Problems **1** and **2** are *optional*, as they should be familiar from QFT I. However, if you are not comfortable with manipulating Gamma matrices, I *encourage* you to complete them. Completing them will result in *bonus points*.

1. The Dirac matrices $\gamma^\mu = (\gamma^0, \gamma^j)$ in the chiral (Weyl) representation are defined as

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \qquad \gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix},$$

where I is the 2×2 identity matrix and σ^j are the Pauli matrices.

(a) With this representation, confirm that $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$.

Solution: To clarify some of the manipulations in these problems, we introduce I_4 as the 4×4 identity, and let $I \to I_2$ be the 2×2 identity. Thus, what is to be shown is $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} I_4$, given the chiral representation

$$\gamma^0 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \qquad \gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}.$$

Recall the properties of the Pauli matrices, $\{\sigma^j, \sigma^k\} = 2\delta^{jk} I_2$.

$$\begin{aligned}
\{\gamma^0, \gamma^0\} &= 2(\gamma^0)^2, \\
&= 2 \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \\
&= 2 \begin{pmatrix} I_2 & 0 \\ 0 & I_2 \end{pmatrix} = 2g^{00}\, I_4, \\
\{\gamma^0, \gamma^j\} &= \gamma^0\gamma^j + \gamma^j\gamma^0, \\
&= \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \\
&= \begin{pmatrix} -\sigma^j & 0 \\ 0 & \sigma^j \end{pmatrix} + \begin{pmatrix} \sigma^j & 0 \\ 0 & -\sigma^j \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 2g^{0j}\, I_4,
\end{aligned}$$

where we note that $g^{00} = +1$ and $g^{0j} = g^{j0} = 0$. Continuing,

$$
\begin{aligned}
\{\gamma^j, \gamma^0\} &= \gamma^j\gamma^0 + \gamma^0\gamma^j = \{\gamma^0, \gamma^j\} = 2g^{0j}\, I_4 = 2g^{j0}\, I_4\,, \\
\{\gamma^j, \gamma^k\} &= \gamma^j\gamma^k + \gamma^k\gamma^j\,, \\
&= \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}, \\
&= \begin{pmatrix} -\sigma^j\sigma^k & 0 \\ 0 & -\sigma^j\sigma^k \end{pmatrix} + \begin{pmatrix} -\sigma^k\sigma^j & 0 \\ 0 & -\sigma^k\sigma^j \end{pmatrix}, \\
&= -\begin{pmatrix} \sigma^j\sigma^k + \sigma^k\sigma^j & 0 \\ 0 & \sigma^j\sigma^k + \sigma^k\sigma^j \end{pmatrix}, \\
&= -\begin{pmatrix} 2\delta^{jk} I_2 & 0 \\ 0 & 2\delta^{jk} I_2 \end{pmatrix}, \\
&= -2\delta^{jk}\, I_4 = 2g^{jk}\, I_4
\end{aligned}
$$

Therefore, we have shown $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\, I_4$

(b) Using the result in (a), show that $\gamma_\mu\gamma^\mu = 4$.

Solution: Contract $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\, I_4$ with $g_{\mu\nu}$,

$$
\begin{aligned}
g_{\mu\nu}\{\gamma^\mu, \gamma^\nu\} &= 2g_{\mu\nu}g^{\mu\nu}\, I_4\,, \\
\{\gamma_\mu, \gamma^\mu\} &= 2g^\mu{}_\mu\, I_4\,, \\
2\gamma_\mu\gamma^\mu &= 2\cdot 4\, I_4\,.
\end{aligned}
$$

So, we conclude $\gamma_\mu\gamma^\mu = 4\, I_4$.

(c) Prove that $\gamma_\mu\gamma^\nu\gamma^\mu = -2\gamma^\nu$ without using an explicit matrix representation.

Solution: Using the anticommutator relation, as well as the result from part (b), we find

$$
\begin{aligned}
\gamma_\mu\gamma^\nu\gamma^\mu &= \gamma_\mu(2g^{\mu\nu}\, I_4 - \gamma^\mu\gamma^\nu)\,, \\
&= 2\gamma_\nu - \gamma_\mu\gamma^\mu\gamma^\nu\,, \\
&= 2\gamma_\nu - 4\gamma^\nu\,, \\
&= -2\gamma_\nu\,.
\end{aligned}
$$

(d) Similarly, prove that $\gamma_\mu\gamma^\nu\gamma^\rho\gamma^\mu = 4g^{\nu\rho}$.

Solution: Using the anticommutator relation, as well as the result from part (c), we find

$$\begin{aligned}\gamma_\mu \gamma^\nu \gamma^\rho \gamma^\mu &= \gamma_\mu \gamma^\nu (2g^{\rho\mu}\, I_4 - \gamma^\mu \gamma^\rho)\,, \\ &= 2\gamma^\rho \gamma^\nu - \gamma_\mu \gamma^\nu \gamma^\mu \gamma^\rho\,, \\ &= 2\gamma^\rho \gamma^\nu + 2\gamma^\nu \gamma^\rho\,, \\ &= 2\{\gamma^\rho, \gamma^\nu\}\,, \\ &= 4g^{\nu\rho}\, I_4\,.\end{aligned}$$

2. Given $\gamma^5 = \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, prove the following trace identities:

(a) $\mathrm{tr}\,(\gamma^\mu \gamma^\nu) = 4g^{\mu\nu}$,

Solution: Taking the trace, we use the cyclic properties of the trace and the anticommutation relations, we have

$$\begin{aligned}\mathrm{tr}\,(\gamma^\mu \gamma^\nu) &= \frac{1}{2}\,\mathrm{tr}(\gamma^\mu \gamma^\nu + \gamma^\mu \gamma^\nu)\,, \\ &= \frac{1}{2}\Big[\,\mathrm{tr}(\gamma^\mu \gamma^\nu) + \mathrm{tr}(\gamma^\mu \gamma^\nu)\Big]\,, \\ &= \frac{1}{2}\,\mathrm{tr}(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu)\,, \\ &= \frac{1}{2}\,\mathrm{tr}(\{\gamma^\mu \gamma^\nu\})\,, \\ &= \frac{1}{2} \cdot 2g^{\mu\nu}\,\mathrm{tr}(I_4)\,, \\ &= 4g^{\mu\nu}\,,\end{aligned}$$

where $\mathrm{tr}(I_4) = 4$.

