1. Consider the leptonic decay of the negatively charged pion, $\pi^- \to \ell^- \bar{\nu}_\ell$ where $\ell = e, \mu$. The electroweak decay amplitude has the form

$$\mathcal{M}(\pi^- \to \ell \bar{\nu}_\ell) \propto f_\pi p^\mu \, \bar{u}_\ell \gamma_\mu (1 - \gamma_5) v_{\bar{\nu}_\ell} \,,$$

where f_π is the pion decay constant. Recall that the decay rate for an $a \to 1 + 2$ reaction in the rest frame of the decaying particle is

$$\Gamma = \frac{|\mathbf{p}|}{32\pi^2 m_a^2} \int \mathrm{d}\Omega \, |\mathcal{M}|^2 \,.$$

   (a) Show that $\pi^- \to \tau^- \nu_\tau$ is forbidden.

   (b) Compute the ratio

$$\mathcal{R} \equiv \frac{\Gamma(\pi^- \to e^- \bar{\nu}_e)}{\Gamma(\pi^- \to \mu^- \bar{\nu}_\mu)} \,.$$

   Compare the ratio to the measured value in the *Review of Particle Physics* by the PDG.

   (c) Compute the ratio $\mathcal{R}$ for $K^- \to \ell^- \bar{\nu}_\ell$, and compare to the measured ratio reported in the RPP. The amplitude is identical to that for the pion with $f_\pi \to f_K$, f_K being the kaon decay constant.

   (d) The weak interaction is known as a V-A interaction, or vector-axial vector. An alternative theory is the pseudoscalar interaction model, where the weak decay amplitude is given by

$$\mathcal{M} \propto g_\pi \bar{u}_\ell \gamma_5 v_{\bar{\nu}_\ell} \,,$$

   where g_π is some coupling. Compute the ratio $\mathcal{R}$ for $\pi^- \to \ell^- \bar{\nu}_\ell$ in this theory. Compare the result to the experimentally measured ratio. What conclusions can you make about this theory?

2. Consider $e^- e^+ \to \mu^- \mu^+$ within the Electroweak (EW) model of leptons. Assume the reaction occurs at a center-of-momentum (CM) energy $\sqrt{s} \gg m_e, m_\mu$.

   (a) Within the EW model, the leading order diagrams which contribute to the $e^- e^+ \to \mu^- \mu^+$ amplitude involve $e^- e^+$ annihilation to virtual γ, Z^0, and H^0 exchange. Argue why the contribution from the H^0 is negligible compared with the γ and Z^0 exchanges.

   (b) Given that the weak mixing angle is $\sin^2 \theta_W \approx 0.222$, and assuming that $s \ll m_Z^2$, show that the quantity

$$R(s) \equiv \frac{s}{(s - m_Z^2) \sin^2 2\theta_W} \approx -\frac{s}{m_Z^2 \sin^2 2\theta_W} \ll 1 \,.$$

   For $\sqrt{s} = 35\,\mathrm{GeV}$, verify this relation numerically.

   (c) Neglecting the H^0 contribution, and assuming that $s \ll m_Z^2$, show that the spin-averaged matrix element for this process is

$$\langle |\mathcal{M}|^2 \rangle = \frac{1}{4} \sum_{s,s'} \sum_{r,r'} |\mathcal{M}_\gamma + \mathcal{M}_Z|^2 \approx \frac{8e^4}{s^2} \left[1 + \frac{1}{2}(g_V^2 + 1) R(s) \right] (p' \cdot k)(p \cdot k')$$
$$+ \frac{8e^4}{s^2} \left[1 + \frac{1}{2}(g_V^2 - 1) R(s) \right] (p' \cdot p)(k' \cdot k) \,,$$

   where p and k are the initial electron and positron momenta, respectively, and p' and k' are the final muon and anti-muon momenta, respectively, and we have neglected terms $R(s)^2$. Here we have introduced $g_V = 1 - 4 \sin^2 \theta_W$ for convenience.

(d) From the amplitude in part (c), show that the unpolarized differential cross-section $\mathrm{d}\sigma/\mathrm{d}\Omega$ is given by

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \approx \frac{\alpha^2}{4s}\left[\left(1 - \frac{g_V^2 s}{2m_Z^2 \sin^2 2\theta_W}\right)(1+\cos^2\theta) - \frac{s}{m_Z^2 \sin^2 2\theta_W}\,\cos\theta\right],$$

where $g_V = 1 - 4\sin^2\theta_W \ll 1$, and θ is the scattering angle in the CM frame.

(e) The *forward-backward asymmetry* A_{FB} is defined as

$$A_{\mathrm{FB}} = \frac{\displaystyle\int_0^1 \mathrm{d}\cos\theta\,\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} - \int_{-1}^0 \mathrm{d}\cos\theta\,\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}}{\displaystyle\int_{-1}^1 \mathrm{d}\cos\theta\,\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}}\,.$$

Determine A_{FB} given the leading order result in part (d).

(f) Plot the theoretical $s \cdot \mathrm{d}\sigma/\mathrm{d}\Omega$ vs. $\cos\theta \in [-1, 1]$ at a CM energy $\sqrt{s} = 35\,\mathrm{GeV}$ for $e^-e^+ \to \mu^-\mu^+$. Plot the y-axis in $\mathrm{nb}\cdot\mathrm{GeV}^2$, restricted to $(s\cdot \mathrm{d}\sigma/\mathrm{d}\Omega)/(\mathrm{nb}\cdot\mathrm{GeV}^2) \in [0.0, 12.0]$. Plot the experimental data for each reaction, measured from the JADE experiment at PETRA, over the theoretical curves. Compare and comment on the quality of the theoretical description of the experimental data, and compare with the leading order QED result from Problem Set 6. **Note:** The data file presents the cross-section as $s \cdot \mathrm{d}\sigma/\mathrm{d}\Omega$. The data files were obtained from the article by the JADE collaboration, https://link.springer.com/article/10.1007/BF01560255.

(g) Make a plot of the ratio of the EW theoretical differential cross-section to the leading order QED prediction from Problem Set 6 as a function of $\cos\theta \in [-1, 1]$ for the CM energy $\sqrt{s} = 35\,\mathrm{GeV}$. Also, plot the experimentally measured differential cross-section to compare with the theoretical ratio. Restrict the y axis between 0.5 and 1.5. Compare and comment on the quality of the theoretical description of the experimental data.