1. Consider a general *binary* reaction $ab \to cd$, where the masses of the particles are m_j and their four-momenta are $p_j = (E_j, \mathbf{p}_j)$ with $E_j^2 = m_j^2 + \mathbf{p}_j^2$ for each $j = \{a, b, c, d\}$. Prove the following results.

   (a) The *Mandelstam invariants* are defined as

$$s = (p_a + p_b)^2 , \qquad t = (p_a - p_c)^2 , \qquad u = (p_a - p_d)^2 .$$

Show that $s + t + u = m_a^2 + m_b^2 + m_c^2 + m_d^2$. **Hint:** Consider conservation of four-momentum.

Solution: Taking the sum $s + t + u$, we find

$$\begin{aligned} s + t + u &= (p_a + p_b)^2 + (p_a - p_c)^2 + (p_a - p_d)^2 , \\ &= \sum_j p_j^2 + 2p_a^2 + 2p_a \cdot p_b - 2p_a \cdot p_c - 2p_a \cdot p_d , \\ &= \sum_j m_j^2 + 2p_a \cdot (p_a + p_b - p_c - p_d) , \\ &= \sum_j m_j^2 , \end{aligned}$$

where in the third line we used conservation of four-momentum states $p_a + p_b = p_c + p_d$.

   (b) Show in the *center-of-momentum* (CM) frame, the frame where $\mathbf{p}_a + \mathbf{p}_b = \mathbf{0}$, that

$$s = (E_a + E_b)^2 = (E_c + E_d)^2 .$$

Show that $s \geq \max((m_a + m_b)^2, (m_c + m_d)^2)$.

Solution: In the CM frame, $\mathbf{p}_a + \mathbf{p}_b = \mathbf{0}$. Therefore,

$$s = (p_a + p_b)^2 = (E_a + E_b)^2 - (\mathbf{p}_a + \mathbf{p}_b)^2 = (E_a + E_b)^2 . \tag{1}$$

Since $\mathbf{p}_a + \mathbf{p}_b = \mathbf{p}_c + \mathbf{p}_d$ by momentum conservation, in the CM frame we also have $\mathbf{p}_c + \mathbf{p}_d = \mathbf{0}$. Therefore, we also find

$$s = (p_c + p_d)^2 = (E_c + E_d)^2 - (\mathbf{p}_c + \mathbf{p}_d)^2 = (E_c + E_d)^2 . \tag{2}$$

Let $\mathbf{p} \equiv \mathbf{p}_a = -\mathbf{p}_b$ and $\mathbf{p}' = \mathbf{p}_c = -\mathbf{p}_d$. The energies of each particle are $E_a = \sqrt{m_a^2 + \mathbf{p}^2}$, $E_b = \sqrt{m_b^2 + \mathbf{p}^2}$, $E_c = \sqrt{m_c^2 + \mathbf{p}'^2}$, and $E_d = \sqrt{m_d^2 + \mathbf{p}'^2}$. The minimum energy for each particle is when $\mathbf{p} = \mathbf{p}' = \mathbf{0}$. So, the minimum $s_{\min.}$ is given by $s_{\min.} = (m_a + m_b)^2$ or $s_{\min.} = (m_c + m_d)^2$. Thus, the minimum s is given by $\max((m_a + m_b)^2, (m_c + m_d)^2)$ since the physical scattering occurs only when the total energy can produce the pair of particles. Note since $|\mathbf{p}|, |\mathbf{p}'| \in [0, \infty)$, s has no maximum bound.

