1. Show that the Lie algebra structure constants c_{jkl}, defined by the Lie bracket $[X^j, X^k] = c_{jkl}X^l$, satisfy the relation $c_{jkm}c_{mln} + c_{klm}c_{mjn} + c_{ljm}c_{mkn} = 0$.

Solution: Here we use the Jacobi identity for the elements of the Lie algebra,

$$\sum_{(j,k,l)} [[X^j, X^k], X^l] = 0\,,$$

where (j,k,l) indicates the cyclic sum. Performing the sum, and using $[X^j, X^k] = c_{jkl}X^l$, we find

$$\begin{aligned}\sum_{(j,k,l)} [[X^j, X^k], X^l] &= [[X^j, X^k], X^l] + [[X^k, X^l], X^j] + [[X^l, X^j], X^k]\,,\\ &= c_{jkm}[X^m, X^l] + c_{klm}[X^m, X^j] + c_{ljm}[X^m, X^k]\,,\\ &= c_{jkm}c_{mln}X^n + c_{klm}c_{mjn}X^n + c_{ljm}c_{mkn}X^n\,,\\ &= \left(c_{jkm}c_{mln} + c_{klm}c_{mjn} + c_{ljm}c_{mkn}\right)X^n\,,\\ &= 0\,.\end{aligned}$$

Since this must be zero for any X^n, we must have $c_{jkm}c_{mln} + c_{klm}c_{mjn} + c_{ljm}c_{mkn} = 0$, as desired.

2. Consider a general Lie algebra $[X^j, X^k] = c_{jkl}X^l$, where $c_{jkl} = -c_{kjl}$. From the structure constants, we may form matrices M^j with matrix elements $(M^j)_{lk} = c_{jkl}$. Note the order of the indices. Show that these matrices furnish a representation of the algebra, i.e., show that $[M^j, M^k] = c_{jkl}M^l$. This representation is called the *adjoint representation*. **Hint:** The Jacobi identity may be helpful.

Solution: Looking at the matrix elements of the commutator of $(M^j)_{lk} = c_{jkl}$,

$$\begin{aligned}([M^j, M^k])_{ln} &= (M^j)_{lm}(M^k)_{mn} - (M^k)_{lm}(M^j)_{mn}\,,\\ &= c_{jml}c_{knm} - c_{kml}c_{jnm}\,,\\ &= c_{ljm}c_{mkn} + c_{klm}c_{mjn}\,,\end{aligned}$$

where we used the antisymmetry of the structure constants. From the Jacobi identity of the structure constants, $c_{jkm}c_{mln} + c_{klm}c_{mjn} + c_{ljm}c_{mkn} = 0$, we have $c_{ljm}c_{mkn} + c_{klm}c_{mjn} = -c_{jkm}c_{mln}$. So, the commutator is

$$\begin{aligned}([M^j, M^k])_{ln} &= c_{ljm}c_{mkn} + c_{klm}c_{mjn}\,,\\ &= -c_{jkm}c_{mln}\,,\\ &= c_{jkm}c_{mnl}\,,\\ &= c_{jkm}(M^m)_{ln}\,.\end{aligned}$$

Therefore $[M^j, M^k] = c_{jkl}M^l$, and we conclude that $(M^j)_{lk} = c_{jkl}$ is a valid representation of the Lie algebra.

3. Suppose X^j is a generator for the Lie algebra $[X^j, X^k] = c_{jkl}X^l$. Show that $X^2 = \sum_j X^j X^j$ commutes with the group generators, and therefore we may write $(X^2)_{ab} = C_2(r)\,\delta_{ab}$ where $C_2(r)$ is a constant called the *quadratic Casimir* of the representation r.

