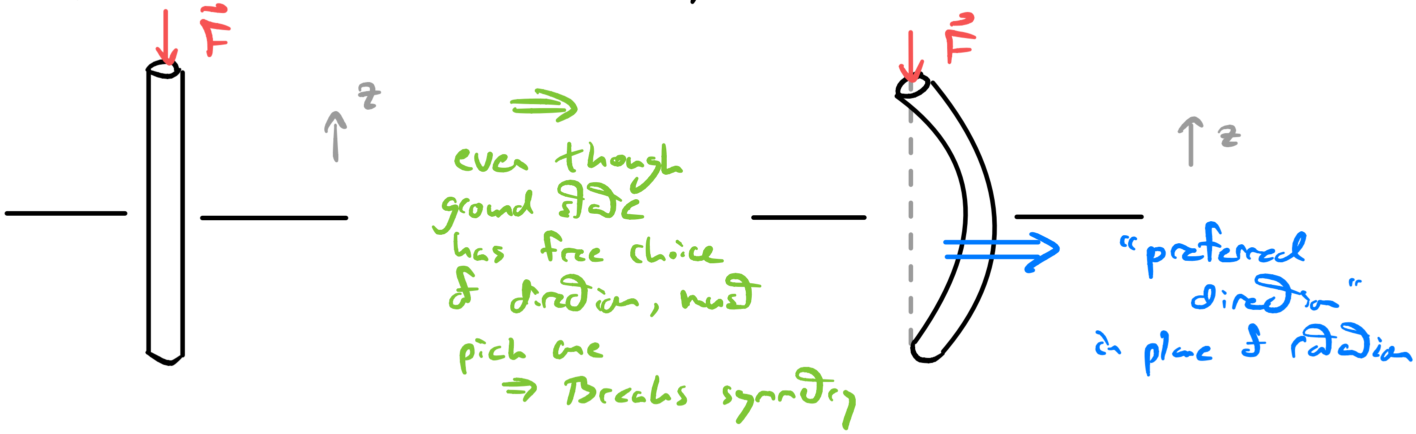
Spontaneous Symmetry Breaking

The procedure in constructing gauge theories has been successful for both electromagnetic and strong forces. In attempting this construction for the weak interaction, we arrive at a problem in that the suspected gauge bosons, the $W^{\pm}$ and Z^0, are massive. We know from QED & QCD that mass terms are not allowed as they break gauge invariance,

e.g.,
$$m_A^2 A_\mu A^\mu \underset{U(1)}{\rightarrow} m_A^2 \left(A_\mu - \tfrac{1}{e}\partial_\mu \alpha\right)\left(A^\mu - \tfrac{1}{e}\partial^\mu \alpha\right)$$
$$\neq m_A^2 A_\mu A^\mu .$$

There is a mechanism, however, to induce mass terms as a consequence of interactions. The underlying feature is that the vacuum is not invariant under the symmetry.

Simple example – Compress a cylindrical stick



This is called "spontaneous symmetry breaking" or "hidden symmetry". Let's start with a simple field theory with discrete symmetry breaking.

Discrete Symmetry Breaking

Real classical scalar field

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi \mp \frac{1}{2}\mu^2\varphi^2 - \frac{\lambda}{4!}\varphi^4 \quad , \lambda > 0$$

two possible signs

- upper sign $\Rightarrow$ μ = mass
- lower sign $\Rightarrow$ apparent instability "tachyonic" mass $i\mu$

then,

$$\mathcal{H} = \frac{1}{2}(\partial_0\varphi)^2 - \frac{1}{2}(\vec{\nabla}\varphi)^2 + U(\varphi)$$

