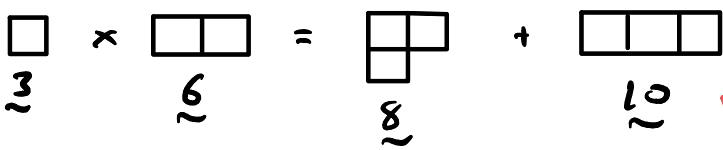
Can look at further reps,

- $\underset{\sim}{3} \times \underset{\sim}{3}^* = \underset{\sim}{1} + \underset{\sim}{8}$



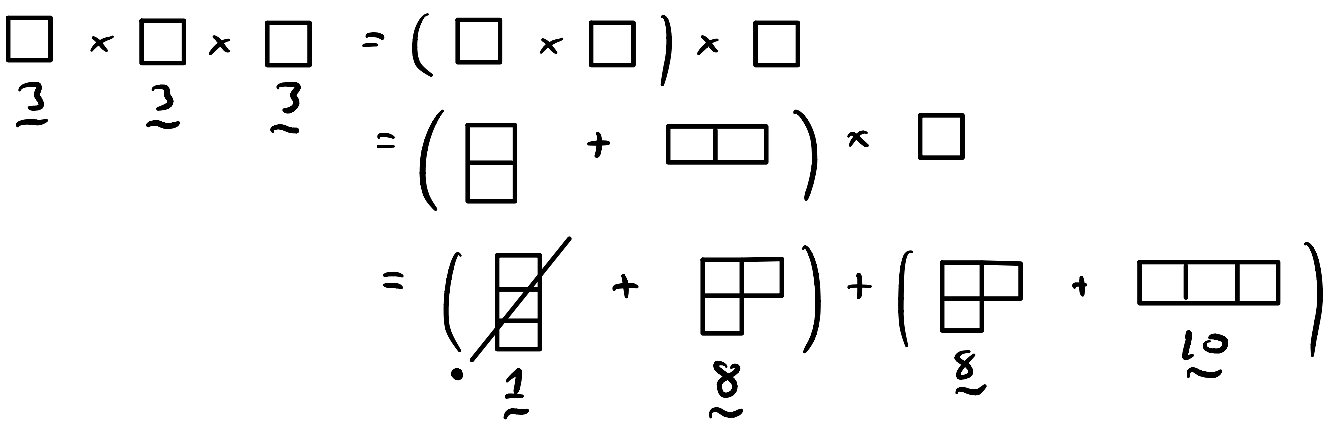
← check

- $\underset{\sim}{3} \times \underset{\sim}{6} = \underset{\sim}{8} + \underset{\sim}{10}$



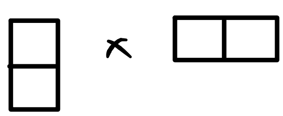
← check

- $\underset{\sim}{3} \times \underset{\sim}{3} \times \underset{\sim}{3} = \underset{\sim}{1} + \underset{\sim}{8} + \underset{\sim}{8} + \underset{\sim}{10}$

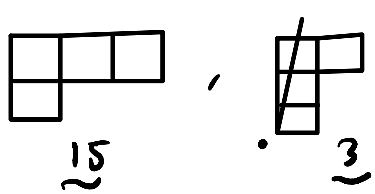


How do we compute more complicated reps?

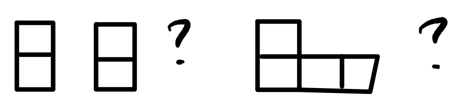
e.g., $\underset{\sim}{3}^* \times \underset{\sim}{6} =$



Could have



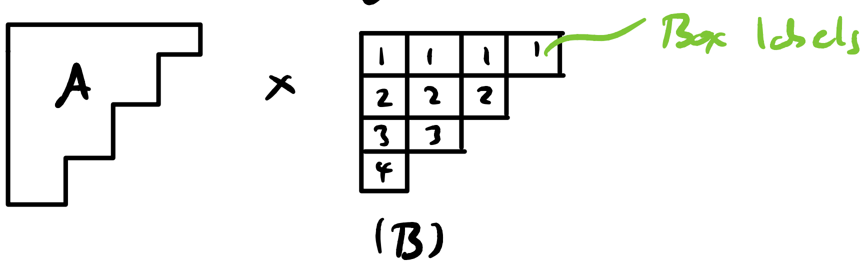
, what about



Rules for shape of Young Tableaux for su(3)

1. No row can be shorter than a lower row
2. No column can be shorter than a column to its right
3. No column can have more than 3 boxes (su(3))

