1. Consider the Lagrange density for *scalar electrodynamics*,

$$\mathcal{L} = (D_\mu\varphi)^\dagger (D^\mu\varphi) - m^2\varphi^\dagger\varphi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - V(\varphi^\dagger\varphi)\,,$$

where m is the mass of the scalar field, $D_\mu = \partial_\mu + iqA_\mu$ where q is the charge of the scalar field, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and $V(\varphi^*\varphi)$ is a U(1) invariant self-interaction term, e.g., $V(\varphi^\dagger\varphi) = \lambda(\varphi^\dagger\varphi)^2$. This theory is invariant under local U(1) gauge transformations. Split the Lagrange density as follows: $\mathcal{L} = \mathcal{L}_{\mathrm{KG}} + \mathcal{L}_{\mathrm{EM}} + \mathcal{L}_{\mathrm{int.}}$, where $\mathcal{L}_{\mathrm{KG}}$ is the usual free complex Klein-Gordon field theory,

$$\mathcal{L}_{\mathrm{KG}} = \partial_\mu\varphi^\dagger\,\partial^\mu\varphi - m^2\varphi^\dagger\varphi\,,$$

and $\mathcal{L}_{\mathrm{EM}}$ is the Lagrange density for the free electromagnetic field,

$$\mathcal{L}_{\mathrm{EM}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,.$$

Determine the interacting Lagrange density $\mathcal{L}_{\mathrm{int.}}$ for scalar electrodynamics.

2. Consider the pair production of pions in electron-positron annihilation, $e^-e^+ \to \pi^-\pi^+$. Assume the reaction occurs at a center-of-momentum (CM) energy $\sqrt{s} \gg m_e$, but is comparable to the mass of the produced pions, $\sqrt{s} \sim m_\pi$. For simplicity, describe the charged pions by quantum scalar electrodynamics (for the Feynman rules, see the notes on `Feynman Rules - SQED`).

(a) Show that the unpolarized differential cross-section to leading order in α is given by

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{8}\frac{\alpha^2\beta_\pi^3}{s}(1-\cos^2\theta) + \mathcal{O}(\alpha^3)\,,$$

where θ is the CM frame scattering angle and β_π is the speed of the pion (recall that $|\mathbf{p}_\pi| = E_\pi\beta_\pi$).

(b) Compute the total cross-section, and compute ratio, $\sigma(e^-e^+ \to \pi^-\pi^+)/\sigma(e^-e^+ \to \mu^-\mu^+)$ where $\sigma(e^-e^+ \to \mu^-\mu^+) = 4\pi\alpha^2/3s$. Compute the theoretical value at $\sqrt{s} = 0.40\,\mathrm{GeV}$ and $.77\,\mathrm{GeV}$, and compare to the experimental R ratio, $R(\sqrt{s} = 0.40\,\mathrm{GeV}) = 0.18 \pm 0.02$ and $R(\sqrt{s} = 0.77\,\mathrm{GeV}) = 9.99 \pm 0.09$. Comment on the comparison. **Hint:** Examining the plots of the R ratio may be helpful, see Fig. 53.2 of https://pdg.lbl.gov/2022/reviews/rpp2022-rev-cross-section-plots.pdf.

3. Consider lepton pair production in electron-positron annihilation within QED, $e^-e^+ \to \ell^-\ell^+$, where $\ell = \mu$ or τ. Assume the reaction occurs at a center-of-momentum (CM) energy $\sqrt{s} \gg m_e$, but is comparable to the mass of the produced leptons, $\sqrt{s} \sim m_\ell$.

(a) Show that the unpolarized differential cross-section $\mathrm{d}\sigma/\mathrm{d}\Omega$, to order α^2, is given by

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2}{4s}\,\beta_\ell\left[1 + \cos^2\theta + (1-\beta_\ell^2)\sin^2\theta\right] + \mathcal{O}(\alpha^3)\,,$$

where β_ℓ is the speed of the produced lepton in the CM frame, and θ is the scattering angle.