with potential

$$U(\varphi) = \pm\frac{1}{2}\mu^2\varphi^2 + \frac{\lambda}{4!}\varphi^4$$

$$= \frac{\lambda}{4!}(\varphi^2 \pm a^2)^2 - \frac{\lambda}{4!}a^4$$

$a = \sqrt{\frac{6\mu^2}{\lambda}}$

This theory has a discrete symmetry

$$\varphi \to -\varphi$$

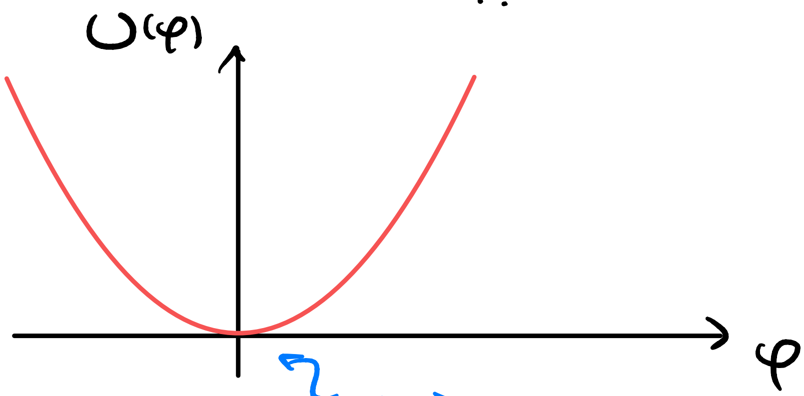
Lowest energy state, "vacuum", has $\varphi = \text{const.}$,

denoted $\langle \varphi \rangle \equiv \langle 0 | \varphi | 0 \rangle$

↳ vacuum expectation value

$\langle \varphi \rangle$ is determined by minimum of $U(\varphi)$.

- For upper sign, $U(\varphi) = \frac{\lambda}{4!}(\varphi^2 + a^2)^2 - \frac{\lambda}{4!}a^4$



$\langle \varphi \rangle = 0$ is min. of $U(\varphi)$ and H.

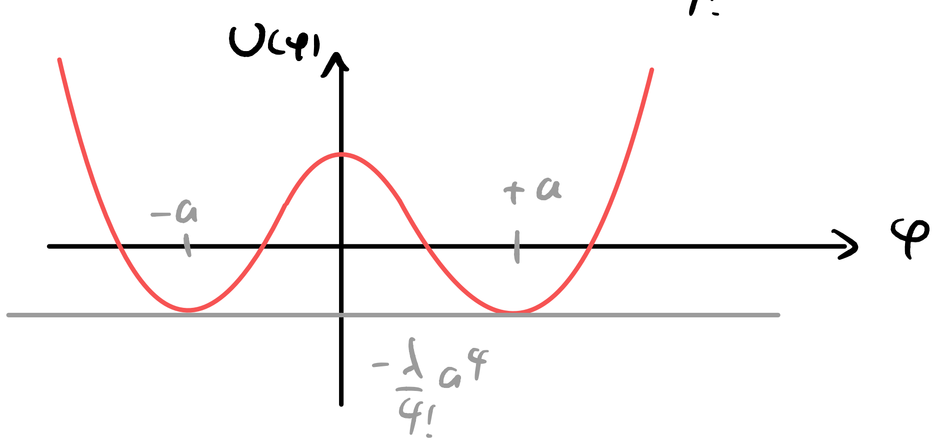
In this case, symmetry is evident in ground state

⇒ Quantization gives usual SHO states,

find particle of mass μ.

- For lower sign, $U(\varphi) = \frac{\lambda}{4!}(\varphi^2 - a^2)^2 + \text{const.}$

↳ irrelevant for physics

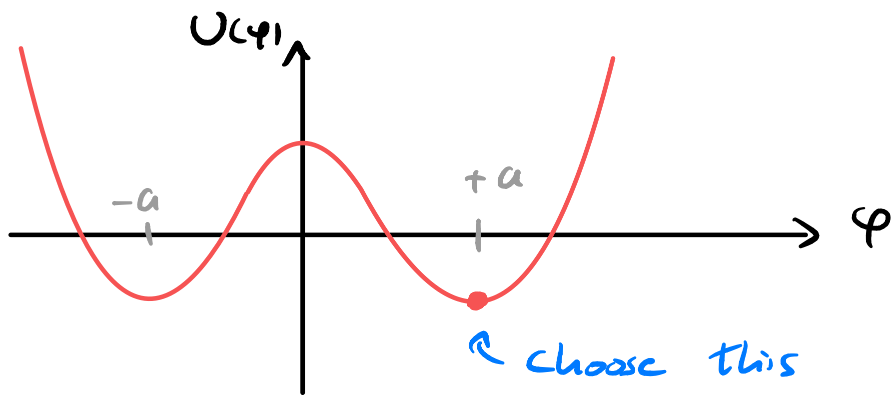


Now find 3 extrema : $\langle\varphi\rangle = 0 \, , \pm a$

There are 2 minima. Due to symmetry, choice of $\langle\varphi\rangle = +a$ or $\langle\varphi\rangle = -a$ is irrelevant.

But, must pick one!

