1. Show that the global U(1) symmetry, $\psi \to e^{i\alpha}\psi$ with $\alpha \in \mathbb{R}$, of the spinor field theory

$$\mathcal{L} = \frac{1}{2} i\bar{\psi}\partial\!\!\!/\psi + \text{h.c.} - m\bar{\psi}\psi\,,$$

leads to a conserved current $\mathcal{J}^\mu = \bar{\psi}\gamma^\mu\psi$. Show explicitly that this current is conserved.

Solution: Under the U(1) transformation, the fermion fields transform as

$$\begin{aligned}\psi \to \psi' &= e^{i\alpha}\psi\,,\\ &= \psi + i\alpha\psi + \mathcal{O}(\alpha^2)\,,\\ &\equiv \psi + \alpha\frac{\delta\psi}{\delta\alpha} + \mathcal{O}(\alpha^2)\,,\end{aligned}$$

and for the antifermion field

$$\begin{aligned}\bar{\psi} \to \bar{\psi}' &= e^{-i\alpha}\bar{\psi}\,,\\ &= \bar{\psi} - i\alpha\bar{\psi} + \mathcal{O}(\alpha^2)\,,\\ &\equiv \bar{\psi} + \alpha\frac{\delta\bar{\psi}}{\delta\alpha} + \mathcal{O}(\alpha^2)\,,\end{aligned}$$

where we defined $\delta\psi/\delta\alpha = i\psi$ and $\delta\bar{\psi}/\delta\alpha = -i\bar{\psi}$. From Noether's theorem, the conserved current is given by

$$\mathcal{J}^\mu = -\frac{\delta\mathcal{L}}{\delta(\partial_\mu\psi)}\frac{\delta\psi}{\delta\alpha} - \frac{\delta\bar{\psi}}{\delta\alpha}\frac{\delta\mathcal{L}}{\delta(\partial_\mu\bar{\psi})}\,.$$

The Lagrange density is $\mathcal{L} = \frac{i}{2}\bar{\psi}\gamma^\mu\partial_\mu\psi - \frac{i}{2}\partial_\mu\bar{\psi}\gamma^\mu\psi - m\bar{\psi}\psi$, so

$$\frac{\delta\mathcal{L}}{\delta(\partial_\mu\psi)} = \frac{i}{2}\bar{\psi}\gamma^\mu\,, \qquad \frac{\delta\mathcal{L}}{\delta(\partial_\mu\bar{\psi})} = -\frac{i}{2}\gamma^\mu\psi\,,$$

which gives the current

$$\begin{aligned}\mathcal{J}^\mu &= -\frac{\delta\mathcal{L}}{\delta(\partial_\mu\psi)}\frac{\delta\psi}{\delta\alpha} - \frac{\delta\bar{\psi}}{\delta\alpha}\frac{\delta\mathcal{L}}{\delta(\partial_\mu\bar{\psi})}\,,\\ &= -\left(\frac{i}{2}\bar{\psi}\gamma^\mu\right)(i\psi) - (-i\bar{\psi})\left(-\frac{i}{2}\gamma^\mu\psi\right)\,,\\ &= \bar{\psi}\gamma^\mu\psi\,.\end{aligned}$$

The current is conserved, $\partial_\mu\mathcal{J}^\mu = 0$, which can be shown explicitly,

$$\begin{aligned}\partial_\mu\mathcal{J}^\mu &= \partial_\mu(\bar{\psi}\gamma^\mu\psi)\,,\\ &= (\partial_\mu\bar{\psi})\gamma^\mu\psi + \bar{\psi}\gamma^\mu(\partial_\mu\psi)\,.\end{aligned}$$

Recall the Dirac equation, $(i\partial\!\!\!/ - m)\psi \implies \gamma^\mu(\partial_\mu\psi) = -im\psi$, as well as the conjugate equation, $(\partial_\mu\bar{\psi})\gamma^\mu = im\bar{\psi}$. So, we have

$$\begin{aligned}\partial_\mu \mathcal{J}^\mu &= (\partial_\mu\bar{\psi})\gamma^\mu\psi + \bar{\psi}\gamma^\mu(\partial_\mu\psi)\,,\\ &= im\bar{\psi}\psi - im\bar{\psi}\psi\,,\\ &= 0\,.\end{aligned}$$

Therefore, the current is conserved.

