1. Show that the Lie algebra structure constants c_{jkl}, defined by the Lie bracket $[X^j, X^k] = c_{jkl}X^l$, satisfy the relation $c_{jkm}c_{mln} + c_{klm}c_{mjn} + c_{ljm}c_{mkn} = 0$.

2. Consider a general Lie algebra $[X^j, X^k] = c_{jkl}X^l$, where $c_{jkl} = -c_{kjl}$. From the structure constants, we may form matrices M^j with matrix elements $(M^j)_{lk} = c_{jkl}$. Note the order of the indices. Show that these matrices furnish a representation of the algebra, i.e., show that $[M^j, M^k] = c_{jkl}M^l$. This representation is called the *adjoint representation*. **Hint:** The Jacobi identity may be helpful.

3. Suppose X^j is a generator for the Lie algebra $[X^j, X^k] = c_{jkl}X^l$. Show that $X^2 = \sum_j X^j X^j$ commutes with the group generators, and therefore we may write $(X^2)_{ab} = C_2(r)\,\delta_{ab}$ where $C_2(r)$ is a constant called the *quadratic Casimir* of the representation r.

4. Let X^j be a generator for a generic $\mathfrak{su}(N)$ Lie algebra, $[X^j, X^k] = c_{jkl}X^l$, and $U(\alpha^j)$ is an element of the corresponding Lie group SU(N), with $U(\alpha^j) = \exp(\alpha^j X_j)$ with $\alpha^j \in \mathbb{R}$. Show that X^j are traceless, antihermitian $N \times N$ matrices.

5. Consider the set of all complex 2×2 matrices M with $\det(M) = i$. Does this set form a group under the usual matrix multiplication? Explain your reasoning.

6. Consider $X_j = -\frac{1}{2}i\sigma_j$ as a bases element of the $\mathfrak{su}(2)$ algebra, $[X_j, X_k] = \epsilon_{jkl}X_l$, where σ_j are the Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Verify the following:

(a) $[\sigma_j, \sigma_k] \equiv \sigma_j\sigma_k - \sigma_k\sigma_j = 2i\epsilon_{jkl}\sigma_l$.

(b) $\{\sigma_j, \sigma_k\} \equiv \sigma_j\sigma_k + \sigma_k\sigma_j = 2\delta_{jk}I_2$.

(c) $\sigma_j\sigma_k = \delta_{jk} + i\epsilon_{jkl}\sigma_l$.

(d) Show that a group element $U(\alpha^j) \in$ SU(2) can be written as

$$U(\alpha^j) = \exp\left(-\frac{1}{2}i\alpha^j\sigma_j\right) = I_2 \cos\left(\frac{1}{2}\alpha\right) - i\frac{\alpha^j\sigma_j}{\alpha}\sin\left(\frac{1}{2}\alpha\right),$$

where $\alpha^2 = \sum_j(\alpha_j)^2$.

7. Consider $X_j = L_j$ as a bases element of the $\mathfrak{so}(3)$ algebra, $[X_j, X_k] = \epsilon_{jkl}X_l$, where L_j are the matrices,

$$L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad L_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad L_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Verify the following:

(a) $[L_j, L_k] = \epsilon_{jkl}L_l$.

(b) $\{L_j, L_k\} \neq N\delta_{jk}$ for any j, k, and N.

(c) Show that a group element $O(\alpha^j) \in$ SO(3) can be written as

$$O(\alpha^j) = \exp\left(\alpha^j L_j\right) = I_3 + \frac{\alpha^j L_j}{\alpha}\sin\alpha + \left(\frac{\alpha^j L_j}{\alpha}\right)^2(1 - \cos\alpha),$$

where $\alpha^2 = \sum_j(\alpha_j)^2$.