$\Rightarrow$ Spontaneously break discrete symmetry



To have QFT, and perturbation theory, must choose stable minimum. Pick $\langle\varphi\rangle = +a$

Then, can write $\quad \varphi(x) = a + \rho(x)$

$$\Rightarrow U(\varphi) \to U(\rho) = \frac{\lambda}{4!}\left(2a\rho + \rho^2\right)^2 + \text{const.}$$

$$= \underbrace{\frac{\lambda a^2}{6}}_{\mu^2}\rho^2 + \frac{\lambda a}{6}\rho^3 + \frac{\lambda}{4!}\rho^4$$

$$= \mu^2\rho^2 + \frac{\lambda a}{6}\rho^3 + \frac{\lambda}{4!}\rho^4$$

So, in terms of perturbative field $\rho(x)$,

we have particles with mass $\underline{\sqrt{2}\mu}$

$$U(\rho) = \frac{1}{2}(2\mu^2)\rho^2 + \frac{\lambda a}{6}\rho^3 + \frac{\lambda}{4!}\rho^4$$

mass term

cubic interaction

But, also have a cubic interaction

⇒ Difficult to see discrete symmetry!

$\varphi \to -\varphi \Rightarrow a+\rho \to -a-\rho$

$\Rightarrow a \to -a,\ \rho \to -\rho$

The 2 forms of $\mathcal{L}$ are equivalent, but a simple field transformation seems to be changing physics!

Total physics is the same, but cannot solve full theory.

1st form of $\mathcal{L}$

⇒ Nonconvergent perturbation theory

⇒ Correct picture of physics

⇒ Shifted field around true vacuum $|a\rangle$ gives perturbation theory.

Continuous Symmetry Breaking

Consider complex scalar field,

$$\mathcal{L} = \partial_\mu \varphi^* \partial^\mu \varphi + \mu^2 \varphi^* \varphi - \frac{\lambda}{3!} (\varphi^* \varphi)^2$$

Note! wrong sign

Recall: $\varphi = \frac{1}{\sqrt{2}}(\varphi_1 + i\varphi_2)$, $\varphi^* = \frac{1}{\sqrt{2}}(\varphi_1 - i\varphi_2)$ $\Big\}$ $\varphi^\dagger \varphi = \frac{1}{2}(\varphi_1^2 + \varphi_2^2)$

So,

$$\mathcal{L} = \frac{1}{2}\partial_\mu \varphi_1 \partial^\mu \varphi_1 + \frac{1}{2}\partial_\mu \varphi_2 \partial^\mu \varphi_2 + \frac{1}{2}\mu^2(\varphi_1^2 + \varphi_2^2) - \frac{\lambda}{4!}(\varphi_1^2 + \varphi_2^2)^2$$

therefore,

$$U(\varphi_1, \varphi_2) = -\frac{1}{2}\mu^2(\varphi_1^2 + \varphi_2^2) + \frac{\lambda}{4!}(\varphi_1^2 + \varphi_2^2)^2$$

$$= \frac{\lambda}{4!}(\varphi_1^2 + \varphi_2^2 - a^2)^2 + \frac{\lambda}{4!} a^4$$

