Problems **1** and **2** are *optional*, as they should be familiar from QFT I. However, if you are not comfortable with manipulating Gamma matrices, I *encourage* you to complete them. Completing them will result in *bonus points*.

1. The Dirac matrices $\gamma^\mu = (\gamma^0, \gamma^j)$ in the chiral (Weyl) representation are defined as

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \qquad \gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix},$$

where I is the 2×2 identity matrix and σ^j are the Pauli matrices.

- **(a)** With this representation, confirm that $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$.
- **(b)** Using the result in (a), show that $\gamma_\mu \gamma^\mu = 4$.
- **(c)** Prove that $\gamma_\mu \gamma^\nu \gamma^\mu = -2\gamma^\nu$ without using an explicit matrix representation.
- **(d)** Similarly, prove that $\gamma_\mu \gamma^\nu \gamma^\rho \gamma^\mu = 4g^{\nu\rho}$.

2. Given $\gamma^5 = \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, prove the following trace identities:

- **(a)** $\mathrm{tr}\,(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}$,
- **(b)** $\mathrm{tr}\,(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})$,
- **(c)** The trace of *any* odd number of gamma matrices is zero.
- **(d)** $\mathrm{tr}\,(\gamma^5) = \mathrm{tr}\,(\gamma^5\gamma^\mu) = \mathrm{tr}\,(\gamma^5\gamma^\mu\gamma^\nu) = \mathrm{tr}\,(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho) = 0$,
- **(e)** $\mathrm{tr}\,(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4i\epsilon^{\mu\nu\rho\sigma}$.

3. The chiral projectors are defined as

$$P_R = \frac{1}{2}(I + \gamma^5), \qquad P_L = \frac{1}{2}(I - \gamma^5),$$

where I is the 4×4 identity matrix. Prove the following properties:

- **(a)** $\gamma^5 P_L = -P_L$, and $\gamma^5 P_R = P_R$,
- **(b)** $(P_{L/R})^2 = P_{L/R}$,
- **(c)** $P_L P_R = P_R P_L = 0$,
- **(d)** $P_L + P_R = I$.

4. Suppose the charge conjugation operator is defined as $C = i\gamma^2\gamma^0$. Confirm that in the Weyl representation,

- **(a)** $C\gamma^\mu C^{-1} = -(\gamma^\mu)^\top$,
- **(b)** $C\gamma^5 C^{-1} = (\gamma^5)^\top$,
- **(c)** $C^{-1} = C^\top = C^\dagger = -C$.

5. A Dirac spinor ψ is called a Majorana spinor if it satisfies the condition $\psi = C\bar{\psi}^\top$, and is called a Weyl spinor if it satisfies either $\psi = P_R\psi$ or $\psi = P_L\psi$. Determine whether or not a spinor can be both Majorana and Weyl.

6. Consider a generic $2 \to n$ reaction $ab \to c_1 c_2 \ldots c_n$ in the *lab frame* or *fixed-target frame*, that is the frame where particle b is at rest and a is the incident beam. Assume $\max(m_a, m_b) < \min(c_1, c_2, \ldots, c_n)$.

(**a**) Show that $s = m_a^2 + m_b^2 + 2m_b\sqrt{m_a^2 + P_{\text{lab.}}^2}$ where $P_{\text{lab.}}$ is the beam momentum.

(**b**) Express the beam kinetic energy, $T_a \equiv E_a - m_a$, in terms of s.

(**c**) What is the minimum kinetic energy of the beam with which the reaction can occur?

7. Consider the following Yukawa theory as a simplified model of an interacting proton p, neutron n, and neutral pion π^0. We assume that the proton and neutron are distinguishable, but mass degenerate. The Lagrange density is given by

$$\mathcal{L} = \sum_f \frac{i}{2}\bar{\psi}_f \partial\!\!\!/ \psi_f + \text{h.c.} - \sum_f M\bar{\psi}_f\psi_f + \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 - \sum_f g\varphi\bar{\psi}_f\gamma^5\psi_f\,,$$

where f is fermion index $f = \{n, p\}$, M is the proton and neutron mass, m is the pion mass, and g is the coupling between proton and pion, as well as the neutron and pion.

(**a**) Consider the elastic reaction

$$n(p, s) + p(k, r) \to n(p', s') + p(k', r')\,,$$

where the arguments are the momenta and the subscripts are the spin-state. Write down the $np \to np$ scattering amplitude to leading order in the coupling g. **Hint:** Only one diagram contributes at $\mathcal{O}(g^2)$. Refer to the summary notes on `Feynman rules - Yukawa Theory`.

(**b**) The *spin-averaged* squared amplitude is defined as

$$\left\langle|\mathcal{M}|^2\right\rangle \equiv \frac{1}{2}\sum_s \frac{1}{2}\sum_r \sum_{s'} \sum_{r'} \left|\mathcal{M}(n_s p_r \to n_{s'} p_{r'})\right|^2.$$

Show that at leading order

$$\left\langle|\mathcal{M}|^2\right\rangle = g^4 \frac{t^2}{(t - m^2)^2} + \mathcal{O}\big(g^6\big)$$

where s, t, and u are the Mandelstam invariants. **Note:** You are encouraged to use a computer algebra software such as `FeynCalc` (https://feyncalc.github.io), which is a `Mathematica` package for symbolic evaluation of Feynman diagrams and algebraic calculations in quantum field theory and elementary particle physics. A useful tutorial can be found here. `Mathematica` is free to all students at William & Mary (see https://software.wm.edu).

(**c**) Compute the unpolarized differential cross-section $\mathrm{d}\sigma/\mathrm{d}t$ in terms of the Mandelstam invariants.