Guide to multiplying two tableaux

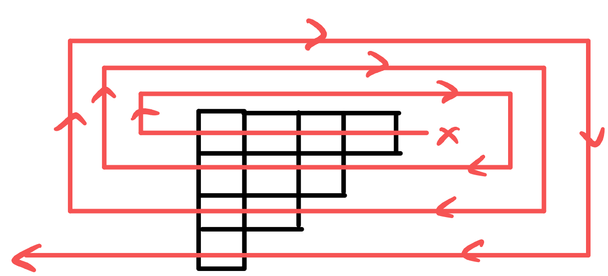


Take Tableux A and add boxes one-by-one from Tableux B, keeping correct shape and obeying 3 language rules

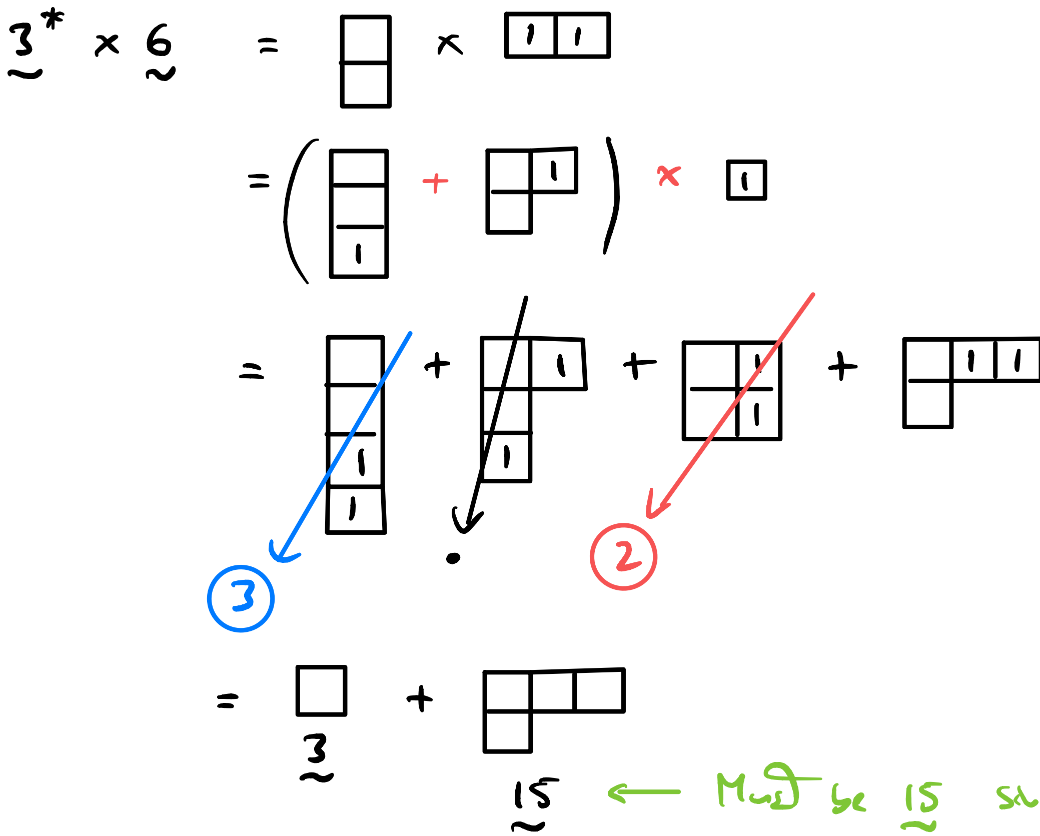
1. From Left to Right, indices must not decrease
2. From top to bottom, indices must decrease
3. From Right to Left in continuous path,