$a^2 = \frac{6\mu^2}{\lambda}$

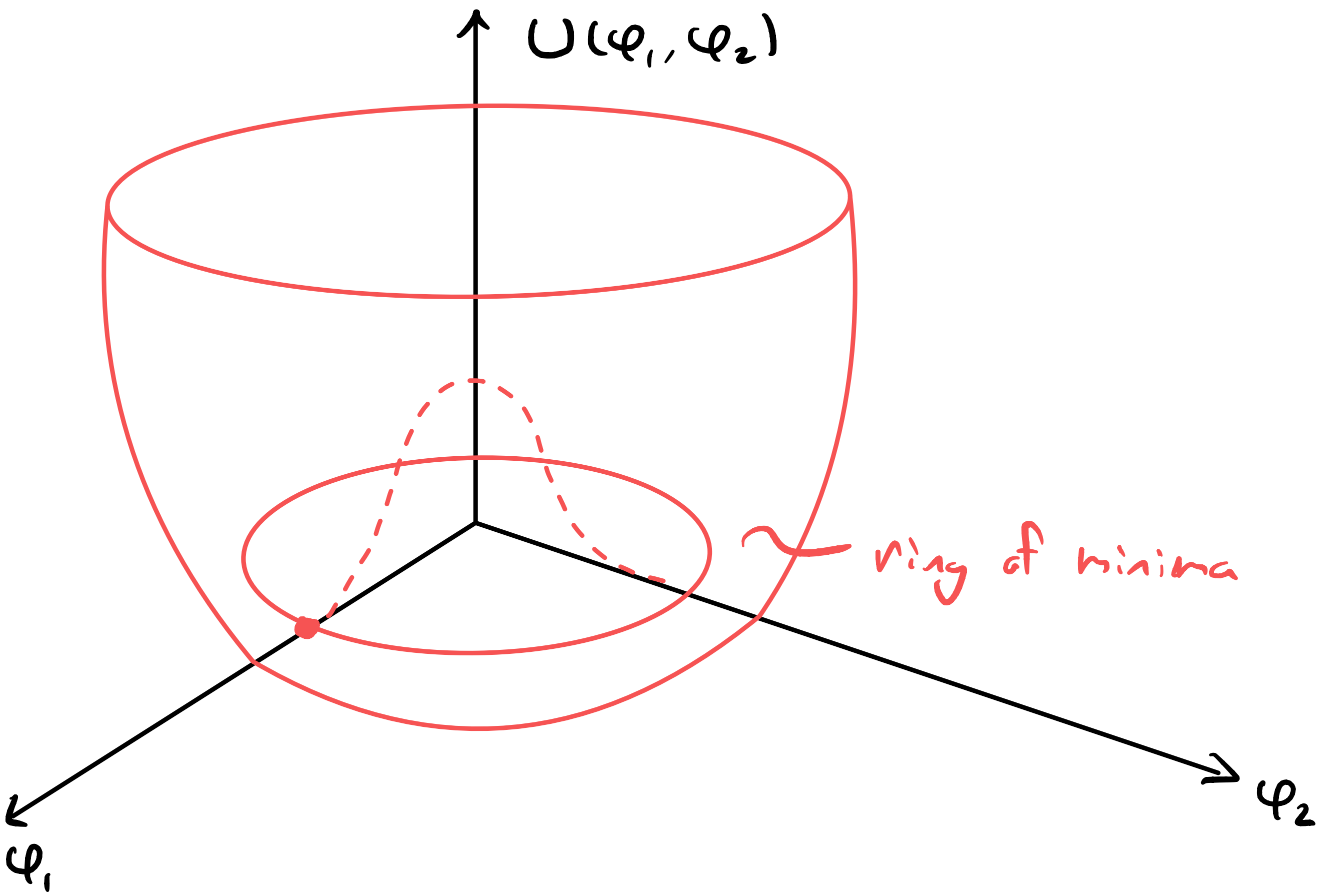
This theory has a global $U(1)$ invariance

$$\varphi \to \varphi' = e^{i\Lambda}\varphi \quad , \Lambda \in \mathbb{R} \text{ const.}$$

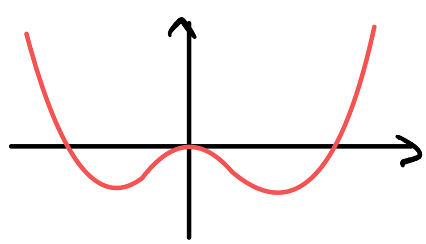
or,

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \to \begin{pmatrix} \varphi_1' \\ \varphi_2' \end{pmatrix} = \begin{pmatrix} \cos\Lambda & -\sin\Lambda \\ \sin\Lambda & \cos\Lambda \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$$

The potential surface looks like



Any vertical section



As before, choice of vacuum is irrelevant to physics, but must be made.

Ring of minima at $\varphi_1^2 + \varphi_2^2 = a^2$

$\Rightarrow$ choose $\langle\varphi_1\rangle = a$, $\langle\varphi_2\rangle = 0$ for definiteness

Spontaneously broken U(1) symmetry

Shift fields to minimum

$$\varphi_1(x) = a + \rho_1(x)$$

$$\varphi_2(x) = \rho_2(x)$$

$$\Rightarrow U(\rho_1, \rho_2) = \frac{\lambda}{4!}\left(2a\rho_1 + \rho_1^2 + \rho_2^2\right)^2$$

$$= \frac{\lambda a^2}{6}\rho_1^2 + \frac{\lambda a}{6}\rho_1(\rho_1^2 + \rho_2^2) + \frac{\lambda}{4!}(\rho_1^2 + \rho_2^2)^2$$

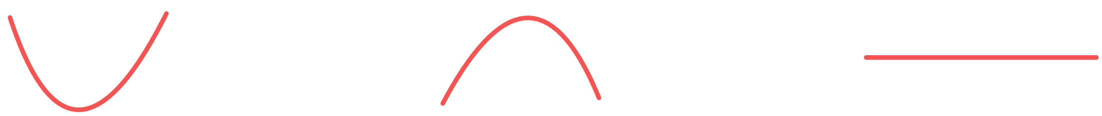
↳ mass term, $\sqrt{2}\mu$, for ρ_1

Notice how hidden symmetry is in this case.