(**d**) Express $\mathrm{d}\sigma/\mathrm{d}\Omega$ in terms of s and the center-of-momentum frame scattering angle θ.

(**e**) Compute the total cross-section as a function of s.

(**f**) Estimate the magnitude of the pion-nucleon coupling g, as well as the quantity $g^2/4\pi$, from the experimentally observed np total cross-section. **Note:** You do not need to fit the data, however feel free to do so. The Review of Particle Physics contains experimental cross-sections for select processes. See the course webpage for the data file.

```
FILE_NAME
 REACTION
 BEAM_MASS TARGET_MASS THRESHOLD FINAL_STATE_MULTIPLICITY
 NUMBER_OF_DATA_POINTS
 POINT_NUMBER PLAB(GEV/C) PLAB_MIN PLAB_MAX SIG(MB) STA_ERR+ STA_ERR- SY_ER+(PCT) SY_ER-(PCT) REFERENCE FLAG
  FORMAT(I5,1X,4F11.5,2F8.4,1X,2F6.1,A)
 ----------------------------------------------------------------------------------------------------------
 NP_TOTAL.DAT
 N P --> ANYTHING
 0.939570    0.938270      0.             0
 578
 1    0.00385    0.00123    0.00531    20360.     100.00  100.00     0.0  0.0  MELKONIAN 49     PR 76, 1744
 2    0.03043    0.03043    0.03043    6202.0     11.200  11.200     0.0  0.0  ENGELKE 63       PR 129, 324
 3    0.03873    0.03873    0.03873    4700.0     40.000  40.000     0.0  0.0  LAMPI 49         PR 76, 188          P
 4    0.04347    0.04347    0.04347    4228.0     18.000  18.000     0.0  0.0  FIELDS 54        PR 94, 389
 5    0.04502    0.04502    0.04502    4060.0     30.000  30.000     0.0  0.0  LAMPI 49         PR 76, 188          P
 6    0.04973    0.04973    0.04973    3675.0     16.000  16.000     0.0  0.0  STORRS 54        PR 95, 1252
 7    0.05000    0.05000    0.05000    3609.5     4.5100  4.5100     2.0  2.0  CIERJACKS 69     PRL 23, 866
 8    0.05020    0.05020    0.05020    3630.0     40.000  40.000     0.0  0.0  LAMPI 49         PR 76, 188          P
 9    0.05100    0.05100    0.05100    3542.9     3.4400  3.4400     2.0  2.0  CIERJACKS 69     PRL 23, 866
 10   0.05200    0.05200    0.05200    3472.1     3.8000  3.8000     2.0  2.0  CIERJACKS 69     PRL 23, 866
 11   0.05300    0.05300    0.05300    3398.6     3.7400  3.7400     2.0  2.0  CIERJACKS 69     PRL 23, 866
 12   0.05311    0.05311    0.05311    3447.0     22.000  22.000     0.0  0.0  DAVIS 71         PR C3, 1798
 13   0.05400    0.05400    0.05400    3326.5     3.6900  3.6900     2.0  2.0  CIERJACKS 69     PRL 23, 866
 14   0.05448    0.05448    0.05448    3330.0     20.000  20.000     0.0  0.0  LAMPI 49         PR 76, 188          P
 15   0.05500    0.05500    0.05500    3266.6     4.2100  4.2100     2.0  2.0  CIERJACKS 69     PRL 23, 866
 16   0.05600    0.05600    0.05600    3206.6     3.6000  3.6000     2.0  2.0  CIERJACKS 69     PRL 23, 866
 17   0.05700    0.05700    0.05700    3138.6     4.1000  4.1000     2.0  2.0  CIERJACKS 69     PRL 23, 866
 18   0.05800    0.05800    0.05800    3079.6     3.5100  3.5100     2.0  2.0  CIERJACKS 69     PRL 23, 866
 19   0.05900    0.05900    0.05900    3015.5     3.9900  3.9900     2.0  2.0  CIERJACKS 69     PRL 23, 866
 20   0.05970    0.05970    0.05970    3004.0     24.000  24.000     0.0  0.0  DAVIS 71         PR C3, 1798
 21   0.06000    0.06000    0.06000    2963.6     3.9500  3.9500     2.0  2.0  CIERJACKS 69     PRL 23, 866
 22   0.06100    0.06100    0.06100    2907.2     3.9000  3.9000     2.0  2.0  CIERJACKS 69     PRL 23, 866
 23   0.06200    0.06200    0.06200    2851.3     3.8600  3.8600     2.0  2.0  CIERJACKS 69     PRL 23, 866
 24   0.06300    0.06300    0.06300    2792.6     3.8000  3.8000     2.0  2.0  CIERJACKS 69     PRL 23, 866
 25   0.06400    0.06400    0.06400    2754.0     4.6200  4.6200     2.0  2.0  CIERJACKS 69     PRL 23, 866
 26   0.06500    0.06500    0.06500    2705.5     3.7400  3.7400     2.0  2.0  CIERJACKS 69     PRL 23, 866
 27   0.06571    0.06571    0.06571    2677.0     22.000  22.000     0.0  0.0  DAVIS 71         PR C3, 1798
 28   0.06600    0.06600    0.06600    2651.5     3.6900  3.6900     2.0  2.0  CIERJACKS 69     PRL 23, 866
 29   0.06700    0.06700    0.06700    2607.4     4.4800  4.4800     2.0  2.0  CIERJACKS 69     PRL 23, 866
 30   0.06800    0.06800    0.06800    2558.2     3.6200  3.6200     2.0  2.0  CIERJACKS 69     PRL 23, 866
 ...
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