(b) $\mathrm{tr}\,(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) = 4(g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho})$,

Solution: Here we use the anticommutation relation inside the trace,

$$\begin{aligned}
\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) &= \mathrm{tr}[\gamma^\mu\gamma^\nu(2g^{\rho\sigma}\, I_4 - \gamma^\sigma\gamma^\rho)]\,, \\
&= 2g^{\rho\sigma}\,\mathrm{tr}(\gamma^\mu\gamma^\nu) - \mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\rho)\,, \\
&= 8g^{\mu\nu}g^{\rho\sigma} - \mathrm{tr}[\gamma^\mu(2g^{\nu\sigma}\, I_4 - \gamma^\sigma\gamma^\nu)\gamma^\rho]\,, \\
&= 8g^{\mu\nu}g^{\rho\sigma} - 2g^{\nu\sigma}\,\mathrm{tr}(\gamma^\mu\gamma^\rho) + \mathrm{tr}(\gamma^\mu\gamma^\sigma\gamma^\nu\gamma^\rho)\,, \\
&= 8g^{\mu\nu}g^{\rho\sigma} - 8g^{\nu\sigma}g^{\mu\rho} + \mathrm{tr}[2g^{\mu\sigma}\, I_4 - \gamma^\sigma\gamma^\mu)\gamma^\nu\gamma^\rho]\,, \\
&= 8g^{\mu\nu}g^{\rho\sigma} - 8g^{\nu\sigma}g^{\mu\rho} + 2g^{\mu\sigma}\,\mathrm{tr}(\gamma^\nu\gamma^\rho) - \mathrm{tr}(\gamma^\sigma\gamma^\mu\gamma^\nu\gamma^\rho)\,, \\
&= 8g^{\mu\nu}g^{\rho\sigma} - 8g^{\nu\sigma}g^{\mu\rho} + 8g^{\mu\sigma}g^{\nu\rho} - \mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma)\,,
\end{aligned}$$

where in the last line we used the cyclic property of the trace. Then, adding this final trace to the left-hand side, we find

$$\begin{aligned}
2\,\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) &= 8g^{\mu\nu}g^{\rho\sigma} - 8g^{\nu\sigma}g^{\mu\rho} + 8g^{\mu\sigma}g^{\nu\rho} \\
&\implies \mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4\,(g^{\mu\nu}g^{\rho\sigma} - g^{\nu\sigma}g^{\mu\rho} + g^{\mu\sigma}g^{\nu\rho})
\end{aligned}$$

(**c**) The trace of *any* odd number of gamma matrices is zero.

Solution: We first prove that $\mathrm{tr}(\gamma^\mu) = 0$, which is obvious in the Weyl basis but is true in general. Recall that $(\gamma_5)^2 = \gamma_5\gamma^5 = 4I_4$. Therefore, the trace can be written as

$$\mathrm{tr}(\gamma^\mu) = \mathrm{tr}(\gamma^\mu I_4) = \mathrm{tr}(\gamma^\mu\gamma^5\gamma^5) = -\,\mathrm{tr}(\gamma^5\gamma^\mu\gamma^5) = -\,\mathrm{tr}(\gamma^\mu\gamma^5\gamma^5) = -\,\mathrm{tr}(\gamma^\mu)\,,$$

where in the fourth equality we used the anticommutation relation $\gamma^\mu\gamma^5 = -\gamma^5\gamma^\mu$ and in the fifth equality results from the cyclic properties of the trace. Therefore, we conclude

$$\mathrm{tr}(\gamma^\mu) = 0\,.$$

A generic trace over an odd number of gamma matrices can be written as a trace over $2n+1$ gamma matrices where $n \in \mathbb{N}$, $\mathrm{tr}(\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_{2n}}\gamma^{\mu_{2n+1}})$. So, inserting $I_4 = (\gamma^5)^2$ at the end gives,

$$\begin{aligned}
\mathrm{tr}(\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_{2n}}\gamma^{\mu_{2n+1}}) &= \mathrm{tr}(\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_{2n}}\gamma^{\mu_{2n+1}} I_4)\,, \\
&= \mathrm{tr}(\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_{2n}}\gamma^{\mu_{2n+1}}\gamma^5\gamma^5)\,, \\
&= (-1)^{2n+1}\,\mathrm{tr}(\gamma^5\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_{2n}}\gamma^{\mu_{2n+1}}\gamma^5)\,, \\
&= -\,\mathrm{tr}(\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_{2n}}\gamma^{\mu_{2n+1}}\gamma^5\gamma^5)\,, \\
&= -\,\mathrm{tr}(\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_{2n}}\gamma^{\mu_{2n+1}})\,,
\end{aligned}$$

where the factor $(-1)^{2n+1} = (-1)$ comes from anticommuting γ^5 to the left through all $2n+1$ gamma matrices. We conclude that $\mathrm{tr}(\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_{2n}}\gamma^{\mu_{2n+1}}) = 0$

(d) $\mathrm{tr}\,(\gamma^5) = \mathrm{tr}\,(\gamma^5\gamma^\mu) = \mathrm{tr}\,(\gamma^5\gamma^\mu\gamma^\nu) = \mathrm{tr}\,(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho) = 0$,

Solution: We begin by first proving $\mathrm{tr}(\gamma^5) = 0$. By definition, $\gamma^5\gamma^\mu = -\gamma^\mu\gamma^5$. Now, taking the trace, and inserting the identity in the form $I_4 = (\gamma^0)^2$,

$$\begin{aligned}
\mathrm{tr}(\gamma^5) &= \mathrm{tr}(I_4\gamma^5)\,,\\
&= \mathrm{tr}(\gamma^0\gamma^0\gamma^5)\,,\\
&= -\,\mathrm{tr}(\gamma^0\gamma^5\gamma^0)\,,\\
&= -\,\mathrm{tr}(\gamma^0\gamma^0\gamma^5)\,,\\
&= -\,\mathrm{tr}(\gamma^5)\,,
\end{aligned}$$

where we anticommuted γ^0 to the right going to line 3, and the used the cyclic property of the trace in line 4. Therefore, we conclude $\mathrm{tr}(\gamma^5) = 0$.

We note that since γ^5 is defined as $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, that $\mathrm{tr}(\gamma^5\gamma^\mu) = \mathrm{tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho) = 0$ since this is the trace of an odd number of gamma matrices.