Notice also no mass term for ρ_2!

Intuitively: "flat" direction in potential

$\Rightarrow$ zero mass mode



mass μ mass $i\mu$ mass 0

The appearance of a massless mode is a characteristic of broken continuous symmetry.

To see in a different way, define

$$\varphi(x) = \frac{1}{\sqrt{2}}\, r(x)\, e^{i\theta(x)}$$

or,
$$\varphi_1(x) = r(x) \cos\theta(x)$$
$$\varphi_2(x) = r(x) \sin\theta(x)$$

in terms of 2 other real fields $r(x), \theta(x)$

$$\Rightarrow \quad \mathcal{L} = \frac{1}{2}\partial_\mu r \partial^\mu r + \frac{1}{2} r^2 \partial_\mu\theta \partial^\mu\theta - U(r)$$

$U(1)$ invariance $\Leftrightarrow$ $U(r,\theta) = U(r)$ only

with $U(r) = \frac{\lambda}{4!}\left(r^2 - a^2\right)^2 + \text{const.}$

Take vacuum at $\langle r \rangle = a$, $\langle \theta \rangle = 0$

Define shifted fields

$$r = a + \rho$$

$$\theta = \theta$$

$$\Rightarrow \mathcal{L} = \frac{1}{2} \partial_\mu \rho \, \partial^\mu \rho + \frac{1}{2} (a+\rho)^2 \partial_\mu \theta \, \partial^\mu \theta - U(a+\rho)$$

Can see that $\theta(x)$ particles are massless because $U(a+\rho)$ independent of θ.

These massless modes are called "Goldstone bosons" or "Nambu-Goldstone bosons".

Consider another example with more than one NG boson.

N real scalar fields $\varphi_j(x)$, $j = 1, \dots, N$

$$\mathcal{L} = \frac{1}{2} \partial_\mu \varphi_j \partial^\mu \varphi_j + \frac{1}{2} \mu^2 \varphi_j \varphi_j - \frac{\lambda}{4!} (\varphi_j \varphi_j)^2$$

$\hookrightarrow$ Notice!

$$\Rightarrow U(\varphi_j) = -\frac{1}{2} \mu^2 \varphi_j \varphi_j + \frac{\lambda}{4!} (\varphi_j \varphi_j)^2$$

$$= \frac{\lambda}{4!} (\varphi_j \varphi_j - a^2)^2 + \text{irrelevant const.}$$

$\hookrightarrow$ $a = \sqrt{\frac{6\mu^2}{\lambda}}$

This theory is invariant under global $O(N)$ rotations

$$\varphi_j \to \varphi_j' = U_{jk}\varphi_k$$

$\hookrightarrow U_{jk} \in O(N)$, $UU^T = \mathbb{1}$

$O(N)$ has $\frac{1}{2}N(N-1)$ generators

$\Rightarrow$ corresponding α^a real const. parameters

Minima of $U(\varphi_i)$ lie on surface $\varphi_i\varphi_j = a^2$.

This is a generalization of ring of minima in $U(1)$ example where $N=2$ & minima formed S^1.

Here, have $O(N)$ and minima form S^{N-1}

Pick a vacuum : $\langle \varphi_N \rangle = a$

$\langle \varphi_j \rangle = 0$ for $j = 1, \dots, N-1$

New feature in this example, vacuum is invariant under a nontrivial continuous symmetry group $O(N-1)$

Shift as before

$$\varphi_N = a + \rho_N(x)$$

$$\varphi_j = \rho_j(x) \quad \text{for } j = 1, \dots, N-1$$

So we find

$$U(\rho_i) = \frac{\lambda}{4!}\left(2a\rho_N + \rho_i\rho_i\right)^2$$

$$= \underbrace{\frac{\lambda a^2}{6}}_{\mu^2}\rho_N^2 + \frac{\lambda a}{6}\rho_N(\rho_i\rho_i) + \frac{\lambda}{4!}(\rho_i\rho_i)^2$$

We see that ρ_N has acquired mass $\sqrt{2}\mu$, but the other $N-1$ particles are NG modes.