2. Derive the classical equations of motion for spinor electrodynamics given the Lagrange density

$$\mathcal{L} = \frac{1}{2}i\bar{\psi}D\!\!\!\!/\,\psi + \text{h.c.} - m\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,,$$

with $D_\mu = \partial_\mu + iqA_\mu$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and the Euler-Lagrange equations

$$\partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\psi)}\right) = \frac{\delta\mathcal{L}}{\delta\psi}\,,\qquad \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\bar{\psi})}\right) = \frac{\delta\mathcal{L}}{\delta\bar{\psi}}\,,\qquad \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu A_\nu)}\right) = \frac{\delta\mathcal{L}}{\delta A_\nu}\,.$$

Solution: Rewriting the Lagrange density as

$$\mathcal{L} = \frac{i}{2}\bar{\psi}\gamma^\mu(\partial_\mu\psi) - \frac{i}{2}(\partial_\mu\bar{\psi})\gamma^\mu\psi - m\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - qA_\mu\bar{\psi}\gamma^\mu\psi\,,$$

we can find the classical equations of motion by direct evaluation. Let us first obtains the equations for the ψ field, which come from the Euler-Lagrange equations as

$$\begin{aligned}\partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\bar{\psi})}\right) &= \partial_\mu\left(-\frac{i}{2}\gamma^\mu\psi\right) = -\frac{i}{2}\gamma^\mu\partial_\mu\psi\,,\\ \frac{\delta\mathcal{L}}{\delta\bar{\psi}} &= \frac{i}{2}\gamma^\mu(\partial_\mu\psi) - m\psi - qA_\mu\gamma^\mu\psi\,.\end{aligned}$$

Combining together, we find the equations

$$\begin{aligned}-\frac{i}{2}\gamma^\mu\partial_\mu\psi &= \frac{i}{2}\gamma^\mu(\partial_\mu\psi) - m\psi - qA_\mu\gamma^\mu\psi\,,\\ &\implies (i\partial\!\!\!/ - m - qA\!\!\!/)\psi = 0\,.\end{aligned}$$

For the $\bar{\psi}$ field,

$$\begin{aligned}\partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\psi)}\right) &= \partial_\mu\left(\frac{i}{2}\bar{\psi}\gamma^\mu\right) = \frac{i}{2}\partial_\mu\bar{\psi}\gamma^\mu\,,\\ \frac{\delta\mathcal{L}}{\delta\psi} &= -\frac{i}{2}(\partial_\mu\bar{\psi})\gamma^\mu - m\bar{\psi} - qA_\mu\bar{\psi}\gamma^\mu\,.\end{aligned}$$

Combining, we find

$$\frac{i}{2}\partial_\mu\bar{\psi}\gamma^\mu = -\frac{i}{2}(\partial_\mu\bar{\psi})\gamma^\mu - m\bar{\psi} - qA_\mu\bar{\psi}\gamma^\mu\,,$$

$$\implies \bar{\psi}(i\overleftarrow{\not\partial} + q\not{A} + m) = 0\,.$$

Finally, for the electromagnetic field, we find

$$\begin{aligned}
\partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu A_\nu)}\right) &= -\frac{1}{4}\partial_\mu\left(2F^{\alpha\beta}\frac{\delta F_{\alpha\beta}}{\delta(\partial_\mu A_\nu)}\right)\,,\\
&= -\frac{1}{2}\partial_\mu\left(F^{\alpha\beta}\frac{\delta}{\delta(\partial_\mu A_\nu)}(\partial_\alpha A_\beta - \partial_\beta A_\alpha)\right)\,,\\
&= -\frac{1}{2}\partial_\mu\left(F^{\alpha\beta}\left(\delta^\mu_\alpha\delta^\nu_\beta - \delta^\mu_\beta\delta^\nu_\alpha\right)\right)\,,\\
&= -\frac{1}{2}\partial_\mu\left(F^{\mu\nu} - F^{\nu\mu}\right)\,,\\
&= -\partial_\mu F^{\mu\nu}\,,
\end{aligned}$$

where in the fourth line we used $F^{\nu\mu} = -F^{\mu\nu}$. For the potential term,

$$\frac{\delta\mathcal{L}}{\delta A_\nu} = -q\bar{\psi}\gamma^\nu\psi\,,$$

from which we arrive at

$$\partial_\mu F^{\mu\nu} = q\bar{\psi}\gamma^\nu\psi\,.$$

Therefore, the classical equations of motion are

$$(i\not\partial - q\not{A} - m)\psi = 0\,,\qquad \bar{\psi}(i\overleftarrow{\not\partial} + q\not{A} + m) = 0\,,\qquad \partial_\mu F^{\mu\nu} = q\bar{\psi}\gamma^\mu\psi\,.$$