   (c) Show in the CM frame that the energy of the particles are

$$E_a = \frac{s + m_a^2 - m_b^2}{2\sqrt{s}} , \quad E_b = \frac{s - m_a^2 + m_b^2}{2\sqrt{s}} , \quad E_c = \frac{s + m_c^2 - m_d^2}{2\sqrt{s}} , \quad E_d = \frac{s - m_c^2 + m_d^2}{2\sqrt{s}} ,$$

and the momenta are

$$|\mathbf{p}_a| = |\mathbf{p}_b| = \frac{1}{2\sqrt{s}}\,\lambda^{1/2}(s, m_a^2, m_b^2)\,, \qquad |\mathbf{p}_c| = |\mathbf{p}_d| = \frac{1}{2\sqrt{s}}\,\lambda^{1/2}(s, m_c^2, m_d^2)\,,$$

where $\lambda(x, y, z) = x^2 + y^2 + z^2 - 2(xy + yz + zx)$ is the Källén triangle function.
Hint: The following equivalent forms of the Källén function may be useful

$$\begin{aligned}
\lambda(x, y, z) &= x^2 + y^2 + z^2 - 2(xy + yz + zx)\,,\\
&= x^2 - 2(y + z)x + (y - z)^2\,,\\
&= [x - (\sqrt{y} + \sqrt{z})^2][x - (\sqrt{y} - \sqrt{z})^2]\,,\\
&= (x - y - z)^2 - 4yz\,.
\end{aligned}$$

Solution: In the CM frame, $\sqrt{s} = E_a + E_b$. Take the square of $(\sqrt{s} - E_a)^2 = E_b^2$ to find

$$\begin{aligned}
E_b^2 &= (\sqrt{s} - E_a)^2\,,\\
&= s + E_a^2 - 2\sqrt{s}E_a\,.
\end{aligned}$$

Now, the on-shell condition gives $E_a^2 = m_a^2 + \mathbf{p}_a^2$ and $E_b^2 = m_b^2 + \mathbf{p}_b^2$, with $\mathbf{p}_a^2 = \mathbf{p}_b^2$ in the CM frame. So, solving for E_a,

$$E_a = \frac{s + m_a^2 - m_b^2}{2\sqrt{s}}\,.$$

From $\sqrt{s} = E_a + E_b$, we then find

$$\begin{aligned}
E_b &= \sqrt{s} - E_a\,,\\
&= \sqrt{s} - \frac{s + m_a^2 - m_b^2}{2\sqrt{s}}\,,\\
&= \frac{s - m_a^2 + m_b^2}{2\sqrt{s}}\,.
\end{aligned}$$

Following similar arguments, we find E_c and E_d.

To find the momentum, use

$$\begin{aligned}
|\mathbf{p}_a|^2 &= E_a^2 - m_a^2\,,\\
&= \left(\frac{s + m_a^2 - m_b^2}{2\sqrt{s}}\right)^2 - m_a^2\,,\\
&= \frac{1}{4s}\Big((s + m_a^2 - m_b^2)^2 - 4sm_a^2\Big)\,,\\
&= \frac{1}{4s}\Big((m_b^2 - s - m_a^2)^2 - 4sm_a^2\Big)\,.
\end{aligned}$$

By the definition of the Källén function, we find

$$|\mathbf{p}_a|^2 = \frac{1}{4s}\,\lambda(s, m_a^2, m_b^2) \tag{3}$$

which is holds since the λ function is symmetric in all arguments. In the CM frame, $|\mathbf{p}_a| = |\mathbf{p}_b|$, therefore

$$|\mathbf{p}_a| = |\mathbf{p}_b| = \frac{1}{2\sqrt{s}}\,\lambda^{1/2}(s, m_a^2, m_b^2)\,.$$

We repeat the above arguments for both $|\mathbf{p}_c|$ and $|\mathbf{p}_d|$, finding the desired results.

(d) Show in the CM frame that

$$t = t_0 - 2|\mathbf{p}_a|\,|\mathbf{p}_c|(1 - \cos\theta)\,,$$

where $t_0 \equiv \Delta^2/4s - (|\mathbf{p}_a| - |\mathbf{p}_c|)^2$ is the maximum value t can take with $\Delta = (m_a^2 - m_b^2) - (m_c^2 - m_d^2)$, and θ is the *scattering angle* defined by

$$\cos\theta \equiv \frac{\mathbf{p}_a \cdot \mathbf{p}_c}{|\mathbf{p}_a||\mathbf{p}_c|}\,.$$

Show that $t_1 \le t \le t_0$ where $t_1 = t_0 - 4|\mathbf{p}_a||\mathbf{p}_c|$ is the minimum value t can take.