Notice that

Number of generators of $O(N) = \frac{1}{2}N(N-1)$

Number of generators of $O(N-1) = \frac{1}{2}(N-1)(N-2)$

$\Rightarrow$ Difference $= N-1$

Number of NG bosons $= N-1$

Not a coincidence

Can understand this as follows:

1. Have theory with potential U invariant under some continuous group G with N generators

2. Vacuum solution is invariant under $H \subset G$ with M generators.

$\Rightarrow$ $N-M$ generators transform the vacuum nontrivially

3. Passing through $\langle \varphi \rangle$ is an $(N-M)$-dimensional surface of constant U.

4. $\Rightarrow$ must be $N-M$ massless modes

= NG modes, one for each dimension.

The general result is called Goldstone's theorem

Goldstone's Theorem

Given a local, positive definite, & Lorentz invariant field theory with a local conserved current j^μ with an associated charge Q Not annihilating the vacuum, then the theory contains massless modes (NG)

This naively looks useless as there are no massless bosons observed in nature. But, there are two interesting cases not covered by usual axioms.

- Chiral symmetry of QCD is spontaneously broken,

$$SU(3)_L \times SU(3)_R \to SU(3)_V$$

So, expect $3^2 - 1 = 8$ Goldstones bosons. But, this symmetry is approximate, so explicit breaking gives masses to the NG modes. These hadrons, the pseudoscalar octet, are unnaturally light compared with other hadrons $\Rightarrow$ They play a special role in chiral symmetry breaking.

- Gauge theories are acceptable field theories that don't obey usual axioms $\to$ Cannot make gauge choice with both positive definiteness & Lorentz invariance. Also, theorem says nothing of observability of the NG modes.

Higgs Mechanism

Next we consider SSB of a local symmetry.

Consider Scalar Electrodynamics with "wrong sign" mass term, which is known as the Abelian Higgs Model.

$$\mathcal{L} = (D_\mu \varphi)^* (D^\mu \varphi) + \mu^2 \varphi^* \varphi - \frac{\lambda}{3!} (\varphi^* \varphi)^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

Notice

where $D_\mu \varphi = \partial_\mu \varphi + i q A_\mu \varphi$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

This theory is locally invariant under U(1) gauge transformations

$$\varphi \to e^{i\alpha(x)} \varphi$$

$$D_\mu \varphi \to e^{i\alpha(x)} \varphi$$

$$A_\mu \to A_\mu - \frac{1}{q} \partial_\mu \alpha$$

Repeat earlier discussion on "polar" fields

$$\varphi(x) = \frac{1}{\sqrt{2}} r(x) e^{i\theta(x)}$$

$$\Rightarrow D_\mu \varphi = \frac{1}{\sqrt{2}} (\partial_\mu r + i r \partial_\mu \theta + i q r A_\mu) e^{i\theta}$$

So,
$$(D_\mu \varphi)^* (D^\mu \varphi) = \frac{1}{2} \partial_\mu r \partial^\mu r + \frac{1}{2} r^2 (\partial_\mu \theta + q A_\mu)^2$$

So,

$$\mathcal{L} = \frac{1}{2}\partial_\mu r\,\partial^\mu r + \frac{1}{2} r^2 (\partial_\mu \theta + q A_\mu)^2 - \frac{\lambda}{4!}(r^2 - a^2)^2 - \frac{1}{4} F_{\mu\nu}F^{\mu\nu}$$

with $a = \sqrt{\frac{6\mu^2}{\lambda}}$ + const.

The minimum of potential give vacua at

$$\langle A_\mu \rangle = 0 \,, \quad \langle \theta \rangle = 0 \,, \quad \langle r \rangle = a$$

Define shifted fields

$$A_\mu(x) = A_\mu(x)$$
$$\theta(x) = \theta(x)$$
$$r(x) = a + \rho(x)$$

$$\Rightarrow \mathcal{L} = \frac{1}{2}\partial_\mu \rho\, \partial^\mu \rho - \frac{1}{2}(a+\rho)^2(\partial_\mu\theta + qA_\mu)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$
$$- \frac{\lambda a^2}{6}\rho^2 - \frac{\lambda a}{6}\rho^3 - \frac{\lambda}{4!}\rho^4 + \text{const.}$$

mass term!