3. An alternative Lagrange density for the classical free electromagnetic field is

$$\mathcal{L}' = -\frac{1}{2}\partial_\mu A_\nu\,\partial^\mu A^\nu\,.$$

(a) Under what assumption does $\mathcal{L}'$ yield the free inhomogeneous Maxwell equations?

Solution: The free inhomogeneous Maxwell equations are $\partial_\mu F^{\mu\nu} = 0$. Since $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, we have

$$\begin{aligned}
0 = \partial_\mu F^{\mu\nu} &= \partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu)\,,\\
&= \partial^\mu\partial_\mu A^\nu - \partial^\nu(\partial_\mu A^\mu)\,,\\
&= \partial^2 A^\nu - \partial^\nu(\partial_\mu A^\mu) = 0\,.
\end{aligned}$$

So, the free inhomogeneous Maxwell equations, in terms of A_μ, are $\partial^2 A^\nu - \partial^\nu(\partial_\mu A^\mu) = 0$.

Now, for the Lagrange density $\mathcal{L}' = -\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu$, we can derive the equations of motion through the Euler-Lagrange equations,

$$\partial_\mu \left(\frac{\delta \mathcal{L}'}{\delta(\partial_\mu A_\nu)} \right) = \frac{\delta \mathcal{L}'}{\delta A_\nu}$$

$$\implies \partial_\mu \left[\frac{\delta}{\delta(\partial_\mu A_\nu)} \left(-\frac{1}{2}\partial_\alpha A_\beta \partial^\alpha A^\beta \right) \right] = \frac{\delta}{\delta A_\nu} \left(-\frac{1}{2}\partial_\alpha A_\beta \partial^\alpha A^\beta \right) ,$$

$$-\frac{1}{2}\partial_\mu \left(\frac{\delta(\partial_\alpha A_\beta)}{\delta(\partial_\mu A_\nu)} \partial^\alpha A^\beta + \partial^\alpha A^\beta \frac{\delta(\partial_\alpha A_\beta)}{\delta(\partial_\mu A_\nu)} \right) = 0 \,,$$

$$-\partial_\mu \left(\delta^\mu_\alpha \delta^\nu_\beta \, \partial^\alpha A^\beta \right) = 0 \,,$$

$$\implies \partial_\mu \partial^\mu A^\nu = 0 \,.$$

Therefore, we find that the equations of motion are $\partial^2 A^\nu = 0$, which differs from $\partial^2 A^\nu - \partial^\nu(\partial_\mu A^\mu) = 0$ by a four-divergence $\partial^\nu(\partial_\mu A^\mu)$. Therefore, the assumption that $\mathcal{L}'$ yields the free inhomogeneous Maxwell equations is the Lorentz gauge condition, $\partial_\mu A^\mu = 0$.

(b) With this assumption, show that $\mathcal{L}'$ differs from $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ by a four-divergence.

Solution: Starting from the definition,

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \,, \\
&= -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) \,, \\
&= -\frac{1}{2}(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\nu A_\mu \partial^\mu A^\nu) \,, \\
&= -\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu + \frac{1}{2}\partial_\mu A_\nu \partial^\nu A^\mu \,, \\
&= \mathcal{L}' + \frac{1}{2}\partial_\mu A_\nu \partial^\nu A^\mu \,.
\end{aligned}$$

Note that $\partial_\mu(A_\nu \partial^\nu A^\mu) = \partial_\mu A_\nu \partial^\nu A^\mu + A_\nu \partial_\mu \partial^\nu A^\mu = \partial_\mu A_\nu \partial^\nu A^\mu + A_\nu \partial^\nu \partial_\mu A^\mu$, where in the last equality we used the fact that the derivatives are symmetric on the second term. Moreover, we can rewrite the second term with $\partial_\nu(A_\nu \partial_\mu A^\mu) = (\partial_\mu A^\mu)^2 + A_\nu \partial^\nu \partial_\mu A^\mu$. Relabeling the summed indices on the second term, $\mu \leftrightarrow \nu$, and combining with the first relation we obtain

$$\partial_\mu(A^\nu \partial^\nu A^\mu - A^\mu \partial_\nu A^\nu) = \partial_\mu A_\nu \partial^\nu A^\mu - (\partial_\mu A^\mu)^2 \,.$$

