1. Consider a general *binary* reaction $ab \to cd$, where the masses of the particles are m_j and their four-momenta are $p_j = (E_j, \mathbf{p}_j)$ with $E_j^2 = m_j^2 + \mathbf{p}_j^2$ for each $j = \{a, b, c, d\}$. Prove the following results.

   (a) The *Mandelstam invariants* are defined as

   $$s = (p_a + p_b)^2\,, \qquad t = (p_a - p_c)^2\,, \qquad u = (p_a - p_d)^2\,.$$

   Show that $s + t + u = m_a^2 + m_b^2 + m_c^2 + m_d^2$.
   Hint: Consider conservation of four-momentum.

   (b) Show in the *center-of-momentum* (CM) frame, the frame where $\mathbf{p}_a + \mathbf{p}_b = \mathbf{0}$, that

   $$s = (E_a + E_b)^2 = (E_c + E_d)^2\,.$$

   Show that $s \geq \max((m_a + m_b)^2, (m_c + m_d)^2)$.

   (c) Show in the CM frame that the energy of the particles are

   $$E_a = \frac{s + m_a^2 - m_b^2}{2\sqrt{s}}\,, \quad E_b = \frac{s - m_a^2 + m_b^2}{2\sqrt{s}}\,, \quad E_c = \frac{s + m_c^2 - m_d^2}{2\sqrt{s}}\,, \quad E_d = \frac{s - m_c^2 + m_d^2}{2\sqrt{s}}\,,$$

   and the momenta are

   $$|\mathbf{p}_a| = |\mathbf{p}_b| = \frac{1}{2\sqrt{s}}\,\lambda^{1/2}(s, m_a^2, m_b^2)\,, \qquad |\mathbf{p}_c| = |\mathbf{p}_d| = \frac{1}{2\sqrt{s}}\,\lambda^{1/2}(s, m_c^2, m_d^2)\,,$$

   where $\lambda(x, y, z) = x^2 + y^2 + z^2 - 2(xy + yz + zx)$ is the Källén triangle function.
   Hint: The following equivalent forms of the Källén function may be useful

   $$\begin{aligned}
   \lambda(x, y, z) &= x^2 + y^2 + z^2 - 2(xy + yz + zx)\,,\\
   &= x^2 - 2(y + z)x + (y - z)^2\,,\\
   &= [x - (\sqrt{y} + \sqrt{z})^2][x - (\sqrt{y} - \sqrt{z})^2]\,,\\
   &= (x - y - z)^2 - 4yz\,.
   \end{aligned}$$

   (d) Show in the CM frame that

   $$t = t_0 - 2|\mathbf{p}_a|\,|\mathbf{p}_c|(1 - \cos\theta)\,,$$

   where $t_0 \equiv \Delta^2/4s - (|\mathbf{p}_a| - |\mathbf{p}_c|)^2$ is the maximum value t can take with $\Delta = (m_a^2 - m_b^2) - (m_c^2 - m_d^2)$, and θ is the *scattering angle* defined by

   $$\cos\theta \equiv \frac{\mathbf{p}_a \cdot \mathbf{p}_c}{|\mathbf{p}_a||\mathbf{p}_c|}\,.$$

   Show that $t_1 \leq t \leq t_0$ where $t_1 = t_0 - 4|\mathbf{p}_a||\mathbf{p}_c|$ is the minimum value t can take.

   (e) Show that in the high-energy limit $|\mathbf{p}_j| \approx E_j \approx \sqrt{s}/2$ for every $j = \{a, b, c, d\}$.

   (f) For the case where all masses are equal, $m_a = m_b = m_c = m_d \equiv m$, write expressions for kinematic quantities in parts **(a)** through **(d)**.

2. The two-body differential *Lorentz invariant phase space* for some initial total momentum $P = (E, \mathbf{P})$ is defined as

$$\mathrm{d}\Phi_2(P \to p_1 + p_2) = \frac{1}{\mathcal{S}} \frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3\, 2E_1} \frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^3\, 2E_2} (2\pi)^4 \delta^{(4)}(P - p_1 - p_2)\,,$$

where $\mathcal{S}$ is a symmetry factor. Perform partial integrations to show that in the CM frame ($\mathbf{P} = \mathbf{0}$) the differential phase space is given by

$$\mathrm{d}\Phi_2(P \to p_1 + p_2) = \frac{1}{\mathcal{S}} \frac{|\mathbf{p}_1|}{4\pi\sqrt{s}} \frac{\mathrm{d}\Omega}{4\pi} \Theta(\sqrt{s} - m_1 - m_2)\,,$$

where $\mathrm{d}\Omega$ is the differential solid angle of $\mathbf{p}_1$, $s = P^2 = E^2$, and $\Theta(x)$ is the Heaviside step function.

3. Consider the binary reaction $ab \to cd$ where each particle is a scalar boson. The differential cross-section is defined as

$$\mathrm{d}\sigma = \frac{1}{\mathcal{F}} |\mathcal{M}|^2 \, \mathrm{d}\Phi_2(p_a + p_b \to p_c + p_d)\,,$$

where $\mathcal{F} = 4\sqrt{(p_a \cdot p_b)^2 - m_a^2 m_b^2}$ is the flux factor. Show that the differential cross-section can be written as

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2 s} \frac{|\mathbf{p}_c|}{|\mathbf{p}_a|} \frac{1}{\mathcal{S}} |\mathcal{M}|^2\,,$$

where the solid angle is defined in the CM frame.

4. Consider the elastic scattering of two scalar particles ($\varphi\varphi \to \varphi\varphi$) of mass m described $\lambda\varphi^4$ theory.

(**a**) At leading order in the coupling λ, the scattering amplitude is given by

$$i\mathcal{M} = -i\lambda + \mathcal{O}(\lambda^2)\,.$$

Compute the total cross-section σ as a function of s.

(**b**) As the energy approaches threshold, $s \to 4m^2$, the total cross-section can be written in terms of the *scattering length* a_0, $\sigma \to 4\pi a_0^2/\mathcal{S}$. Determine a_0 in terms of the coupling λ.

(**c**) The *partial wave expansion* is defined as

$$\mathcal{M}(s, \theta) = \sum_{\ell=0}^{\infty} (2\ell + 1)\, \mathcal{M}_\ell(s)\, P_\ell(\cos\theta)\,,$$

where ℓ is the angular momentum, θ is the scattering angle defined in the CM frame, and $P_\ell(z)$ are the Legendre polynomials. Given the scattering amplitude at leading order in λ, calculate the *partial wave amplitudes* $\mathcal{M}_\ell$ for every ℓ.

Hint: The following properties of the Legendre polynomials may be useful. Given the first two polynomials, $P_0(z) = 1$ and $P_1(z) = z$, all remaining P_ℓ can be generated through the Bonnet recursion relation for $\ell > 1$,

$$\ell P_\ell(z) = z(2\ell - 1)\, P_{\ell-1}(z) - (\ell - 1)\, P_{\ell-2}(z)\,.$$

The polynomial are orthogonal over $-1 \leq z \leq +1$,

$$\int_{-1}^{+1} \mathrm{d}z\, P_{\ell'}(z) P_\ell(z) = \frac{2}{2\ell + 1} \delta_{\ell'\ell}\,.$$