Notice that $\frac{1}{2}(a+\rho)^2(\partial_\mu\theta + qA_\mu)^2 \supset \frac{1}{2}a^2q^2 A_\mu A^\mu$
$$+ aq\,\partial_\mu\theta\, A^\mu$$

So, apparent mass terms are as follows

$$m_\rho = \sqrt{2}\,\mu \,, \quad m_A = a q \,, \quad m_\theta = 0$$

But, there is a hard-to-interpret mixing term

$$a^2 q\, \partial_\mu \theta A^\mu \quad ???$$

To gain intuition about this term, take advantage of freedom to make gauge transformations.

Recall that gauge transformations don't change physics!

So,

$$\mathcal{L} \supset (\partial_\mu \theta + q A_\mu)^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

$$\xrightarrow[\text{by } U]{\text{gauge transformation}} (\partial_\mu \theta^U + q A_\mu^U)^2 - \frac{1}{4} F_{\mu\nu}^U F^{U\,\mu\nu}$$

where explicitly,

$$A_\mu \to A_\mu^U = A_\mu - \frac{1}{q} \partial_\mu \alpha$$

$$\rho \to \rho$$

$$\theta \to \theta^U = \theta + \alpha$$

Now, can pick any $\alpha(x)$. Let's pick $\underline{\alpha(x) = \theta(x)}$

So,

$$\theta^U = 0$$

$$q A_\mu^U = q A_\mu - \partial_\mu \theta$$

Therefore,

$$\mathcal{L}^{U} \supset (qA_\mu^{U})^2 - \frac{1}{4}F_{\mu\nu}^{U}F^{U\mu\nu}$$

$$\Rightarrow \mathcal{L} = \frac{1}{2}\partial_\mu\rho\,\partial^\mu\rho + \frac{1}{2}(a+\rho)^2 q^2 A_\mu^{U}A^{U\mu}$$
$$- \frac{1}{4}F_{\mu\nu}^{U}F^{U\mu\nu} - U(a+\rho)$$

In this gauge, which is physically the same as others, we see that

(1) θ has "disappeared" $\Rightarrow$ No NG boson!

(2) gauge field is massive!

$$m_A = qa$$

The gauge transformation has removed confusing mixing terms. Sometimes people say "massless vector has eaten the NG boson, thereby gains mass." Really we picked "crappy" coordinates, and true physics is in "good" coordinates.

"Before"			"After"	
	2 scalars r, θ	$\Rightarrow$		1 scalar ρ
	2 polarizations of A_μ			3 polarizations of A_μ
	4 dof	$\Rightarrow$		4 dof

This special choice of gauge is called unitary gauge.

The mechanism is called the Higgs Mechanism.

Nobel '13 $\boxed{\text{Englert, Higgs}}$, Brout[+]
Guralnick[+], Hagen, Kibble[+]
$\Big\}$ all in 1964

of course, first formulated by Anderson in 1962.

The leftover boson ρ (massive) is called Higgs Boson

Sometimes, all bosons involved, $(\rho, \theta) = \varphi$

are called Higgs bosons.

The same mechanism works in Nonabelian case

$$\mathcal{L} \supset (D_\mu \varphi)^* (D^\mu \varphi)$$

$$(D_\mu \varphi)_j = \partial_\mu \varphi_j + i g A_\mu^a (T_a)_{jk} \varphi_k$$

So,

$$\mathcal{L} \supset (-ig A_\mu^a (T_a)_{jk} \varphi_k^*)(ig A^{\mu a} (T_a)_{jl} \varphi_l)$$

if $\varphi \to \langle \varphi \rangle$

this becomes $\mathcal{L} \gtrsim \underbrace{\langle \varphi \rangle^* \langle \varphi \rangle}_{\sim m^2} A_\mu A^\mu$

Can pick the unitary gauge to kill mixing terms.
Note that remaining unbroken generators (if any) can have an unbroken gauge group with corresponding massless gauge fields.

The quantization of the Abelian Higgs model follows as before, except we need to define the quantization with respect to the true vacuum, after SSB.
The gauge-field piece is

$$iS \supset i\int d^4x \, \frac{1}{2} A_\mu \left[(\partial^2 + m_A^2) g^{\mu\nu} + \partial^\mu \partial^\nu \right] A_\nu$$

$$\equiv i\int d^4x \left(-\frac{1}{2} A_\mu (D^{-1})^{\mu\nu} A_\nu \right)$$

There are no inversion problems as for massless vector bosons,

So,
$$\boxed{D_{\mu\nu} = \frac{-i}{q^2 - m_A^2} \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{m_A^2} \right)}$$

with $\boxed{m_A = qa}$

propagator in unitary gauge

The unitary gauge is convenient for tree level calculations. For loop corrections, the $q^\mu q^\nu$ term complicates things (UV behavior). The core idea was to remove the mixing term $\partial_\mu \theta A^\mu$ by fixing the gauge such that the field θ is part of the gauge transformation.