Therefore, the remaining identity to show is $\mathrm{tr}(\gamma^5\gamma^\mu\gamma^\nu) = 0$. Note that if $\mu = \nu$, then $(\gamma^\mu)^2 = \pm I_4$ where the '+' is for $\mu = 0$, and '−' otherwise. So, if $\mu = \nu$, then $\mathrm{tr}(\gamma^5\gamma^\mu\gamma^\nu) \to \pm\,\mathrm{tr}(\gamma^5 I_4) = 0$ by the first identity proved in this solution. What remains is the case where $\mu \neq \nu$. We insert an identity of the form $I_4 = \pm(\gamma^\rho)^2$, where we are free to choose $\rho \neq \mu$ and $\rho \neq \nu$, so that $\{\gamma^\rho, \gamma^\mu\} = \{\gamma^\rho, \gamma^\nu\} = 0$. Taking the trace,

$$\begin{aligned}
\mathrm{tr}(\gamma^5\gamma^\mu\gamma^\nu) &= \mathrm{tr}(I_4\gamma^5\gamma^\mu\gamma^\nu)\,,\\
&= \pm\,\mathrm{tr}(\gamma^\rho\gamma^\rho\gamma^5\gamma^\mu\gamma^\nu)\,,\\
&= (\pm 1)(-1)^3\,\mathrm{tr}(\gamma^\rho\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho)\,,\\
&= (\pm 1)(-1)^3\,\mathrm{tr}(\gamma^\rho\gamma^\rho\gamma^5\gamma^\mu\gamma^\nu)\,,\\
&= (-1)^3\,\mathrm{tr}(\gamma^5\gamma^\mu\gamma^\nu) = -\,\mathrm{tr}(\gamma^5\gamma^\mu\gamma^\nu)\,,
\end{aligned}$$

where in the third line we anticommuted γ^ρ to the right three times, and in the fourth used the cyclic property of the trace. We conclude that $\mathrm{tr}(\gamma^5\gamma^\mu\gamma^\nu) = 0$ for all μ, ν.

(e) $\mathrm{tr}\,(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = -4i\epsilon^{\mu\nu\rho\sigma}$.

Solution: Using the anticommutation relation on the last two gamma matrices, we find

$$\begin{aligned}
\mathrm{tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) &= \mathrm{tr}[\gamma^5\gamma^\mu\gamma^\nu(2g^{\rho\sigma}\,I_4 - \gamma^\sigma\gamma^\rho)]\,,\\
&= 2g^{\rho\sigma}\,\mathrm{tr}(\gamma^5\gamma^\mu\gamma^\nu) - \mathrm{tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\rho)\,,\\
&= -\,\mathrm{tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\rho)\,,
\end{aligned}$$

where we used the result from part d that $\mathrm{tr}(\gamma^5\gamma^\mu\gamma^\nu) = 0$. If $\rho = \sigma$, then $(\gamma^\rho)^2 = \pm I_4$

where the '+' is for $\rho = 0$ and '−' otherwise. Thus, if $\rho = \sigma$, we have $\mathrm{tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) \to \mp\,\mathrm{tr}(\gamma^5\gamma^\mu\gamma^\nu) = 0$. So, we conclude that $\mathrm{tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma)$ is antisymmetric in ρ and σ. We can repeat this argument for any pair of indices, ultimately concluding that $\mathrm{tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma)$ is completely antisymmetric in all μ, ν, ρ, σ indices. In 4D spacetime, the only Lorentz tensor that is completely antisymmetric is the Levi-Civita, therefore we conclude

$$\mathrm{tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = A\epsilon^{\mu\nu\rho\sigma}\,,$$

where A is an undetermined constant and $\epsilon^{\mu\nu\rho\sigma}$ is defined such that $\epsilon^{0123} = +1$.

To determine the constant, we can take any particular combination of Lorentz indices. Let us take $(\mu, \nu, \rho, \sigma) = (0, 1, 2, 3)$, so that

$$\mathrm{tr}(\gamma^5\gamma^0\gamma^1\gamma^2\gamma^3) = A\epsilon^{0123} = A\,.$$

Using the definition $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, we evaluate the trace,

$$\begin{aligned}
\mathrm{tr}(\gamma^5\gamma^0\gamma^1\gamma^2\gamma^3) &= i\,\mathrm{tr}(\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3)\,,\\
&= (-)^3 i\,\mathrm{tr}(\gamma^0\gamma^0\gamma^1\gamma^2\gamma^3\gamma^1\gamma^2\gamma^3)\,,\\
&= (-1)^3(-1)^2 i\,\mathrm{tr}(\gamma^0\gamma^0\gamma^1\gamma^1\gamma^2\gamma^3\gamma^2\gamma^3)\,,\\
&= (-1)^3(-1)^2(-1) i\,\mathrm{tr}(\gamma^0\gamma^0\gamma^1\gamma^1\gamma^2\gamma^2\gamma^3\gamma^3)\,,\\
&= (-1)^3(-1)^2(-1) i\,\mathrm{tr}((+I_4)(-I_4)(-I_4)(-I_4))\,,\\
&= -i\,\mathrm{tr}(I_4)\,,\\
&= -i4\,,
\end{aligned}$$

where in the second, third, and fourth lines we anticommutation relations to arrange identical gamma matrices into pairs, and in the fifth line we used that $(\gamma^0)^2 = +I_4$ while $(\gamma^j)^2 = -I_4$. We conclude that $A = -4i$, so that

$$\mathrm{tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = -4i\epsilon^{\mu\nu\rho\sigma}$$

3. The chiral projectors are defined as

$$P_R = \frac{1}{2}(I_4 + \gamma^5)\,, \qquad P_L = \frac{1}{2}(I_4 - \gamma^5)\,,$$

where I_4 is the 4×4 identity matrix. Prove the following properties:

(a) $\gamma^5 P_L = -P_L$, and $\gamma^5 P_R = P_R$,

Solution: Note that $(\gamma^5)^2 = I_4$, thus,

$$\begin{aligned}\gamma^5 P_{L/R} &= \frac{1}{2}\gamma^5(I_4 \mp \gamma^5)\,, \\ &= \frac{1}{2}(\gamma^5 \mp (\gamma^5)^2)\,, \\ &= \frac{1}{2}(\gamma^5 \mp I_4)\,, \\ &= \mp\frac{1}{2}(I_4 \mp \gamma^5)\,, \\ &= \mp P_{L/R}\,.\end{aligned}$$

(b) $(P_{L/R})^2 = P_{L/R}$,

Solution: Taking the square of the projectors,

$$\begin{aligned}(P_{L/R})^2 &= \left(\frac{1}{2}(I_4 \mp \gamma^5)\right)^2 , \\ &= \frac{1}{4}(I_4 \mp \gamma^5)(I_4 \mp \gamma^5)\,, \\ &= \frac{1}{4}(I_4 \mp \gamma^5 \mp \gamma^5 + (\gamma^5)^2)\,, \\ &= \frac{1}{2}(I_4 \mp \gamma^5) = P_{L/R}\end{aligned}$$

(c) $P_L P_R = P_R P_L = 0$,

Solution: Taking product

$$\begin{aligned}P_{L/R}P_{R/L} &= \frac{1}{2}(I_4 \mp \gamma^5) \cdot \frac{1}{2}(I_4 \pm \gamma^5)\,, \\ &= \frac{1}{2}(I_4 \mp \gamma^5 \pm \gamma^5 - (\gamma^5)^2)\,, \\ &= \frac{1}{2}(I_4 - I_4) = 0\,.\end{aligned}$$

Therefore, we conclude $P_L P_R = P_R P_L = 0$.