$\# 1s \geq \# 2s \geq \# 3s \geq \cdots$

at each point in path



Example

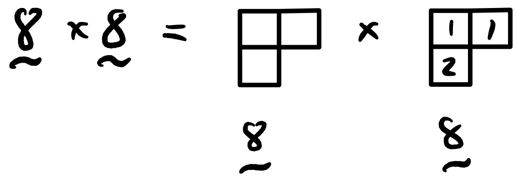


← Must be 15 since $3 \times 6 = 18$ and $3 + 15 = 18$ ✓

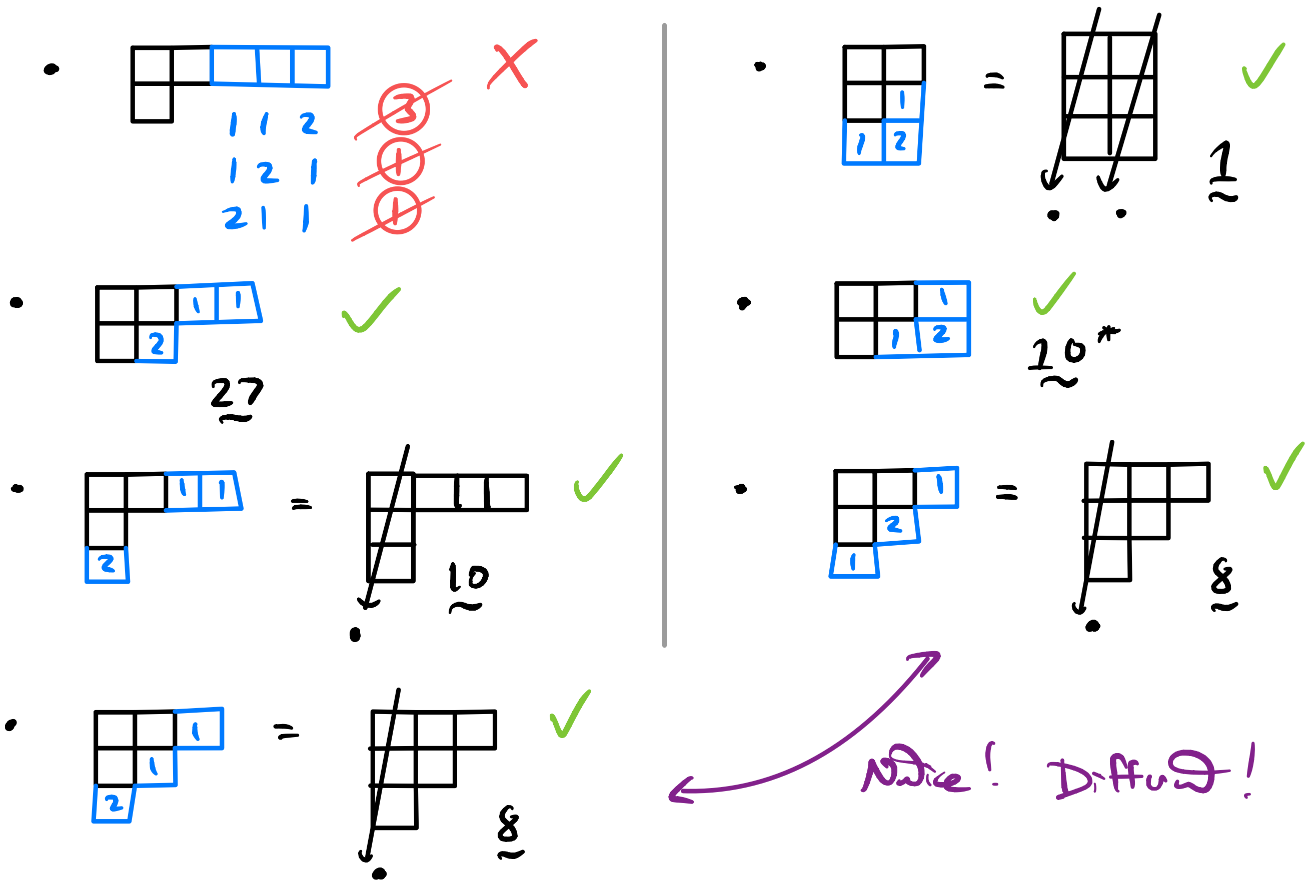
Is there any easy way to get dimension?!