$$\varphi = \frac{1}{\sqrt{2}} r e^{i\theta} \xrightarrow[\text{SSB}]{} \frac{1}{\sqrt{2}}(a+r)\, e^{i\theta}$$

$$\text{unitary gauge} \Rightarrow \quad \varphi \to e^{i\alpha}\varphi \quad , \text{where } \alpha = -\theta$$

Can also choose to fix gauge by using R_ξ procedure.
Recall that after SSB

$$\mathcal{L} = \frac{1}{2}\partial_\mu \rho \partial^\mu \rho - \frac{1}{2}(a+\rho)^2(\partial_\mu \theta + q A_\mu)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

$$- \frac{\lambda a^2}{6}\rho^2 - \frac{\lambda a}{6}\rho^3 - \frac{\lambda}{4!}\rho^4 + \text{const.}$$

Need to remove mixing

$$\text{Mixing term} \quad \mathcal{L} \supset -q a^2 A_\mu (\partial_\mu \theta + q A_\mu)$$

A suitable gauge choice is $G(A_\mu) = \partial^\mu A_\mu + q a \xi \theta = 0$

Implement with gauge-fixing term

$\rightarrow$ gives mass for θ

$$\mathcal{L}_{g.f.} = -\frac{1}{2\xi}\left(\partial^\mu A_\mu + \xi q a \theta\right)^2 \qquad -\frac{1}{2}\xi(qa)^2\theta^2$$

$\hookrightarrow$ mixing term

$$= -qa(\partial_\mu A^\mu)\theta$$

$$= +qa A^\mu \partial_\mu \theta$$

So, adding this to Lagrange density removes mixing term. Note from Faddeev-Popov need to add ghost term too. So, got Higgs boson ρ with same mass $\sqrt{2}\mu$, and θ, the would-be NG mode, now has a gauge-dependent mass, $m_\theta^2 = \xi(qa)^2$. The gauge propagator is now

$$\boxed{D_{\mu\nu} = \frac{-i}{q^2 - m_A^2}\left(g_{\mu\nu} - (1-\xi)\frac{q_\mu q_\nu}{q^2 - \xi m_A^2}\right)}$$

$\hookrightarrow$ R_ξ-gauge propagator

The unitary gauge is recovered as $\xi \to \infty$.

Interestingly, the ghost fields do not couple to A_μ field directly, but to ρ-field.

Indeed, $G(A_\mu) = \partial^\mu A_\mu + q a \xi \theta = 0$

& under infinitesimal gauge transformation

$$\varphi \to (1 + i\alpha)\varphi$$

$$\Rightarrow \quad \delta A_\mu = -\frac{1}{q}\partial_\mu \alpha$$

but also, since $\varphi = \frac{1}{\sqrt{2}}(a+\rho)e^{i\theta} \simeq \frac{1}{\sqrt{2}}(a+\rho+ia\theta)$

$$\Rightarrow \delta\rho = -a\alpha\theta$$

and $\quad \delta\theta = (a+\rho)\alpha$

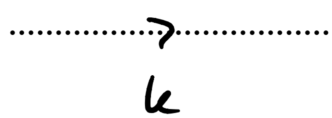
so,

$$\frac{\delta G}{\delta\alpha} \sim -\partial^2 - \xi m_A^2(1+\rho/a)$$

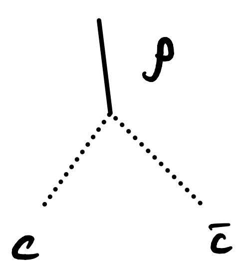
so, ghost term

$$\det\left(q\frac{\delta G}{\delta\alpha}\right) = \int \mathcal{D}c\,\mathcal{D}\bar{c}\; e^{i\int d^4x\, \bar{c}\left[-\partial^2 - \xi m_A^2 - \xi q^2 a \rho\right]c}$$

so, Feynman Rules are



$\sim \dfrac{i}{k^2 - \xi m_A^2}$



$\sim -i\xi q^2 a$

See that ghost has gauge-dependent mass, and interaction with Higgs. In unitarity gauge, $\xi \to \infty$, then the ghost is ∞-heavy, and decouples from theory. Likewise, the NG boson decouples, leaving what we have before.