(b) Show that the total $e^-e^+ \to \ell^-\ell^+$ cross-section at leading order is

$$\sigma = \frac{4\pi\alpha^2}{3s}\sqrt{1 - \frac{4m_\ell^2}{s}}\left(1 + \frac{2m_\ell^2}{s}\right) + \mathcal{O}(\alpha^3)\,.$$

(c) Take the ultrarelativistic limit of the results of **(a)** and **(b)**, that is $m_\ell^2/s \to 0$, to recover the following high-energy results,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2}{4s}(1+\cos^2\theta) + \mathcal{O}(\alpha^3)\,, \quad \text{and} \quad \sigma = \frac{4\pi\alpha^2}{3s} + \mathcal{O}(\alpha^3)\,.$$

(d) Plot the $\mathcal{O}(\alpha^2)$ theoretical $s \cdot \mathrm{d}\sigma/\mathrm{d}\Omega$ vs. $\cos\theta \in [-1, 1]$ at a CM energy $\sqrt{s} = 35\,\mathrm{GeV}$ for both $e^-e^+ \to \mu^-\mu^+$ and $e^-e^+ \to \tau^-\tau^+$ (make a separate plot for each reaction). Plot the y-axis in $\mathrm{nb} \cdot \mathrm{GeV}^2$, restricted to $(s \cdot \mathrm{d}\sigma/\mathrm{d}\Omega)/\,(\mathrm{nb} \cdot \mathrm{GeV}^2) \in [0.0, 12.0]$. Plot the experimental data for each reaction, measured from the JADE experiment at PETRA, over the theoretical curves. Compare and comment on the quality of the theoretical description of the experimental data. **Note:** The data file presents the cross-section as $s \cdot \mathrm{d}\sigma/\mathrm{d}\Omega$. The data files were obtained from the article by the JADE collaboration, https://link.springer.com/article/10.1007/BF01560255.

(e) Make a plot of the ratio of the experimentally measured differential cross-section to the leading order QED prediction for each reaction as a function of $\cos\theta \in [-1, .1]$ for the CM energy $\sqrt{s} = 35\,\mathrm{GeV}$. Restrict the y axis between 0.5 and 1.5. Compare and comment on the quality of the theoretical description of the experimental data.

4. In supersymmetry (SUSY), each fermion has a scalar partner, and each gauge boson has a fermionic partner. For example, the supersymmetric partner of the muon is the spin-0 *smuon* ($\tilde{\mu}$), and the partner of the photon is the spin-1/2 *photino* ($\tilde{\gamma}$). These particles have yet to be discovered in nature, yet we can place bounds on some of their properties by performing precision experiments such as measuring the anomalous magnetic moment of the muon, $a_\mu \equiv (g_\mu - 2)/2$. In this problem, we will estimate bounds on the masses of these hypothetical particles.

   Let us consider a simple supersymmetric extension of the Standard Model which includes the smuon and the photino. The Lagrange density for this model is given by

$$\mathcal{L}_{\mathrm{SUSY}} = \mathcal{L}_{\mathrm{SM}} + \frac{i}{2}\bar{\chi}\partial\!\!\!/\chi + \mathrm{h.c.} - m_{\tilde{\gamma}}\bar{\chi}\chi$$
$$+ (D_\nu\varphi)^\dagger(D^\nu\varphi) - m_{\tilde{\mu}}^2\varphi^\dagger\varphi - q\varphi\bar{\psi}\chi + \mathrm{h.c.}\,,$$

   where $D_\nu = \partial_\nu + iqA_\nu$, A_ν is the photon field, ψ is the muon field, φ is the smuon field with mass term $m_{\tilde{\mu}}$, and χ is the photino field with mass term $m_{\tilde{\gamma}}$. The coupling q is the electric charge of the fields, e.g., $q = -e$ for the muon where e is the fundamental charge which is related to the fine-structure constant via $\alpha = e^2/4\pi \sim 1/137$. The smuon has the same electric charge as its Standard Model counterpart.

(a) Determine the Feynman rules for the SUSY model. That is, draw a diagram an associated factor for the smuon propagator, the photino propagator, and any interaction vertices with these two particles. **Hint:** You do not need to derive these using generating functionals, use your knowledge of other well-known field theories to determine the various quantities.

(b) Draw the leading order contribution of this SUSY model to the $\mu\mu\gamma$ vertex function, $-iq\Gamma^\mu$, and write down the mathematical expression using the Feynman rules derived in part **(a)**. **Do Not** evaluate any integrals. Label all momenta and Lorentz indices. For the momenta, let p be the initial muon momentum, p' the final muon momentum, and q be the momentum transfer by the EM field, $q = p' - p$. The muon is on-mass shell, $p^2 = p'^2$. **Hint:** There is only a single contributing diagram.