— Yes! See later

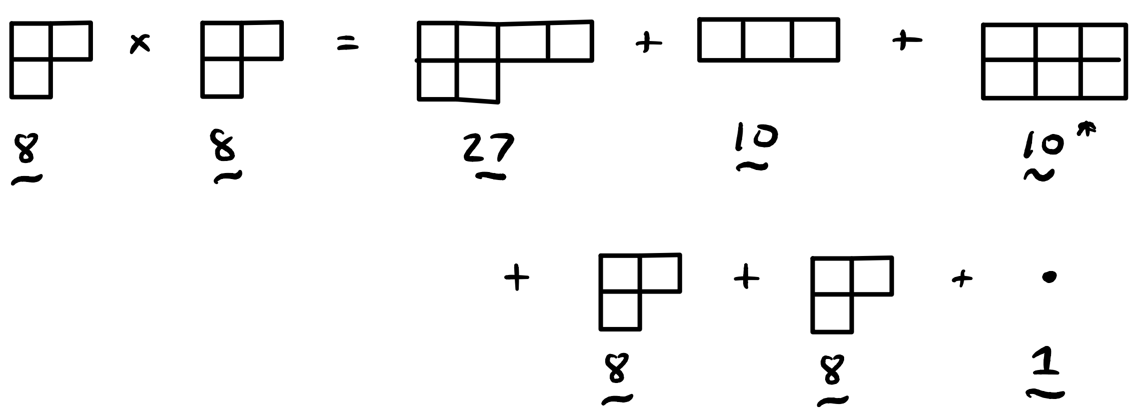
Example



let's look at options



So,



SU(3) Dimensionality Formula

To find the dimension of an SU(3) Tableaux, let

$a_1 =$ # of boxes 1st row exceeds 2nd

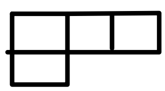
$a_2 =$ # of boxes in 2nd row

Then,

$$\boxed{N(a_1, a_2) = \frac{1}{2}(a_1+1)(a_2+1)(a_1+a_2+2)}$$

SU(3) only...

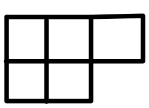
Example



$a_1 = 2, \ a_2 = 1$

$$\Rightarrow N(2,1) = \frac{1}{2}(3)(2)(5)$$
$$= 15 \checkmark$$

Example

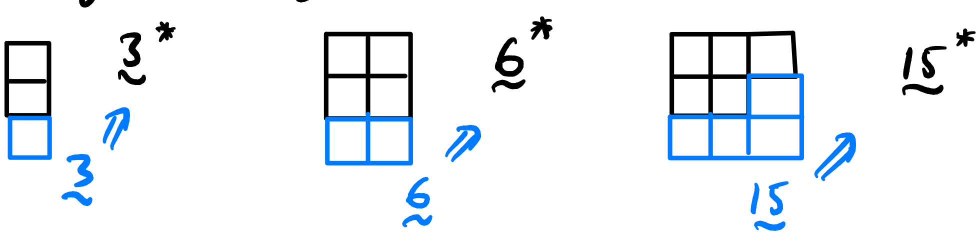


$a_1 = 1, \ a_2 = 2$

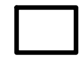
$$\Rightarrow N(1,2) = \frac{1}{2}(2)(3)(5)$$
$$= 15^*$$

How do we know?

Image gives conjugate rep – Make complete box 3 tall

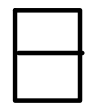


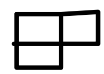
Explicit forms of some su(3) Reps

We have $\underset{\sim}{3} \longleftrightarrow \frac{1}{2}(\lambda_a)_{jk} \longleftrightarrow$  $\underset{\sim}{3}$

8 matrices

3×3

$\underset{\sim}{3}^* \longleftrightarrow -\frac{1}{2}(\lambda_a)_{jk} \longleftrightarrow$  $\underset{\sim}{3}^*$

For the $\underset{\sim}{N^2-1}$ of $su(N)$ [for $su(3)$, $\underset{\sim}{8}$ = ] there is a trick. This is the adjoint rep of $su(N)$

denote it by $(T_a)_{bc}$

8 matrices

8×8

Claim: $\boxed{(T_a)_{bc} = -C_{abc}}$, w/ $[X_a, X_b] = C_{abc} X_c$

where $\sum_{(a,b,d)} C_{abe} C_{edf} = 0$

(Jacobi)

Check:

$$\left([T_a, T_b]\right)_{df} = (T_a)_{de}(T_b)_{ef} - (T_b)_{de}(T_a)_{ef}$$

$$= +C_{ade}C_{bef} - C_{bde}C_{aef}$$

$$= -C_{dae}C_{bef} - C_{bde}C_{aef}$$

antisym

Jacobi

$$= +C_{abe}C_{def}$$

$$= -C_{abe}C_{edf}$$

$$= +C_{abe}(T_e)_{df}$$ ∎

For $su(3)$, structure constants are $i f_{abc}$

$$\Rightarrow (T_a)_{bc} = -i f_{abc} \quad \text{for } \underset{\sim}{8} \text{ of } su(3)$$

$\boxed{\begin{array}{l}\text{Compare to } su(2)\text{: structure constants are } i\epsilon_{jkl} \\ \qquad \Rightarrow (T_j)_{kl} = -i\epsilon_{jkl} \quad \underset{\sim}{3} \quad \text{“}L_j\text{”}\end{array}}$

Constructing an arbitrary rep is non-trivial, but there are methods for doing so.