Solution: From the definition, $t = (p_a - p_c)^2$, we find

$$\begin{aligned}
t &= (p_a - p_c)^2\,,\\
&= (E_a - E_c)^2 - (\mathbf{p}_a - \mathbf{p}_c)^2\,,\\
&= (E_a - E_c)^2 - \mathbf{p}_a^2 - \mathbf{p}_c^2 + 2|\mathbf{p}_a||\mathbf{p}_c|\cos\theta\,,\\
&= (E_a - E_c)^2 - \mathbf{p}_a^2 - \mathbf{p}_c^2 + (2|\mathbf{p}_a||\mathbf{p}_c| - 2|\mathbf{p}_a||\mathbf{p}_c|) + 2|\mathbf{p}_a||\mathbf{p}_c|\cos\theta\,,\\
&= (E_a - E_c)^2 - (|\mathbf{p}_a| - |\mathbf{p}_c|)^2 - 2|\mathbf{p}_a||\mathbf{p}_c|(1 - \cos\theta)\,.
\end{aligned}$$

The maximum value of t occurs when $\cos\theta = 1$, so we define

$$\begin{aligned}
t_0 &\equiv t|_{\cos\theta = 1}\,,\\
&= (E_a - E_c)^2 - (|\mathbf{p}_a| - |\mathbf{p}_c|)^2\,.
\end{aligned}$$

so that

$$t = t_0 - 2|\mathbf{p}_a||\mathbf{p}_c|(1 - \cos\theta)\,.$$

Now, we can manipulate t_0 using the expressions for energies,

$$E_a - E_c = \frac{1}{2\sqrt{s}}((m_a^2 - m_b^2) - (m_c^2 - m_d^2))\,,$$
$$\equiv \frac{\Delta}{2\sqrt{s}}\,,$$

so $t_0 = \Delta^2/4s - (|\mathbf{p}_a| - |\mathbf{p}_c|)^2$. Note that when $\mathbf{p}_a = \mathbf{p}_c = \mathbf{0}$, then $\Delta^2 \geq 0$. For non-zero $\mathbf{p}_a$ and $\mathbf{p}_c$, t_0 can be either positive or negative depending on the scattering process (e.g. if $m_a = m_b = m_c = m_d$, then $t_0 \leq 0$.) The minimum value of t coincides with $\cos\theta = -1$, or

$$t_1 \equiv t|_{\cos\theta=-1}\,,$$
$$= t_0 - 4|\mathbf{p}_a||\mathbf{p}_c|\,.$$

So, $t_1 \leq t \leq t_0$.

(e) Show that in the high-energy limit $|\mathbf{p}_j| \approx E_j \approx \sqrt{s}/2$ for every $j = \{a, b, c, d\}$.

Solution: Consider first particle a. It's CM frame energy is

$$E_a = \frac{s + m_a^2 - m_b^2}{2\sqrt{s}}\,,$$
$$= \frac{\sqrt{s}}{2} + \frac{m_a^2 - m_b^2}{2\sqrt{s}}\,,$$
$$= \frac{\sqrt{s}}{2} + \mathcal{O}(s^{-1/2})\,,$$

and for the momentum, using the second form of the Källén function, we find

$$|\mathbf{p}_a| = \frac{\sqrt{s}}{2}\sqrt{1 - \frac{2(m_a^2 + m_b^2)}{s} + \frac{(m_a^2 - m_b^2)^2}{s^2}}\,,$$
$$= \frac{\sqrt{s}}{2} - \frac{m_a^2 + m_b^2}{2\sqrt{s}} + \mathcal{O}(s^{-3/2})\,,$$
$$= \frac{\sqrt{s}}{2} + \mathcal{O}(s^{-1/2})\,,$$

where in the second line we performed a series expansion about $1/s = 0$. Therefore, both $|\mathbf{p}_a|$ and E_a scale as

$$E_a = |\mathbf{p}_a| = \frac{\sqrt{s}}{2} + \mathcal{O}(s^{-1/2})\,,$$

as $s \to \infty$. Repeating this analysis for particles b, c, and d, we find that at high-energy $|\mathbf{p}_j| \approx E_j \approx \sqrt{s}/2$ for each $j = \{a, b, c, d\}$.