Solution: We want to show that $[X^2, X^k] = 0$ where $X^2 = \sum_j X^j X^j$ and $[X^j, X^k] = c_{jkl}X^l$. So, taking the commutator

$$\begin{aligned}
[X^2, X^k] &= \sum_j [X^j X^j, X^k]\,, \\
&= \sum_j X^j [X^j, X^k] + \sum_j [X^j, X^k] X^j\,,
\end{aligned}$$

where we used $[AB, C] = A[B, C] + [A, C]B$. Now, we use $[X^j, X^k] = c_{jkl}X^l$, noting l is being summed over implicitly. So, we find

$$\begin{aligned}
[X^2, X^k] &= \sum_j X^j [X^j, X^k] + \sum_j [X^j, X^k] X^j\,, \\
&= \sum_{j,l} X^j (c_{jkl}X^l) + \sum_{j,l} (c_{jkl}X^l) X^j\,, \\
&= \sum_{j,l} c_{jkl}X^j X^l + \sum_{j,l} c_{jkl}X^l X^j\,, \\
&= \sum_{j,l} c_{jkl}X^j X^l + \sum_{j,l} c_{lkj}X^j X^l\,, \\
&= \sum_{j,l} c_{jkl}X^j X^l - \sum_{j,l} c_{jkl}X^j X^l = 0\,,
\end{aligned}$$

where in the fourth line we interchanged the summed indices, and in going to the last line we noted that $c_{lkj} = -c_{jkl}$. Therefore, X^2 commutes with all the generators X^j. Therefore, we can write $(X^2)_{ab} = C_2(r)\delta_{ab}$, where $C_2(r)$ is some constant which depends on the representation r, and a, b span the dimension of the representation, $a, b = 1, \dots, r$.

4. Let X^j be a generator for a generic $\mathfrak{su}(N)$ Lie algebra, $[X^j, X^k] = c_{jkl}X^l$, and $U(\alpha^j)$ is an element of the corresponding Lie group $\mathrm{SU}(N)$, with $U(\alpha^j) = \exp(\alpha^j X_j)$ with $\alpha^j \in \mathbb{R}$. Show that X^j are traceless, antihermitian $N \times N$ matrices.

Solution: Since $U(\alpha^j) \in \mathrm{SU}(N)$, then we require

$$U(\alpha^j)^\dagger U(\alpha^j) = U(\alpha^j) U(\alpha^j)^\dagger = I_N\,,$$

where I_N is the $N \times N$ identity. Furthermore, $\det(U(\alpha^j)) = 1$. From the properties of matrix exponentials, $\exp(\alpha^j X_j)^\dagger = \exp(\alpha^j X_j^\dagger)$. Let us Taylor expand the product $U(\alpha^j)^\dagger U(\alpha^j)$ about

$\alpha^j = 0$,

$$I_N = U(\alpha^j)^\dagger U(\alpha^j) = \Big(I_N + \alpha^j X_j^\dagger + \mathcal{O}(\alpha^2)\Big)\left(I_N + \alpha^j X_j + \mathcal{O}(\alpha^2)\right) ,$$
$$= I_N + \alpha^j X_j + \alpha^j X_j^\dagger + \mathcal{O}(\alpha^2)\,,$$
$$= I_N + \alpha^j (X_j + X_j^\dagger) + \mathcal{O}(\alpha^2)\,,$$

Since this must hold order-by-order in α, we have $X_j + X_j^\dagger = 0$, or $X_j = -X_j^\dagger$, proving that the generators are antihermitian. Next, recall for matrix exponentials $\det(\exp^A) = \exp(\mathrm{tr}(A))$ where A is an $N \times N$ matrix. Since $\det(U(\alpha^j)) = 1$, we have the following

$$1 = \det(U(\alpha^j)) = \det\left(\exp(\alpha^j X_j)\right) ,$$
$$= \exp\left(\mathrm{tr}(\alpha^j X_j)\right) ,$$
$$= \exp\left(\alpha^j\, \mathrm{tr}(X_j)\right) .$$

Since this must hold for any α^j, we conclude that $\mathrm{tr}(X_j) = 0$.

5. Consider the set of all complex 2×2 matrices M with $\det(M) = i$. Does this set form a group under the usual matrix multiplication? Explain your reasoning.