(d) $P_L + P_R = I_4$.

Solution: Taking $P_L + P_R$, we find

$$P_L + P_R = \frac{1}{2}(I_4 - \gamma^5) + \frac{1}{2}(I_4 + \gamma^5)\,,$$
$$= \frac{1}{2}\left(2I_4 - \gamma^5 + \gamma^5\right)\,,$$
$$= I_4\,.$$

4. Suppose the charge conjugation operator is defined as $C = i\gamma^2\gamma^0$. Confirm that in the Weyl representation,

(a) $C^{-1} = C^\top = C^\dagger = -C$.

Solution: Given $C = i\gamma^2\gamma^0$, we first check if the matrix C is unitary, $C^\dagger C = CC^\dagger = I_4$. Note the following useful property,

$$(\gamma^\mu)^2 = \frac{1}{2}\{\gamma^\mu, \gamma^\mu\} \text{ (no sum on } \mu)\,,$$
$$= \frac{1}{2} \cdot 2g^{\mu\mu}\, I_4 \text{ (no sum on } \mu)\,,$$
$$= g^{\mu\mu}\, I_4 \text{ (no sum on } \mu)\,,$$

so $(\gamma^0)^2 = I_4$ and $(\gamma^j)^2 = -I_4$.

$$C^\dagger = (i\gamma^2\gamma^0)^\dagger = -i(\gamma^0)^\dagger(\gamma^2)^\dagger = -i\gamma^0(-\gamma^2) = i\gamma^0\gamma^2 = -i\gamma^2\gamma^0 = -C$$

So,

$$C^\dagger C = (-i\gamma^2\gamma^0)(i\gamma^2\gamma^0) = \gamma^2\gamma^0\gamma^2\gamma^0 = -\gamma^2\gamma^0\gamma^0\gamma^2 = -\gamma^2\gamma^2 = +I_4$$

Since C is unitary, we conclude that $C^\dagger = C^{-1}$.

$$C^\top = (C^\dagger)^* = -C^* = -(i\gamma^2\gamma^0)^* = -(-i)(\gamma^2)^*(\gamma^0)^*\,.$$

In the Weyl basis, γ^0 is real, so $(\gamma^0)^* = \gamma^0$, and $(\gamma^2)^*$ is

$$(\gamma^2)^* = \begin{pmatrix} 0 & (\sigma^2)^* \\ -(\sigma^2)^* & 0 \end{pmatrix} = -\begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix} = -\gamma^2\,,$$

where we used the fact that $(\sigma^2)^* = -\sigma^2$ since the non-zero entries of σ^2 are purely imaginary. Therefore,

$$C^\top = -(-i)(\gamma^2)^*(\gamma^0)^* = i(-\gamma^2)\gamma^0 = -i\gamma^2\gamma^0 = -C\,.$$

We thus conclude that C is a real matrix, $C^* = C$, and that

$$C^{-1} = C^\top = C^\dagger = -C\,,$$

as desired.

(b) $C\gamma^\mu C^{-1} = -(\gamma^\mu)^\top$,

Solution: To prove this, first recall that $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$. Since γ^0 is real in the Weyl basis, then we conclude $(\gamma^\mu)^\top = \gamma^0(\gamma^\mu)^*\gamma^0$, where the $*$ denote complex conjugation. Now, with $C = i\gamma^2\gamma^0 = -i\gamma^0\gamma^2$, with $C^{-1} = C^\dagger = -i\gamma^2\gamma^0 = -C$, multiply $(\gamma^\mu)^\top = \gamma^0(\gamma^\mu)^*\gamma^0$ on the left with C^{-1} and on the right with C,

$$\begin{aligned} C^{-1}(\gamma^\mu)^\top C &= C^\dagger \left(\gamma^0(\gamma^\mu)^*\gamma^0\right) C\,, \\ &= \left(-i\gamma^2\gamma^0\right)\left(\gamma^0(\gamma^\mu)^*\gamma^0\right)\left(-i\gamma^0\gamma^2\right), \\ &= -\gamma^2(\gamma^\mu)^*\gamma^2\,, \end{aligned}$$

where in the last line we used $(\gamma^0)^2 = I_4$. For $\mu = 0, 1, 3$, then $(\gamma^\mu)^* = \gamma^\mu$ since they are real in the Weyl basis. Then, $\gamma^\mu\gamma^2 = -\gamma^2\gamma^\mu$ for $\mu \neq 2$. Since $(\gamma^2)^2 = -I_4$, we find $C^{-1}(\gamma^\mu)^\top C = -\gamma^2(-\gamma^2\gamma^\mu) = (-1)^2(-\gamma^\mu) = -\gamma^\mu$ for $\mu \neq 2$. When $\mu = 2$, then $(\gamma^2)^* = -\gamma^2$. So, $C^{-1}(\gamma^\mu)^\top C = -\gamma^2(-\gamma^2)\gamma^2 = -\gamma^2$. Therefore, we conclude for all μ, $C^{-1}(\gamma^\mu)^\top C = -\gamma^\mu$. Now, multiply on the left by C, and on the right by C^{-1},

$$CC^{-1}(\gamma^\mu)^\top CC^{-1} = -C\gamma^\mu C^{-1}\,, \implies C\gamma^\mu C^{-1} = -(\gamma^\mu)^\top\,,$$

which was to be proved.

(c) $C\gamma^5 C^{-1} = (\gamma^5)^\top$,

Solution: Here, let us take the commutator of C and γ^5,

$$\begin{aligned} [C, \gamma^5] &= C\gamma^5 - \gamma^5 C\,, \\ &= i\gamma^2\gamma^0\gamma^5 - i\gamma^5\gamma^2\gamma^0\,, \\ &= i\gamma^2\gamma^0\gamma^5 - i\gamma^2\gamma^0\gamma^5\,, \\ &= 0 \end{aligned}$$

since $\{\gamma^5, \gamma^\mu\} = 0$. Thus, $C\gamma^5 C^{-1} = \gamma^5$. Note that $(\gamma^5)^\dagger = \gamma^5$, and in the Weyl basis $\gamma^5 = (\gamma)^*$. Thus, we conclude $C\gamma^5 C^{-1} = (\gamma^5)^\top$.

5. A Dirac spinor ψ is called a Majorana spinor if it satisfies the condition $\psi = C\bar{\psi}^\top$, and is called a Weyl spinor if it satisfies either $\psi = P_R\psi$ or $\psi = P_L\psi$. Determine whether or not a spinor can be both Majorana and Weyl.