(f) For the case where all masses are equal, $m_a = m_b = m_c = m_d \equiv m$, write expressions for kinematic quantities in parts **(a)** through **(d)**.

Solution: By direct substitution into the general formulae we derived,

$$E \equiv E_a = E_b = E_c = E_d = \frac{\sqrt{s}}{2}\,,$$

and

$$|\mathbf{p}| \equiv |\mathbf{p}_a| = |\mathbf{p}_b| = |\mathbf{p}_c| = |\mathbf{p}_d| = \frac{1}{2}\sqrt{s - 4m^2}\,,$$

with $s = 4E^2$, and since $E = \sqrt{m^2 + \mathbf{p}^2} \geq m$, so $s \geq 4m^2$. For t, $t_0 = 0$, and

$$t = -2\mathbf{p}^2(1 - \cos\theta)\,.$$

2. The two-body differential *Lorentz invariant phase space* for some initial total momentum $P = (E, \mathbf{P})$ is defined as

$$\mathrm{d}\Phi_2(P \to p_1 + p_2) = \frac{1}{\mathcal{S}} \frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3\, 2E_1} \frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^3\, 2E_2} (2\pi)^4 \delta^{(4)}(P - p_1 - p_2)\,,$$

where $\mathcal{S}$ is a symmetry factor. Perform partial integrations to show that in the CM frame ($\mathbf{P} = \mathbf{0}$) the differential phase space is given by

$$\mathrm{d}\Phi_2(P \to p_1 + p_2) = \frac{1}{\mathcal{S}} \frac{|\mathbf{p}_1|}{4\pi\sqrt{s}} \frac{\mathrm{d}\Omega}{4\pi} \Theta(\sqrt{s} - m_1 - m_2)\,,$$

where $\mathrm{d}\Omega$ is the differential solid angle of $\mathbf{p}_1$, $s = P^2 = E^2$, and $\Theta(x)$ is the Heaviside step function.

Assume we are integrating against a test function $f(\mathbf{p}_1, \mathbf{p}_2)$. Since the phase space is Lorentz invariant, we can evaluate in any reference frame. We choose the CM frame. The four-dimensional Dirac delta can be written as

$$\delta^{(4)}(P - p_1 - p_2) = \delta^{(4)}(E - E_1 - E_2)\,\delta^{(3)}(\mathbf{p}_1 + \mathbf{p}_2)\,,$$

where we used $\mathbf{P} = \mathbf{0}$.
So, we can integrate over the measure $\mathrm{d}^3\mathbf{p}_2$, eliminating the spatial momentum Dirac delta functions,

$$\mathrm{d}\Phi_2(P \to p_1 + p_2) = \frac{1}{(4\pi)^2} \frac{1}{\mathcal{S}} \frac{\mathrm{d}^3\mathbf{p}_1}{E_1 E_2} \delta^{(4)}(E - E_1 - E_2)\,.$$

Note that since $\mathbf{p}_1 = -\mathbf{p}_2$, $E_1 = \sqrt{m_1^2 + \mathbf{p}_1^2}$ and $E_2 = \sqrt{m_2^2 + \mathbf{p}_1^2}$. The remaining delta function can be evaluated by a change of variables to $|\mathbf{p}_1|$,

$$\delta(E - E_1 - E_2) = \left| \frac{\partial(E - E_1 - E_2)}{\partial|\mathbf{p}_1|} \right|^{-1} \delta(|\mathbf{p}_1| - |\mathbf{p}_1^\star|)\,,$$

$$= \frac{E_1 E_2}{|\mathbf{p}_1|\sqrt{s}}\, \delta(|\mathbf{p}_1| - |\mathbf{p}_1^\star|)$$