Solution: Let G be the set of all 2×2 matrices with $\det(M) = i$. Let us assume that $M \in$ G, some group where $\det(M) = i$. If M_1 and M_2 are elements of the group, then the product $M_1 \cdot M_2$ should close under the group multiplication, that is $M_3 = M_1 \cdot M_2 \in$ G. Since M_3 is in G, then $\det(M_3) = i$. But, consider $\det(M_3) = \det(M_1 \cdot M_2) = \det(M_1)\det(M_2)$, from the properties of determinants. So, $\det(M_3) = \det(M_1)\det(M_2) = (i)(i) = -1 \neq i$, which contradicts our assumption. Therefore, the product of two group elements does not close under group multiplication, and thus G does not form a group.

Alternatively, assume the existence of an inverse matrix $M^{-1} \in$ G. The determinant of an inverse matrix is $\det(M^{-1}) = 1/\det(M) = 1/i = -i \neq i$. Therefore, we conclude that such an inverse matrix does not exist, and therefore G is not a group.

6. Consider $X_j = -\frac{1}{2} i\sigma_j$ as a bases element of the $\mathfrak{su}(2)$ algebra, $[X_j, X_k] = \epsilon_{jkl} X_l$, where σ_j are the Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

Verify the following:

(a) $[\sigma_j, \sigma_k] \equiv \sigma_j\sigma_k - \sigma_k\sigma_j = 2i\epsilon_{jkl}\sigma_l$.

Solution: We compute the following products,

$$\sigma_1\sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i\sigma_3 \,,$$

$$\sigma_2\sigma_1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = -i\sigma_3 \,,$$

$$\sigma_2\sigma_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = i\sigma_1 \,,$$

$$\sigma_3\sigma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = -i\sigma_1 \,,$$

$$\sigma_3\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\sigma_2 \,,$$

$$\sigma_1\sigma_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i\sigma_2 \,,$$

as well as the squares of the Pauli matrices

$$\sigma_1^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2 \,,$$

$$\sigma_2^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2 \,,$$

$$\sigma_3^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2 \,.$$

Therefore, we have $\sigma_j^2 = I_2$ and $\sigma_j\sigma_k = -\sigma_k\sigma_j$ for $j \neq k$. So, the commutator $[\sigma_j, \sigma_j] = 0$, while $[\sigma_1, \sigma_2] = +2i\sigma_3$, $[\sigma_2, \sigma_3] = +2i\sigma_1$, and $[\sigma_3, \sigma_1] = +2i\sigma_2$. The commutators are completely antisymmetric, thus we can write it in terms of the Levi-Civita ϵ_{jkl} tensor, $[\sigma_j, \sigma_k] = 2i\epsilon_{jkl}\sigma_l$.

(b) $\{\sigma_j, \sigma_k\} \equiv \sigma_j\sigma_k + \sigma_k\sigma_j = 2\delta_{jk}I_2$.

Solution: From the results of part (a), we find that $\sigma_j\sigma_k + \sigma_k\sigma_j = 0$ for $j \neq k$. Therefore, $\{\sigma_k, \sigma_k\} = \sigma_j\sigma_k + \sigma_k\sigma_j = \delta_{jk}(2\sigma_j\sigma_j) = 2\delta_{jk}I_2$.

(c) $\sigma_j\sigma_k = \delta_{jk}I_2 + i\epsilon_{jkl}\sigma_l$.

Solution: Let us add the two results $[\sigma_j, \sigma_k] = 2i\epsilon_{jkl}\sigma_l$ and $\{\sigma_j, \sigma_k\} = 2\delta_{jk}I_2$,

$$[\sigma_j, \sigma_k] + \{\sigma_j, \sigma_k\} = 2\sigma_j\sigma_k = 2i\epsilon_{jkl}\sigma_l + 2\delta_{jk}I_2 \,.$$

Therefore, we immediately find that $\sigma_j\sigma_k = \delta_{jk}I_2 + i\epsilon_{jkl}\sigma_l$.