Solution: Let us define $\psi^c \equiv C\bar{\psi}^\top$, $\psi_L \equiv P_L\psi$, and $\psi_R \equiv P_R\psi$. We take the charge conjugation of a chiral fermion. For example, let us take $(\psi_L)^c = C\bar{\psi}_L^\top$. Since $\bar{\psi} = \psi^\dagger\gamma^0$, we have

$$\begin{aligned}(\psi_L)^c &= C\bar{\psi}_L^\top\,,\\ &= C(\psi_L^\dagger\gamma^0)^\top\,,\\ &= C((P_L\psi)^\dagger\gamma^0)^\top\,,\\ &= C(\psi^\dagger P_L\gamma^0)^\top\,,\\ &= C(\gamma^0)^\top(P_L)^\top\psi^*\,,\end{aligned}$$

where we used that $\psi^\dagger = (\psi^\top)^*$ and $P_L^\dagger = P_L$ since $(\gamma^5)^\dagger = \gamma^5$. Now, we note that

$$\begin{aligned}(P_L)^\top &= \frac{1}{2}(I_4 - \gamma^5)^\top\,,\\ &= \frac{1}{2}(I_4 - (\gamma^5)^\top)\,,\\ &= \frac{1}{2}(I_4 - C^{-1}\gamma^5C)\,,\\ &= C^{-1}\left[\frac{1}{2}(I_4 - \gamma^5)\right]C\,,\\ &= C^{-1}P_LC\,,\end{aligned}$$

where we used $C^{-1}\gamma^5C = (\gamma^5)^\top$ and $C^{-1}C = I_4$. So, we have

$$\begin{aligned}(\psi_L)^c &= C(\gamma^0)^\top C^{-1}P_LC\psi^*\,,\\ &= -\gamma^0P_LC\psi^*\,,\end{aligned}$$

where we used $C(\gamma^0)^\top C^{-1} = -\gamma^0$. Recall that $\gamma^0\gamma^5 = -\gamma^5\gamma^0$, so $\gamma^0P_L = P_R\gamma^0$. Thus, we have

$$\begin{aligned}(\psi_L)^c &= -\gamma^0P_LC\psi^*\,,\\ &= -P_R\gamma^0C\psi^*\,.\end{aligned}$$

Finally, we use again $C^{-1}\gamma^0C = -(\gamma^0)^\top \implies \gamma^0C = -C(\gamma^0)^\top$, so that

$$\begin{aligned}(\psi_L)^c &= -P_R\gamma^0C\psi^*\,,\\ &= P_RC(\gamma^0)^\top\psi^*\,,\\ &= P_RC(\psi^\dagger\gamma^0)^\top\,,\\ &= P_RC\bar{\psi}^\top\,,\\ &= P_R\psi^c\,,\\ &= (\psi^c)_R\,.\end{aligned}$$

We find that $(\psi_L)^c = (\psi^c)_R$. Similar arguments show that $(\psi_R)^c = (\psi^c)_L$. We conclude under charge conjugation the chirality flips for a Weyl fermion. So, if we consider a four-component spinor, $\psi = (\psi_L, \psi_R)^\top$, then charge conjugation flips the chirality, but the spinor is simply a rotated version. The two-component spinors themselves are not eigenstates of both the Majorana and Weyl equation.

6. Consider a generic $2 \to n$ reaction $ab \to c_1 c_2 \ldots c_n$ in the *lab frame* or *fixed-target frame*, that is the frame where particle b is at rest and a is the incident beam. Assume $\max(m_a, m_b) < \min(m_{c_1}, m_{c_2}, \ldots, m_{c_n})$.

(a) Show that $s = m_a^2 + m_b^2 + 2m_b\sqrt{m_a^2 + P_{\text{lab.}}^2}$ where $P_{\text{lab.}}$ is the beam momentum.

Solution: By definition, $s = (p_a + p_b)^2 = m_a^2 + m_b^2 + 2p_a \cdot p_b$, where $p_a^2 = m_a^2$ and $p_b^2 = m_b^2$. Since in the lab frame, $\mathbf{p}_b = \mathbf{0}$ and $E_b = m_b$, we have $p_a \cdot p_b = E_a E_b = m_b\sqrt{m_a^2 + |\mathbf{p}_a|^2}$. Since the beam momentum is $|\mathbf{p}_a| = P_{\text{lab.}}$, we find the desired result,

$$s = m_a^2 + m_b^2 + 2m_b\sqrt{m_a^2 + P_{\text{lab.}}^2} \, .$$

(b) Express the beam kinetic energy, $T_a \equiv E_a - m_a$, in terms of s.

Solution: From the previous result, $s = m_a^2 + m_b^2 + 2m_b E_a$ where $E_a = \sqrt{m_a^2 + P_{\text{lab.}}^2}$, we have

$$E_a = \frac{s - m_a^2 - m_b^2}{2m_b} \, ,$$

in the lab frame. Then, the kinetic energy is

$$T_a = E_a - m_a = \frac{s - m_a^2 - m_b^2}{2m_b} - m_a = \frac{s - (m_a + m_b)^2}{2m_b} \, .$$

(c) What is the minimum kinetic energy of the beam with which the reaction can occur?

Solution: Since $\max(m_a, m_b) < \min(m_{c_1}, m_{c_2}, \ldots, m_{c_n})$, the minimum s is

$$s^{(\text{min})} = (m_{c_1} + m_{c_2} + \cdots m_{c_n})^2 = \left(\sum_{j=1}^{n} m_{c_j}\right)^2 \, .$$

So, the minimum kinetic energy is

$$\begin{aligned} T_a^{(\text{min})} &= \frac{s^{(\text{min})} - (m_a + m_b)^2}{2m_b} \, , \\ &= \frac{1}{2m_b}\left[\left(\sum_{j=1}^{n} m_{c_j}\right)^2 - (m_a + m_b)^2\right] \, . \end{aligned}$$

7. Consider the following Yukawa theory as a simplified model of an interacting proton p, neutron n, and neutral pion π^0. We assume that the proton and neutron are distinguishable, but mass degenerate. The Lagrange density is given by

$$\mathcal{L} = \sum_f \frac{i}{2}\bar{\psi}_f \partial\!\!\!/ \psi_f + \text{h.c.} - \sum_f M\bar{\psi}_f\psi_f + \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 - \sum_f g\varphi\bar{\psi}_f\gamma^5\psi_f\,,$$

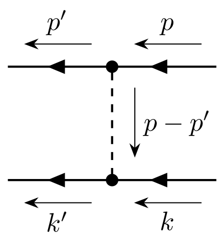
where f is fermion index $f = \{n, p\}$, M is the proton and neutron mass, m is the pion mass, and g is the coupling between proton and pion, as well as the neutron and pion.

