1. Can the following hadrons, in principle, exist within QCD? (a) qq, (b) $qq\bar{q}$, (c) $qq\bar{q}\bar{q}$, (d) gg, (f) qqg, (f) $q\bar{q}g$, (g) $qqqq\bar{q}$. **Hint:** Consider $\mathrm{SU}(3)_c$ symmetry transformations of observable hadrons. Gluons transform under the adjoint representation of $\mathrm{SU}(3)_c$.

2. Consider a non-abelian gauge field $A_\mu \equiv A_\mu^j T_j$, where $T_j \in \mathfrak{su}(N)$ are generators satisfying the Lie algebra $[T_j, T_k] = ic_{jkl}T_l$ with c_{jkl} being structure constants and $j, k, l = 1, 2, \ldots, N^2 - 1$. Under a local gauge transformation, $U = \exp(i\alpha^j(x)T_j)$ where $\alpha_j(x) \in \mathbb{R}$ for every j, the gauge fields transform as

$$A_\mu \to UA_\mu U^{-1} + \frac{i}{g}\left(\partial_\mu U\right) U^{-1}\,.$$

Show that under infinitesimal transformations, $\alpha^a(x) \ll 1$, the gauge fields transform as

$$A_\mu^j \to A_\mu^j - \frac{1}{g}\partial_\mu \alpha^j(x) - c_{jkl}\,\alpha^k A_\mu^l + \mathcal{O}(\alpha^2)\,.$$

3. The $SU(3)_c$ Yang-Mills Lagrange density for interacting gluon fields is given by $\mathcal{L}_{\mathrm{YM}} = -\frac{1}{2}\,\mathrm{tr}\left(G_{\mu\nu}G^{\mu\nu}\right)$, where the field-strength tensor is defined as $G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig_s[A_\mu, A_\nu]$ with $A_\mu = A_\mu^a\,\lambda_a/2$ are the gluon gauge fields and λ_a are the Gell-Mann matrices. Write the Lagrange density as a free part $\mathcal{L}_{\mathrm{YM}}^{(\mathrm{free})}$ and an interacting part $\mathcal{L}_{\mathrm{YM}}^{(\mathrm{int})}$ which depends on the strong coupling g_s.

4. Consider a $q\bar{q}$ meson within an exact flavor SU(3) quark model, i.e., $q = u, d, s$. Assume the meson is flavor neutral. A generic wave function for this meson is given by

$$\left|n\,^{2S+1}L_J, m_J\right\rangle_{q\bar{q}} = \sum_{m_L, m_S}\left\langle Lm_L; Sm_S|Jm_J\right\rangle \sum_{s,\bar{s}}\left\langle\frac{1}{2}s; \frac{1}{2}\bar{s}|Sm_S\right\rangle \times \int\frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\,\varphi_{n,L}(p)\,Y_{Lm_L}(\hat{\mathbf{p}})\left|q_s(\mathbf{p})\bar{q}_{\bar{s}}(-\mathbf{p})\right\rangle\,,$$

where n is the radial quantum number, S is the total intrinsic spin, L is the orbital angular momentum, J is the total angular momentum, m_J is the total angular momentum projection on some fixed z-axis, m_L is the orbital angular momentum projection, m_S is the total intrinsic spin projection, $\varphi_{n,L}$ is the momentum-space radial wave function, and Y_{Lm_L} are spherical harmonics. The quarks are spin-1/2 fermions with spin s and $\bar{s}$ for the q and $\bar{q}$, respectively. The two-quark state is defined in the center-of-momentum frame as the usual direct product $\left|q_s(\mathbf{p})\bar{q}_{\bar{s}}(-\mathbf{p})\right\rangle \equiv \left|q_s(\mathbf{p})\right\rangle \otimes \left|\bar{q}_{\bar{s}}(-\mathbf{p})\right\rangle$.

(a) Determine the allowed values of S.

(b) Show that under parity $\mathcal{P}$, the $q\bar{q}$ meson has an eigenvalue

$$\mathcal{P}\left|n\,^{2S+1}L_J, m_J\right\rangle_{q\bar{q}} = (-1)^{L+1}\left|n\,^{2S+1}L_J, m_J\right\rangle_{q\bar{q}}\,.$$

Hint: Recall that $\mathcal{P}\left|q_s(\mathbf{p})\right\rangle = \eta_q\left|q_s(-\mathbf{p})\right\rangle$ and $\eta_{\bar{q}} \equiv -\eta_q$.

(c) Show that under charge conjugation $\mathcal{C}$, the $q\bar{q}$ meson has an eigenvalue

$$\mathcal{C}\left|n\,^{2S+1}L_J, m_J\right\rangle_{q\bar{q}} = (-1)^{L+S}\left|n\,^{2S+1}L_J, m_J\right\rangle_{q\bar{q}}\,.$$

Hint: Recall that $\mathcal{C}\left|q_s(\mathbf{p})\right\rangle = \left|\bar{q}_s(\mathbf{p})\right\rangle$, and under interchange $\mathrm{P}_{12}\left|q_1 q_2\right\rangle = -\left|q_2 q_1\right\rangle$.

(d) Determine *all* allowed J^{PC} quantum numbers for of the meson for $L \leq 3$. List all J^{PC} that are forbidden for $J \leq 3$ (observed mesons with these quantum numbers are called *exotic*, as they are not allowed in the quark model).

(e) List *one* example (if one exist) of an observed unflavored meson for each J^{PC} supermultiplet by searching the Particle Data Group database (https://pdglive.lbl.gov) for *light unflavored mesons*. Are there any examples of observed mesons with exotic quantum numbers?