where $|\mathbf{p}_1^\star|$ is the solution to $E-E_1-E_2=0$. So, converting the measure to spherical coordinates, we find

$$\begin{aligned} d\Phi_2(P \to p_1+p_2) &= \frac{1}{(4\pi)^2}\frac{1}{\mathcal{S}}\frac{d^3\mathbf{p}_1}{E_1E_2}\,\frac{E_1E_2}{|\mathbf{p}_1|\sqrt{s}}\,\delta(|\mathbf{p}_1|-|\mathbf{p}_1^\star|)\,, \\ &= \frac{1}{(4\pi)^2}\frac{1}{\mathcal{S}}\frac{d\Omega d|\mathbf{p}_1|\,|\mathbf{p}_1|^2}{E_1E_2}\,\frac{E_1E_2}{|\mathbf{p}_1|\sqrt{s}}\,\delta(|\mathbf{p}_1|-|\mathbf{p}_1^\star|)\,, \\ &= \frac{1}{\mathcal{S}}\frac{|\mathbf{p}_1^\star|}{4\pi\sqrt{s}}\frac{d\Omega}{4\pi}\,\Theta(\sqrt{s}-m_1-m_2)\,, \end{aligned}$$

where the integral over the delta function yields the Heaviside function, enforcing the total energy to be greater than the threshold. Since the $\star$ is a label, we arrive at the desired result.

3. Consider the binary reaction $ab \to cd$ where each particle is a scalar boson. The differential cross-section is defined as

$$d\sigma = \frac{1}{\mathcal{F}}\,|\mathcal{M}|^2\,d\Phi_2(p_a+p_b \to p_c+p_d)\,,$$

where $\mathcal{F} = 4\sqrt{(p_a\cdot p_b)^2 - m_a^2m_b^2}$ is the flux factor. Show that the differential cross-section can be written as

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\,\frac{|\mathbf{p}_c|}{|\mathbf{p}_a|}\,\frac{1}{\mathcal{S}}\,|\mathcal{M}|^2\,,$$

where the solid angle is defined in the CM frame.

Solution: From Problem 2, we have an expression for the phase space in the CM frame,

$$d\Phi_2(P \to p_1+p_2) = \frac{1}{\mathcal{S}}\frac{|\mathbf{p}_c|}{16\pi^2\sqrt{s}}\,d\Omega\,,$$

where we leave the Heaviside function implicit. Therefore, we only need to express the flux factor in the CM frame. Note that $s=(p_a+p_b)^2 = m_a^2+m_b^2+2p_a\cdot p_b$. So, $(p_a\cdot p_b)^2 = (s-m_a^2-m_b^2)^2/4$, therefore

$$\begin{aligned} \mathcal{F} &= 4\sqrt{(p_a\cdot p_b)^2 - m_a^2m_b^2}\,, \\ &= 4\sqrt{\frac{(s-m_a^2-m_b^2)^2}{4} - m_a^2m_b^2}\,, \\ &= 2\lambda^{1/2}(s,m_a^2,m_b^2)\,. \end{aligned}$$

Since $2\sqrt{s}|\mathbf{p}_a| = \lambda^{1/2}(s,m_a^2,m_b^2)$, we find $\mathcal{F} = 4\sqrt{s}|\mathbf{p}_a|$. Combining the pieces, we find the desired result

$$\begin{aligned} d\sigma &= \frac{1}{4\sqrt{s}|\mathbf{p}_a|}\,|\mathcal{M}|^2\,\frac{1}{\mathcal{S}}\frac{|\mathbf{p}_c|}{16\pi^2\sqrt{s}}\,d\Omega\,, \\ &= \frac{1}{64\pi^2 s}\frac{|\mathbf{p}_c|}{|\mathbf{p}_a|}\frac{1}{\mathcal{S}}|\mathcal{M}|^2\,d\Omega\,. \end{aligned}$$

4. Consider the elastic scattering of two scalar particles ($\varphi\varphi \to \varphi\varphi$) of mass m described $\lambda\varphi^4$ theory.