(a) Consider the elastic reaction

$$n(p, s) + p(k, r) \to n(p', s') + p(k', r')\,,$$

where the arguments are the momenta and the subscripts are the spin-state. Write down the $np \to np$ scattering amplitude to leading order in the coupling g. **Hint:** Only one diagram contributes at $\mathcal{O}(g^2)$. Refer to the summary notes on `Feynman rules - Yukawa Theory`.

Solution: From the Feynman rules, we find a single diagram at $\mathcal{O}(g^2)$,

$i\mathcal{M} =$  $+\,\mathcal{O}(g^4),$

$$= \bar{u}_{s'}(p')(-ig\gamma^5)u_s(p)\,\frac{i}{(p-p')^2 - m^2}\,\bar{u}_{r'}(k')(-ig\gamma^5)u_r(k) + \mathcal{O}(g^4)\,,$$

$$= -\frac{ig^2}{t - m^2}[\bar{u}_{s'}(p')\gamma^5 u_s(p)][\bar{u}_{r'}(k')\gamma^5 u_r(k)] + \mathcal{O}(g^4)$$

(b) The *spin-averaged* squared amplitude is defined as

$$\left\langle|\mathcal{M}|^2\right\rangle \equiv \frac{1}{2}\sum_s \frac{1}{2}\sum_r \sum_{s'}\sum_{r'} \left|\mathcal{M}(n_s p_r \to n_{s'}p_{r'})\right|^2.$$

Show that at leading order

$$\left\langle|\mathcal{M}|^2\right\rangle = g^4\frac{t^2}{(t-m^2)^2} + \mathcal{O}\big(g^6\big)$$

where s, t, and u are the Mandelstam invariants. **Note:** You are encouraged to use a computer algebra software such as `FeynCalc` (https://feyncalc.github.io), which is a `Mathematica` package for symbolic evaluation of Feynman diagrams and algebraic calculations in quantum field theory and elementary particle physics. A useful tutorial can be found here. `Mathematica` is free to all students at William & Mary (see https://software.wm.edu).

Solution: By definition,

$$\langle|\mathcal{M}|^2\rangle \equiv \frac{1}{2}\sum_s \frac{1}{2}\sum_r \sum_{s'}\sum_{r'} \Big|\mathcal{M}\Big|^2 ,$$

$$\equiv \frac{1}{4}\sum_{s,s'}\sum_{r,r'} \mathcal{M}^\dagger \mathcal{M}$$

Now,

$$\mathcal{M}^\dagger\mathcal{M} = \frac{g^4}{(t-m^2)^2}\,[\bar{u}_{s'}(p')\gamma^5 u_s(p)]^\dagger[\bar{u}_{r'}(k')\gamma^5 u_r(k)]^\dagger[\bar{u}_{s'}(p')\gamma^5 u_s(p)][\bar{u}_{r'}(k')\gamma^5 u_r(k)] + \mathcal{O}(g^6)$$

Note that

$$\begin{aligned}
[\bar{u}_{s'}\gamma_5 u_s]^\dagger &= u_s^\dagger \gamma_5^\dagger \bar{u}_{s'}^\dagger ,\\
&= u_s^\dagger \gamma_5^\dagger (u_{s'}^\dagger \gamma^0)^\dagger ,\\
&= u_s^\dagger \gamma_5 (\gamma^0)^\dagger u_{s'} ,\\
&= u_s^\dagger \gamma_5 \gamma^0 u_{s'} ,\\
&= -u_s^\dagger \gamma^0 \gamma_5 u_{s'} ,\\
&= -\bar{u}_s \gamma_5 u_{s'} ,
\end{aligned}$$

where $\gamma_5^\dagger = \gamma_5$, $(\gamma^0)^\dagger = \gamma^0$, and $\{\gamma^0, \gamma^5\} = 0$

$$\begin{aligned}
\sum_{s,s'} \bar{u}_s(p)\gamma^5 u_{s'}(p')\,\bar{u}_{s'}(p')\gamma^5 u_s(p) &= \sum_{s,s'} \mathrm{tr}\left(\bar{u}_s(p)\gamma^5 u_{s'}(p')\,\bar{u}_{s'}(p')\gamma^5 u_s(p)\right) ,\\
&= \mathrm{tr}\left(\gamma^5 \sum_{s'} u_{s'}(p')\,\bar{u}_{s'}(p')\gamma^5 \sum_s u_s(p)\bar{u}_s(p)\right) ,\\
&= \mathrm{tr}\left(\gamma^5(\not{p}' + M)\gamma^5(\not{p} + M)\right) ,\\
&= 4(M^2 - p'\cdot p) = 2t ,
\end{aligned}$$

where $t = (p'-p)^2 = 2M^2 - 2p'\cdot p$. Similarly, we find

$$\begin{aligned}
\sum_{r,r'} \bar{u}_r(k)\gamma^5 u_{r'}(k')\,\bar{u}_{r'}(k')\gamma^5 u_r(k) &= \mathrm{tr}\left(\gamma^5(\not{k}' + M)\gamma^5(\not{k} + M)\right) ,\\
&= 4(M^2 - k'\cdot k) = 2t .
\end{aligned}$$

Putting all the pieces together, we find

$$\langle|\mathcal{M}|^2\rangle = \frac{1}{4}\frac{g^4}{(t-m^2)^2}(2t)^2 + \mathcal{O}(g^6) = g^4\,\frac{t^2}{(t-m^2)^2} + \mathcal{O}(g^6) ,$$

as desired.

(c) Compute the unpolarized differential cross-section $\mathrm{d}\sigma/\mathrm{d}t$ in terms of the Mandelstam invariants.

Solution: The unpolarized differential cross section for equal mass scattering is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2 s}\left\langle|\mathcal{M}|^2\right\rangle .$$

To find $\mathrm{d}\sigma/\mathrm{d}t$, note that for these kinematics $t = -2|\mathbf{p}|^2(1-\cos\theta)$, where $|\mathbf{p}|$ is the magnitude of the relative momentum of the neutron in the CM frame. So, $\mathrm{d}t = 2|\mathbf{p}|^2\,\mathrm{d}\cos\theta$, and $\mathrm{d}\Omega = 2\pi\,\mathrm{d}\cos\theta$. Finally, $|\mathbf{p}|^2 = (s-4M^2)/4$. So, the unpolarized $\mathrm{d}\sigma/\mathrm{d}t$ is

$$\begin{aligned}\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{\mathrm{d}\Omega}{\mathrm{d}t}\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} &= \frac{\pi}{|\mathbf{p}|^2}\frac{1}{64\pi^2 s}\left\langle|\mathcal{M}|^2\right\rangle ,\\ &= \frac{1}{16\pi s(s-4M^2)}\left\langle|\mathcal{M}|^2\right\rangle .\end{aligned}$$

Therefore, given that the spin-averaged matrix element squared is

$$\left\langle|\mathcal{M}|^2\right\rangle = g^4\frac{t^2}{(t-m^2)^2} + \mathcal{O}(g^6)\,,$$

we have

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{g^4}{16\pi}\frac{1}{s(s-4M^2)}\frac{t^2}{(t-m^2)^2} + \mathcal{O}(g^6)$$

(d) Express $\mathrm{d}\sigma/\mathrm{d}\Omega$ in terms of s and the center-of-momentum frame scattering angle θ.