(d) Show that a group element $U(\alpha^j) \in \mathrm{SU}(2)$ can be written as

$$U(\alpha^j) = \exp\left(-\frac{1}{2}i\alpha^j\sigma_j\right) = I_2 \cos\left(\frac{1}{2}\alpha\right) - i\frac{\alpha^j\sigma_j}{\alpha}\sin\left(\frac{1}{2}\alpha\right),$$

where $\alpha^2 = \sum_j (\alpha_j)^2$.

Solution: Let us Taylor expand about $\alpha^j = 0$,

$$\begin{aligned}\exp\left(-\frac{1}{2}i\alpha^j\sigma_j\right) &= \sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{1}{2}i\alpha^j\sigma_j\right)^n, \\ &= \sum_{n=0}^{\infty}\frac{1}{(2n)!}\left(-\frac{1}{2}i\alpha^j\sigma_j\right)^{2n} + \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\left(-\frac{1}{2}i\alpha^j\sigma_j\right)^{2n+1},\end{aligned}$$

where we split the sum into even and odd terms. Now, $(-i\alpha^j\sigma_j/2)^{2n} = (-1)^n(\alpha^j\sigma_j/2)^{2n}$, while $(-i\alpha^j\sigma_j/2)^{2n+1} = -i(-1)^n(\alpha^j\sigma_j/2)^{2n+1}$. Now, we evaluate $(\alpha^j\sigma_j)^2$,

$$\begin{aligned}(\alpha^j\sigma_j)^2 &= (\alpha^j\sigma_j)(\alpha^k\sigma_k), \\ &= \alpha^j\alpha^k(\sigma_j\sigma_k), \\ &= \alpha^j\alpha^k(\delta_{jk}I_2 + i\epsilon_{jkl}\sigma_l), \\ &= \alpha^j\alpha^j I_2 = \alpha^2 I_2.\end{aligned}$$

So, $(\alpha^j\sigma_j)^{2n} = (\alpha)^{2n} I_2$, and $(\alpha^j\sigma_j)^{2n+1} = (\alpha^j\sigma_j)^{2n}(\alpha^j\sigma_j) = (\alpha)^{2n}(\alpha^j\sigma_j)$. So, the exponential expansion is

$$\begin{aligned}\exp\left(-\frac{1}{2}i\alpha^j\sigma_j\right) &= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(\frac{1}{2}\alpha^j\sigma_j\right)^{2n} - i\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\frac{1}{2}\alpha^j\sigma_j\right)^{2n+1}, \\ &= I_2\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(\frac{\alpha}{2}\right)^{2n} - i\left(\frac{\alpha^j\sigma_j}{\alpha}\right)\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\frac{\alpha}{2}\right)^{2n+1}, \\ &= I_2\cos\left(\frac{\alpha}{2}\right) - i\frac{\alpha^j\sigma_j}{\alpha}\sin\left(\frac{\alpha}{2}\right).\end{aligned}$$

7. Consider $X_j = L_j$ as a bases element of the $\mathfrak{so}(3)$ algebra, $[X_j, X_k] = \epsilon_{jkl}X_l$, where L_j are the matrices,

$$L_1 = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{pmatrix}, \qquad L_2 = \begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ -1 & 0 & 0\end{pmatrix}, \qquad L_3 = \begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}.$$

Verify the following:

(a) $[L_j, L_k] = \epsilon_{jkl}L_l$.

Solution: Notice that $(L^j)_{lk} = \epsilon_{jkl}$. So, the commutator is

$$\begin{aligned}([L^j, L^k])_{ln} &= (L^j)_{lm}(L^k)_{mn} - (L^k)_{lm}(L^j)_{mn} \,, \\ &= \epsilon_{jml}\epsilon_{knm} - \epsilon_{kml}\epsilon_{jnm} \,.\end{aligned}$$