   (a) At leading order in the coupling λ, the scattering amplitude is given by

$$i\mathcal{M} = -i\lambda + \mathcal{O}(\lambda^2)\,.$$

   Compute the total cross-section σ as a function of s.

> **Solution:** For equal mass scattering, $|\mathbf{p}_a| = |\mathbf{p}_c|$. So, the differential cross section is
>
> $$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2 s}\,\frac{1}{2}\,|\mathcal{M}|^2\,, \\ = \frac{\lambda^2}{128\pi^2 s} + \mathcal{O}(\lambda^3)\,.$$
>
> Integrating, we have
>
> $$\sigma = \int \mathrm{d}\Omega\,\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\,, \\ = \frac{\lambda^2}{32\pi s} + \mathcal{O}(\lambda^3)\,.$$

   (b) As the energy approaches threshold, $s \to 4m^2$, the total cross-section can be written in terms of the *scattering length* a, $\sigma \to 4\pi a_0^2/\mathcal{S}$. Determine a_0 in terms of the coupling λ.

> **Solution:** As $s \to m^2$, then
>
> $$\sigma \to \frac{1}{\mathcal{S}}\frac{\lambda^2}{16\pi(4m^2)} + \mathcal{O}(\lambda^3) = \frac{4\pi a_0^2}{\mathcal{S}}\,.$$
>
> So, we find
>
> $$a_0 = \frac{\lambda}{16\pi m} + \mathcal{O}(\lambda^3)\,.$$

   (c) The *partial wave expansion* is defined as

$$\mathcal{M}(s,\theta) = \sum_{\ell=0}^{\infty}(2\ell+1)\,\mathcal{M}_\ell(s)\,P_\ell(\cos\theta)\,,$$

   where ℓ is the angular momentum, θ is the scattering angle defined in the CM frame, and $P_\ell(z)$ are the Legendre polynomials. Given the scattering amplitude at leading order in λ, calculate the *partial wave amplitudes* $\mathcal{M}_\ell$ for every ℓ.

   Hint: The following properties of the Legendre polynomials may be useful. Given the first two polynomials, $P_0(z) = 1$ and $P_1(z) = z$, all remaining P_ℓ can be generated through the Bonnet recursion relation for $\ell > 1$,

$$\ell P_\ell(z) = z(2\ell-1)\,P_{\ell-1}(z) - (\ell-1)\,P_{\ell-2}(z)\,.$$

The polynomial are orthogonal over $-1 \leq z \leq +1$,

$$\int_{-1}^{+1} \mathrm{d}z \, P_{\ell'}(z) P_\ell(z) = \frac{2}{2\ell+1} \delta_{\ell'\ell} \, .$$

Solution: From the orthogonality of P_ℓ, we find

$$\mathcal{M}_\ell(s) = \frac{1}{2} \int_{-1}^{+1} \mathrm{d}\cos\theta \, P_\ell(\cos\theta) \, \mathcal{M}(s,\theta) \, .$$

Now, $\mathcal{M} = -\lambda + \mathcal{O}(\lambda^2)$, which is a constant at leading order. Recognizing that $1 = P_0(\cos\theta)$, we find

$$\begin{aligned}
\mathcal{M}_\ell(s) &= -\frac{\lambda}{2} \int_{-1}^{+1} \mathrm{d}\cos\theta \, P_\ell(\cos\theta) + \mathcal{O}(\lambda^2) \, , \\
&= -\frac{\lambda}{2} \int_{-1}^{+1} \mathrm{d}\cos\theta \, P_\ell(\cos\theta) P_0(\cos\theta) + \mathcal{O}(\lambda^2) \, , \\
&= -\frac{\lambda}{2} \frac{2}{2\ell+1} \delta_{\ell 0} + \mathcal{O}(\lambda^2) \, , \\
&= -\lambda \, \delta_{\ell 0} + \mathcal{O}(\lambda^2)
\end{aligned}$$

So, the scattering amplitude is the S wave ($\ell = 0$) amplitude, all other $\ell \neq 0$ partial wave amplitudes are identically zero at this order.