1. Consider the Abelian Higgs model, with the symmetric Lagrange density given by

$$\mathcal{L} = (D_\mu\varphi)^*(D^\mu\varphi) + \mu^2\varphi^*\varphi - \frac{\lambda}{3!}(\varphi^*\varphi)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,,$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $D_\mu = \partial_\mu + iqA_\mu$. This theory has an unstable extremum at $\varphi = 0$, and is invariant under global U(1) transformations, $\varphi \to e^{i\alpha(x)}\varphi$ and $A_\mu \to A_\mu - q^{-1}\partial_\mu\alpha$.

(**a**) Let $\varphi(x) = \frac{1}{\sqrt{2}}r(x)\,e^{i\theta(x)}$ where $r(x)$, $\theta(x)$ are real scalar fields. Show that the Lagrange density in terms of these fields is

$$\mathcal{L} = \frac{1}{2}\partial_\mu r\,\partial^\mu r + \frac{1}{2}r^2(\partial_\mu\theta + qA_\mu)^2 - \frac{\lambda}{4!}(r^2 - a^2)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \text{const.}\,,$$

where we have ignored non-dynamical constants. What is a in terms of the theory parameters μ, λ, and q?

(**b**) Expand the theory about the true vacuum of the theory, $r(x) \to a + \rho(x)$, $\theta(x) \to \theta(x)$, $A_\mu(x) \to A_\mu(x)$. Write each term of the new Lagrange density in terms of the fields ρ, A_μ, and θ. Determine the mass of each field in terms of the parameters of the theory μ, λ, and q, and the vacuum expectation value a.

(**c**) Eliminate the mixing term, $\partial_\mu\theta A^\mu$, by choosing the *unitary gauge*, $A_\mu \to A_\mu - q^{-1}\partial_\mu\theta$. Write down each term of the Lagrange density in this gauge.

(**d**) Write the Feynman rules for the Abelian Higgs model in the unitary gauge.

2. Recall that the decay rate for an $X \to n$ reaction in the X particle's rest frame is

$$\mathrm{d}\Gamma(X \to 1 + \cdots + n) = \frac{1}{2m_X}\langle|\mathcal{M}|^2\rangle\,\mathrm{d}\Phi_n\left(p_X - \sum_{j=1}^{n} p_j\right)\,,$$

where the n-body differential phase space is defined by

$$\mathrm{d}\Phi_n\left(p_X - \sum_{j=1}^{n} p_j\right) \equiv (2\pi)^4\delta^{(4)}\left(p_X - \sum_{j=1}^{n} p_j\right)\frac{1}{\mathcal{S}}\prod_{k=1}^{n}\frac{\mathrm{d}^3\mathbf{p}_k}{(2\pi)^3 2E_k}\,,$$

with each momentum being $p_j = (E_j, \mathbf{p}_j)$ and $\mathcal{S}$ is a symmetry factor for identical particles. Show that for a two-body X decay, the decay rate is given by

$$\mathrm{d}\Gamma(X \to 1 + 2) = \frac{1}{32\pi^2}\langle|\mathcal{M}|^2\rangle\frac{|\mathbf{p}|}{m_X^2}\,\mathrm{d}\Omega\,,$$

where $|\mathbf{p}| = \lambda^{1/2}(m_X^2, m_1^2, m_2^2)/2m_X$ and Ω are the momentum and the solid angle of particle 1.