So, substituting this into $\mathcal{L} = \mathcal{L}' + \frac{1}{2}\partial_\mu A_\nu \partial^\nu A^\mu$, we have

$$\mathcal{L} = \mathcal{L}' + \frac{1}{2}\partial_\mu(A^\nu \partial^\nu A^\mu - A^\mu \partial_\nu A^\nu) + \frac{1}{2}(\partial_\mu A^\mu)^2 \,.$$

So, $\mathcal{L}$ differs from $\mathcal{L}'$ by a four-divergence so long as we restrict $\mathcal{L}$ to the Lorentz gauge, $\partial_\mu A^\mu = 0$.

4. Verify that the field strength tensor $F_{\mu\nu}$ can be computed through the commutator $iqF_{\mu\nu} = [D_\mu, D_\nu]$.

Solution: Evaluating the commutator against some test function φ,

$$\begin{aligned}
[D_\mu, D_\nu]\varphi &= [\partial_\mu + iqA_\mu, \partial_\nu + iqA_\nu]\varphi\,, \\
&= [\partial_\mu, \partial_\nu]\varphi + iq[\partial_\mu, A_\nu]\varphi + iq[A_\mu, \partial_\nu]\varphi - q^2[A_\mu, A_\nu]\varphi\,, \\
&= iq[\partial_\mu, A_\nu]\varphi + iq[A_\mu, \partial_\nu]\varphi\,, \\
&= iq\left(\partial_\mu(A_\nu\varphi) - A_\nu\partial_\mu\varphi + A_\mu\partial_\nu\varphi - \partial_\nu(A_\mu\varphi)\right)\,,
\end{aligned}$$

where in going to the third line we used that $\partial_\mu\partial_\nu\varphi = \partial_\nu\partial_\mu\varphi$, and $[A_\mu, A_\nu] = 0$. Now, $\partial_\mu(A_\nu\varphi) = (\partial_\mu A_\nu)\varphi + A_\nu\partial_\mu\varphi$ and $\partial_\nu(A_\mu\varphi) = (\partial_\nu A_\mu)\varphi + A_\mu\partial_\nu\varphi$. So,

$$\begin{aligned}
\frac{1}{iq}[D_\mu, D_\nu]\varphi &= \partial_\mu(A_\nu\varphi) - A_\nu\partial_\mu\varphi + A_\mu\partial_\nu\varphi - \partial_\nu(A_\mu\varphi)\,, \\
&= (\partial_\mu A_\nu)\varphi + A_\nu\partial_\mu\varphi - A_\nu\partial_\mu\varphi + A_\mu\partial_\nu\varphi - (\partial_\nu A_\mu)\varphi - A_\mu\partial_\nu\varphi\,, \\
&= (\partial_\mu A_\nu - \partial_\nu A_\mu)\varphi\,, \\
&= F_{\mu\nu}\varphi\,,
\end{aligned}$$

so we conclude that $iqF_{\mu\nu} = [D_\mu, D_\nu]$.

5. Show that the radiative transition, $e^- \to e^- + \gamma$, is forbidden in vacuum.

Solution: Let us defined the following kinematics,

$$e^-(p) \to e^-(p') + \gamma(k)\,,$$

where $p = (E, \mathbf{p})$, $p' = (E', \mathbf{p}')$, and $k = (\omega, \mathbf{k})$ are the four-momenta of the incoming electron, outgoing electron, and outgoing photon, respectively. In vacuum, each of these particles are on their mass-shell, $p^2 = p'^2 = m_e^2$, and $k^2 = 0$. The S matrix element is given by

$$S(e^- \to e^-\gamma) = (2\pi)^4\delta^{(4)}(p - p' - k)\, i\mathcal{M}(e^- \to e^-\gamma)\,,$$

where the delta function enforces conservation of four-momentum, $p = p' + k$ and $\mathcal{M}$ is the amplitude. The leading order amplitude is non-zero, given by $i\mathcal{M} = -ie\bar{u}(p')\not{\epsilon}u(p) + \mathcal{O}(e^2)$.