(c) Let's assume that $m_{\tilde{\mu}} \sim m_{\tilde{\gamma}} \sim \Lambda_{\mathrm{SUSY}}$, where Λ_{SUSY} is a typical scale of SUSY interactions. One can show, assuming that $m_\mu \ll \Lambda_{\mathrm{SUSY}}$, that the SUSY contribution to the muon anomaly a_μ is given by

$$\delta a_\mu^{\mathrm{SUSY}} = \mathcal{C}\,\frac{\alpha}{4\pi}\,\frac{m_\mu}{\Lambda_{\mathrm{SUSY}}}\,,$$

where $\mathcal{C}$ is a constant of $\mathcal{O}(1)$. Constrain the SUSY scale Λ_{SUSY} using the experimental and theoretical values of a_μ. Currently, the best experimental estimate for a_μ was recently measured to be $a_\mu^{(\mathrm{ex.})} = 116\,592\,059(22) \times 10^{-11}$ by The Muon $g-2$ Collaboration (see D. P. Aguillard et al., Phys. Rev. Lett. **131**, 161802). Within the Standard Model, the best theoretical estimate for the anomaly is $a_\mu^{(\mathrm{th.})} = 116\,591\,810(43) \times 10^{-11}$ from The Muon $g-2$ Theory Initiative (see Physics Reports 887 (2020) 1-166). Comment on why is the assumption $m_\mu \ll \Lambda_{\mathrm{SUSY}}$ justifiable, and whether or not SUSY particles (within this model) can be detectable at the Large Hadron Collider.

(d) **Challenge (Optional):** Determine the constant $\mathcal{C}$ in $\delta a_\mu^{\mathrm{SUSY}}$ by evaluating the Feynman amplitude. To do this, perform the following steps

- Let P be the total momentum, $P = p + p'$, such that $P \cdot q = 0$. In the $q \to 0$ limit $P^2 = 4m_\mu^2$. Substitute $p = (P-q)/2$ and $p' = (P+q)/2$ in the Feynman integral. The vertex can then be written as a function $\Gamma^\mu(p', p) = \Gamma^\mu(P, q)$.
- Use the projection formula

$$\delta a_\mu = \frac{1}{12m^2} \operatorname{tr}\left[\left(m_\mu^2 \gamma_\nu - P_\nu \not{P} - \frac{3}{2} m_\mu P_\nu\right) V^\nu + \frac{m_\mu}{4}\left(\frac{\not{P}}{2} + m_\mu\right)[\gamma_\nu, \gamma_\rho]\left(\frac{\not{P}}{2} + m_\mu\right)\delta V^{\rho,\nu}\right],$$

  where $V^\nu = V^\nu(P) = \Gamma^\nu(P, 0)$ and

$$\delta V^{\rho,\nu} = \delta V^{\rho,\nu}(P) = \left.\frac{\partial \Gamma^\nu}{\partial q_\rho}\right|_{q=0}.$$

  Show that $\delta V^{\rho,\nu} = 0$ for this amplitude, and evaluate the remaining trace.
- Use the Feynman parameterization to combine the denominators of the Feynman integral. The relevant formula is

$$\frac{1}{A^2 B} = 2\int_0^1 \mathrm{d}x \, \frac{x}{[x(A-B)+B]^3}.$$

- Perform the convergent Feynman integrals using the following formulae

$$\int \mathrm{d}^4 k \, \frac{1}{[k^2 + 2P\cdot k - M^2]^3} = -\frac{i\pi^2}{2}\frac{1}{P^2 + M^2},$$

$$\int \mathrm{d}^4 k \, \frac{k^\mu}{[k^2 + 2P\cdot k - M^2]^3} = -\frac{i\pi^2}{2}\frac{P^\mu}{P^2 + M^2},$$

$$\int \mathrm{d}^4 k \, \frac{k^2 - (k\cdot p)^2/m^2}{[k^2 + 2P\cdot k - M^2]^3} = \frac{6i\pi^2 m^2 \alpha^2}{P^2 + M^2},$$

  where $P = \alpha p$ with α being some factor independent of k.
- Simplify the expression by setting $m_{\tilde{\gamma}} = m_{\tilde{\mu}} \equiv \Lambda_{\mathrm{SUSY}}$, and assuming $m_\mu \ll \Lambda_{\mathrm{SUSY}}$. The remaining Feynman parameter integral over x can then be analytically evaluated.