Solution:

$$\begin{aligned}\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} &= \frac{g^4}{64\pi^2 s}\frac{t^2}{(t-m^2)^2} + \mathcal{O}(g^6)\,,\\ &= \frac{g^4}{64\pi^2 s}\frac{4|\mathbf{p}|^4(1-\cos\theta)^2}{(-2|\mathbf{p}|^2(1-\cos\theta)-m^2)^2} + \mathcal{O}(g^6)\,,\\ &= \frac{g^4}{64\pi^2 s}\left(\frac{1-\cos\theta}{\zeta(s)-\cos\theta}\right)^2 + \mathcal{O}(g^6)\,,\end{aligned}$$

where in the last line we defined for convenience,

$$\zeta(s) \equiv 1 + \frac{m^2}{2|\mathbf{p}|^2} = 1 + \frac{2m^2}{s-4M^2}\,.$$

(e) Compute the total cross-section as a function of s.

Solution: The total cross-section is defined as

$$\sigma = \int \mathrm{d}\Omega \, \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = 2\pi \int_{-1}^{+1} \mathrm{d}\cos\theta \, \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \,, \\ = \frac{g^4}{32\pi s} \int_{-1}^{+1} \mathrm{d}\cos\theta \left(\frac{1-\cos\theta}{\zeta(s)-\cos\theta} \right)^2 + \mathcal{O}(g^6) \,.$$

Note that for physical scattering, $\zeta(s) > 1$ for all $s \geq 4M^2$. So, the integral yields

$$\int_{-1}^{+1} \mathrm{d}\cos\theta \left(\frac{1-\cos\theta}{\zeta(s)-\cos\theta} \right)^2 = \frac{2}{\zeta(s)+1} \left[2\zeta(s) + \left(\zeta^2(s) - 1\right) \log\left(\frac{\zeta(s)-1}{\zeta(s)+1} \right) \right] \,.$$

So, the total cross-section is

$$\sigma = \frac{g^4}{16\pi s} \frac{1}{\zeta(s)+1} \left[2\zeta(s) + \left(\zeta^2(s) - 1\right) \log\left(\frac{\zeta(s)-1}{\zeta(s)+1} \right) \right] \,, \\ = \left(\frac{g^2}{4\pi} \right)^2 \frac{\pi}{s(\zeta(s)+1)} \left[2\zeta(s) + \left(\zeta^2(s) - 1\right) \log\left(\frac{\zeta(s)-1}{\zeta(s)+1} \right) \right] \,, \\ = \left(\frac{g^2}{4\pi} \right)^2 \frac{\pi}{s} \left[\frac{2\zeta(s)}{\zeta(s)+1} + (\zeta(s) - 1) \log\left(\frac{\zeta(s)-1}{\zeta(s)+1} \right) \right] \,.$$

(f) Estimate the magnitude of the pion-nucleon coupling g, as well as the quantity $g^2/4\pi$, from the experimentally observed np total cross-section. **Note:** You do not need to fit the data, however feel free to do so. The Review of Particle Physics contains experimental cross-sections for select processes. See the course webpage for the data file.

Solution: The summary data from the Review of Particle Physics is given in terms of the lab frame beam momentum $P_{\text{lab.}}$, which is related to s via

$$s = 2M^2 + 2M\sqrt{M^2 + P_{\text{lab.}}^2} \,.$$

So, the total cross-section is given by

$$\sigma(P_{\text{lab.}}) = \left(\frac{g^2}{4\pi} \right)^2 (\hbar c)^2 \, \mathcal{I}\left(2M^2 + 2M\sqrt{M^2 + P_{\text{lab.}}^2} \right) \,,$$

where $\mathcal{I}(s)$ is a distribution function

$$\mathcal{I}(s) \equiv \frac{\pi}{s} \left[\frac{2\zeta(s)}{\zeta(s)+1} + (\zeta(s) - 1) \log\left(\frac{\zeta(s)-1}{\zeta(s)+1} \right) \right] \,,$$

and $s = 2M^2 + 2M\sqrt{M^2 + P_{\text{lab.}}^2}$. The factor $(\hbar c)^2$ converts the cross-section in natural units to millibarn (mb), with $(\hbar c)^2 \approx 0.389\,\mathrm{GeV}^2 \cdot \mathrm{mb}$.
Note that the first inelastic threshold in this theory is π^0 production, $np \to np\pi^0$, when $s = (2M+m)^2$. Therefore, we must only include data when $(2M)^2 \leq s < (2M+m)^2$, or

in terms of $P_{\text{lab.}}$,

$$P_{\text{lab.}} < \sqrt{\left(\frac{(2M+m)^2 - 2M^2}{2M}\right)^2 - M^2}\,,$$
$$= \frac{1}{2M}\sqrt{m(4M+m)(2M+m)^2}$$

Since $M \approx 0.940\,\text{GeV}$ and $m \approx 0.140\,\text{GeV}$, then $P_{\text{lab.}} \lesssim 0.790\,\text{GeV}/c$.

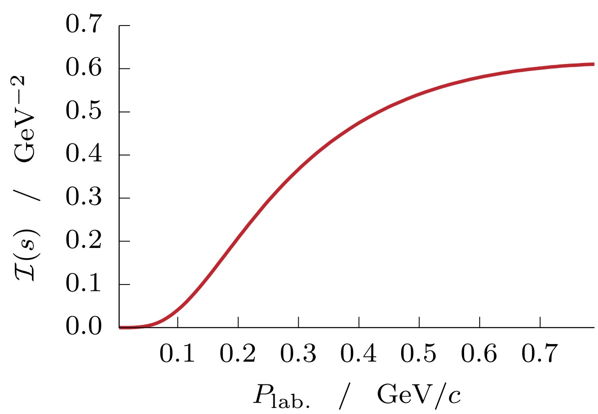


Figure 1: The function $\mathcal{I}$ as a function of $P_{\text{lab.}}$

We can gain some insight into this theoretical cross-section by plotting the function $\mathcal{I}$ as a function of $P_{\text{lab.}}$. In Fig 1, we see that the function increases as $P_{\text{lab.}}$ increases. This is not the behavior of the data, which can be seen in Fig. 2, which rises dramatically as we approach threshold. Therefore, we suspect that this calculation is incomplete, either in the need for higher orders (which seems problematic from the want of a perturbative expansion if the threshold behavior is so different), or that the theory itself is incomplete (indeed, np scattering contains charge exchange, but even still this theory is not well-defined perturbatively.)