Recall the Jacobi identity for the structure constants, here the Levi-Civita, $\epsilon_{jkm}\epsilon_{mln} + \epsilon_{klm}\epsilon_{mjn} + \epsilon_{ljm}\epsilon_{mkn} = 0$. We can use the antisymmetry of ϵ_{jkl} to write

$$\begin{aligned}-\epsilon_{jkm}\epsilon_{mln} &= \epsilon_{klm}\epsilon_{mjn} + \epsilon_{ljm}\epsilon_{mkn} \,, \\ &= \epsilon_{jml}\epsilon_{knm} - \epsilon_{kml}\epsilon_{jnm} \,, \\ &= (L^j)_{lm}(L^k)_{mn} - (L^k)_{lm}(L^j)_{mn} \,,\end{aligned}$$

where we identified the difference in Levi-Civita's as the commutator of $[L^j, L^k]$. So, we find

$$\begin{aligned}(L^j)_{lm}(L^k)_{mn} - (L^k)_{lm}(L^j)_{mn} &= -\epsilon_{jkm}\epsilon_{mln} \,, \\ &= \epsilon_{jkm}\epsilon_{mnl} \,, \\ &= \epsilon_{jkm}(L^m)_{ln} \,.\end{aligned}$$

We conclude that $[L^j, L^k] = \epsilon_{jkl}L^l$.

(b) $\{L_j, L_k\} \neq N\delta_{jk}$ for any j, k, and N.

Solution: The anticommutator $\{L_j, L_k\} = L_jL_k + L_kL_j$. Since $(L^j)_{lk} = \epsilon_{jkl}$, we have

$$\begin{aligned}(L^j)_{lm}(L^k)_{mn} + (L^k)_{lm}(L^j)_{mn} &= \epsilon_{jml}\epsilon_{knm} + \epsilon_{kml}\epsilon_{jnm} \,, \\ &= -\epsilon_{jlm}\epsilon_{knm} - \epsilon_{klm}\epsilon_{jnm} \,,\end{aligned}$$

where in the last line we used the antisymmetry properties of the permutation tensor. Now, we use the property $\epsilon_{jlm}\epsilon_{knm} = \delta_{jk}\delta_{ln} - \delta_{jn}\delta_{kl}$. So, we have

$$\begin{aligned}(L^j)_{lm}(L^k)_{mn} + (L^k)_{lm}(L^j)_{mn} &= -\epsilon_{jlm}\epsilon_{knm} - \epsilon_{klm}\epsilon_{jnm} \,, \\ &= -(\delta_{jk}\delta_{ln} - \delta_{jn}\delta_{kl}) - (\delta_{kj}\delta_{ln} - \delta_{kn}\delta_{lj}) \,, \\ &= -2\delta_{jk}\delta_{ln} + \delta_{jn}\delta_{kl} + \delta_{kn}\delta_{lj} \,.\end{aligned}$$

Thus, we see that $\{L_j, L_k\} \neq N\delta_{jk}$ for any j, k, and N.

(c) Show that a group element $O(\alpha^j) \in \mathrm{SO}(3)$ can be written as

$$O(\alpha^j) = \exp\left(\alpha^j L_j\right) = I_3 + \frac{\alpha^j L_j}{\alpha}\,\sin\alpha + \left(\frac{\alpha^j L_j}{\alpha}\right)^2 (1 - \cos\alpha) \,,$$

where $\alpha^2 = \sum_j (\alpha_j)^2$.

Solution: Taylor expanding about $\alpha^j = 0$, we find

$$\begin{aligned}\exp\left(\alpha^j L_j\right) &= \sum_{n=0}^{\infty}\frac{1}{n!}\left(\alpha^j L_j\right)^n \,, \\ &= \sum_{n=0}^{\infty}\frac{1}{(2n)!}\left(\alpha^j L_j\right)^{2n} + \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\left(\alpha^j L_j\right)^{2n+1} \,,\end{aligned}$$

where we split the sum into even and odd terms. Let us evaluate $(\alpha^j L_j)^2$,