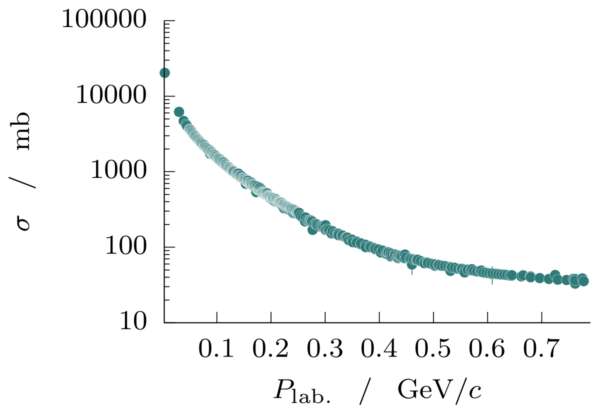


Figure 2: np cross-section as a function of $P_{\text{lab.}}$.

Nevertheless, we can still obtain a rough estimate for the coupling can then be given by comparing the theoretical cross-section to the measured cross-section at some $P_{\text{lab.}}$. Since the near threshold region is not captured, let's consider taking a data point near $P_{\text{lab.}}$. For example, point 440 from the RPP table, $P\text{lab.} = 0.696\,\text{GeV}/c$ with $\sigma_{\text{exp.}} = (38.963 \pm 0.169)\,\text{mb}$. At this momentum, $s_{\text{exp.}} = 3.97\,\text{GeV}^2$, and $\mathcal{I}(s_{\text{exp.}}) = 0.601\,\text{GeV}^{-2}$. So,

$$
\begin{aligned}
\left(\frac{g^2}{4\pi}\right)^2 &\approx \frac{\sigma_{\text{exp.}}}{(\hbar c)^2\,\mathcal{I}(s_{\text{exp.}})}\,, \\
&\approx \frac{38.963\,\text{mb}}{(0.389\,\text{GeV}^2\cdot\text{mb})(0.601\,\text{GeV}^{-2})}\,, \\
&\approx 166\,.
\end{aligned}
$$

So, $g^2/4\pi \approx 13$, and $g \approx 12.8$. The coupling is extremely large, which clearly indicates that this theory does not admit a valid interpretation as a perturbation series.

```
FILE_NAME
 REACTION
 BEAM_MASS TARGET_MASS THRESHOLD FINAL_STATE_MULTIPLICITY
 NUMBER_OF_DATA_POINTS
 POINT_NUMBER PLAB(GEV/C) PLAB_MIN PLAB_MAX SIG(MB) STA_ERR+ STA_ERR- SY_ER+(PCT) SY_ER-(PCT) REFERENCE FLAG
  FORMAT(I5,1X,4F11.5,2F8.4,1X,2F6.1,A)
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 578
 1    0.00385    0.00123    0.00531    20360.     100.00  100.00     0.0  0.0  MELKONIAN 49     PR 76, 1744
 2    0.03043    0.03043    0.03043    6202.0     11.200  11.200     0.0  0.0  ENGELKE 63       PR 129, 324
 3    0.03873    0.03873    0.03873    4700.0     40.000  40.000     0.0  0.0  LAMPI 49         PR 76, 188        P
 4    0.04347    0.04347    0.04347    4228.0     18.000  18.000     0.0  0.0  FIELDS 54        PR 94, 389
 5    0.04502    0.04502    0.04502    4060.0     30.000  30.000     0.0  0.0  LAMPI 49         PR 76, 188        P
 6    0.04973    0.04973    0.04973    3675.0     16.000  16.000     0.0  0.0  STORRS 54        PR 95, 1252
 7    0.05000    0.05000    0.05000    3609.5     4.5100  4.5100     2.0  2.0  CIERJACKS 69     PRL 23, 866
 8    0.05020    0.05020    0.05020    3630.0     40.000  40.000     0.0  0.0  LAMPI 49         PR 76, 188        P
 9    0.05100    0.05100    0.05100    3542.9     3.4400  3.4400     2.0  2.0  CIERJACKS 69     PRL 23, 866
 10   0.05200    0.05200    0.05200    3472.1     3.8000  3.8000     2.0  2.0  CIERJACKS 69     PRL 23, 866
 11   0.05300    0.05300    0.05300    3398.6     3.7400  3.7400     2.0  2.0  CIERJACKS 69     PRL 23, 866
 12   0.05311    0.05311    0.05311    3447.0     22.000  22.000     0.0  0.0  DAVIS 71         PR C3, 1798
 13   0.05400    0.05400    0.05400    3326.5     3.6900  3.6900     2.0  2.0  CIERJACKS 69     PRL 23, 866
 14   0.05448    0.05448    0.05448    3330.0     20.000  20.000     0.0  0.0  LAMPI 49         PR 76, 188        P
 15   0.05500    0.05500    0.05500    3266.6     4.2100  4.2100     2.0  2.0  CIERJACKS 69     PRL 23, 866
 16   0.05600    0.05600    0.05600    3206.6     3.6000  3.6000     2.0  2.0  CIERJACKS 69     PRL 23, 866
 17   0.05700    0.05700    0.05700    3138.6     4.1000  4.1000     2.0  2.0  CIERJACKS 69     PRL 23, 866
 18   0.05800    0.05800    0.05800    3079.6     3.5100  3.5100     2.0  2.0  CIERJACKS 69     PRL 23, 866
 19   0.05900    0.05900    0.05900    3015.5     3.9900  3.9900     2.0  2.0  CIERJACKS 69     PRL 23, 866
 20   0.05970    0.05970    0.05970    3004.0     24.000  24.000     0.0  0.0  DAVIS 71         PR C3, 1798
 21   0.06000    0.06000    0.06000    2963.6     3.9500  3.9500     2.0  2.0  CIERJACKS 69     PRL 23, 866
 22   0.06100    0.06100    0.06100    2907.2     3.9000  3.9000     2.0  2.0  CIERJACKS 69     PRL 23, 866
 23   0.06200    0.06200    0.06200    2851.3     3.8600  3.8600     2.0  2.0  CIERJACKS 69     PRL 23, 866
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 26   0.06500    0.06500    0.06500    2705.5     3.7400  3.7400     2.0  2.0  CIERJACKS 69     PRL 23, 866
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 29   0.06700    0.06700    0.06700    2607.4     4.4800  4.4800     2.0  2.0  CIERJACKS 69     PRL 23, 866
 30   0.06800    0.06800    0.06800    2558.2     3.6200  3.6200     2.0  2.0  CIERJACKS 69     PRL 23, 866
 ...
```