$$\begin{aligned}[(\alpha^j L_j)^2]_{ln} &= [(\alpha^j L_j)(\alpha^k L_k)]_{ln} \,, \\ &= \alpha^j\alpha^k\,(L_j)_{lm}(L_k)_{mn} \,, \\ &= \alpha^j\alpha^k\,\epsilon_{jml}\epsilon_{knm} \,, \\ &= -\alpha^j\alpha^k\,\epsilon_{jlm}\epsilon_{knm} \,, \\ &= -\alpha^j\alpha^k\,(\delta_{jk}\delta_{ln} - \delta_{jn}\delta_{kl}) \,, \\ &= -(\alpha^2\delta_{ln} - \alpha_n\alpha_l) \,,\end{aligned}$$

where in the fourth line we used $\epsilon_{jml} = -\epsilon_{jlm}$, and in the fifth line we used the property $\epsilon_{jlm}\epsilon_{knm} = \delta_{jk}\delta_{ln} - \delta_{jn}\delta_{kl}$. Next, we evaluate $(\alpha^j L_j)^3$,

$$\begin{aligned}[(\alpha^j L_j)^3]_{lp} &= [(\alpha^j L_j)(\alpha^k L_k)(\alpha^r L_r)]_{lp} \,, \\ &= \alpha^j\alpha^k\alpha^r\,(L_j)_{lm}(L_k)_{mn}(L_r)_{np} \,, \\ &= -(\alpha^2\delta_{ln} - \alpha^n\alpha^l)\alpha^r(L_r)_{np} \,, \\ &= -\alpha^2\alpha^r(L_r)_{lp} + \alpha^n\alpha^l\alpha^r(L_r)_{np} \,, \\ &= -\alpha^2\alpha^r(L_r)_{lp} \,, \\ &= -\alpha^2(\alpha^j L_j)_{lp} \,,\end{aligned}$$

where we used that $\alpha^n\alpha^r(L_r)_{np} = \alpha^n\alpha^r\epsilon_{rpn} = 0$ since we have a symmetric sum over a completely antisymmetric object. So, we find that $(\alpha^j L_j)^3 = -\alpha^2(\alpha^j L_j)$. Finally, we evaluate $(\alpha^j L_j)^4$ as

$$\begin{aligned}(\alpha^j L_j)^4 &= (\alpha^j L_j)^3(\alpha^j L_j) \,, \\ &= -\alpha^2(\alpha^k L_k)(\alpha^j L_j) = -\alpha^2(\alpha^j L_j)^2 \,.\end{aligned}$$

Therefore, the sum over even terms can be expressed in terms of $(\alpha^j L_j)^2$, while the odd terms can be written in terms of $(\alpha^j L_j)$. Specifically,

$$\begin{aligned}(\alpha^j L_j)^{2n} &= (-1)^{n+1}\alpha^{2n}\left(\frac{\alpha^j L_j}{\alpha}\right)^2 \,, \quad n > 0 \\ (\alpha^j L_j)^{2n+1} &= (-1)^{n}\alpha^{2n+1}\left(\frac{\alpha^j L_j}{\alpha}\right) \,, \quad n \geq 0 \,.\end{aligned}$$

Substituting these expressions into the series expansion,

$$\begin{aligned}
\exp\left(\alpha^j L_j\right) &= I_3 + \sum_{n=1}^{\infty} \frac{1}{(2n)!}\left(\alpha^j L_j\right)^{2n} + \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}\left(\alpha^j L_j\right)^{2n+1}\,, \\
&= I_3 - \left(\frac{\alpha^j L_j}{\alpha}\right)^2 \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!}\alpha^{2n} + \left(\frac{\alpha^j L_j}{\alpha}\right) \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\alpha^{2n+1}\,, \\
&= I_3 + \left(\frac{\alpha^j L_j}{\alpha}\right)^2 \left(1 - \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\alpha^{2n}\right) + \left(\frac{\alpha^j L_j}{\alpha}\right) \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\alpha^{2n+1}\,, \\
&= I_3 + \left(\frac{\alpha^j L_j}{\alpha}\right)^2 (1-\cos\alpha) + \left(\frac{\alpha^j L_j}{\alpha}\right)\sin\alpha\,.
\end{aligned}$$