Let us examine conservation of four-momentum, which in terms of its components are $E = E' + \omega$ and $\mathbf{p} = \mathbf{p}' + \mathbf{k}$. Let us choose to evaluate the amplitude in the rest frame of the initial electron, so $\mathbf{p} = \mathbf{0}$, and $E = m_e$. Therefore, by conservation of energy and momentum, we have $m_e = E' + \omega$ and $\mathbf{p}' = -\mathbf{k}$, respectively. Since the particles are on-shell, we further have $E' = \sqrt{m_e^2 + \mathbf{p}'^2}$ and $\omega = |\mathbf{k}|$. Combining these results, conservation of energy imposes the condition

$$m_e = \sqrt{m_e^2 + \mathbf{k}^2} + |\mathbf{k}|\,,$$

This condition is only true if $\mathbf{k} = \mathbf{0}$, that is there is no photon emitted. We conclude that conservation of momentum forbids this reaction, giving $S(e^- \to e^-\gamma) = 0$.

6. Consider "Bhabha scattering", $e^-e^+ \to e^-e^+$, within QED in the high-energy limit, i.e., the ultra-relativistic limit $m_e^2/s \to 0$. Compare the experimentally measured unpolarized differential cross-section to the theoretical prediction at leading order in $\alpha = e^2/4\pi$, the fine-structure constant.

(a) Working in the center-of-momentum (CM) frame, what are the energies and momenta of each particle in the reaction as a function of the Mandelstam invariant s? What is the invariant momentum transfer, t, as a function of s and $\cos\theta$ where θ is the scattering angle?

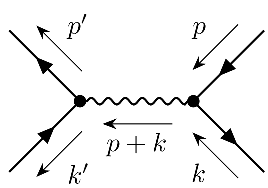
> **Solution:** Let us define the following kinematics in the CM frame, consulting the results from `Problem Set 2`,
>
> $$e^-(p) + e^+(k) \to e^-(p') + e^+(k')\,,$$
>
> where $p = (E, \mathbf{p})$ and $p' = (E, \mathbf{p}')$ are incoming and outgoing electron four-momenta, respectively, and $k = (E, \mathbf{k})$ and $k' = (E, \mathbf{k}')$ are incoming and outgoing positron four-momenta, respectively. Note that since all particles have equal mass m_e, they all have identical energy, $E = \sqrt{s}/2$. Further, all the momenta obey $\mathbf{p} = -\mathbf{k}$ and $\mathbf{p}' = -\mathbf{k}'$, so their magnitudes are also identical,
>
> $$|\mathbf{p}| = |\mathbf{k}| = |\mathbf{p}'| = |\mathbf{k}'| = \frac{1}{2}\sqrt{s - 4m_e^2} = \frac{\sqrt{s}}{2} + \mathcal{O}(m_e^2/s)\,,$$
> $$= E + \mathcal{O}(m_e^2/s)$$
>
> So, each particles energy and momenta are $E = |\mathbf{p}| = \sqrt{s}/2$ as $m_e^2/s \to 0$.
>
> Also, note that the Mandelstam variables are $s = 2p\cdot k$, $t = 2p\cdot p'$, and $u = 2p\cdot k'$, with the constraint $s + t + u = 0$ in the ultrarelativistic limit, so we have
>
> $$t = -2|\mathbf{p}|^2(1-\cos\theta) = -2E^2(1-\cos\theta) = -\frac{s}{2}(1-\cos\theta)\,.$$

(b) Compute the unpolarized differential cross-section $\mathrm{d}\sigma/\mathrm{d}\Omega$, where Ω is the solid angle of the electron in the e^-e^+ CM frame, to order α^2 in terms of the Mandelstam invariants s and t.

> **Solution:** Here we consider the reaction,
>
> $$e^-(p, s) + e^+(k, r) \to e^-(p', s') + e^+(k', r')\,.$$
>
> where the momenta are as defined in part **(a)**, with s and s' are the spin projections (helicities) for the incoming and outgoing electrons, while r and r' are the helicities of the incoming and outgoing positrons.
>
> The unpolarized differential cross section is given by
>
> $$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi s}\left\langle|\mathcal{M}|^2\right\rangle$$
>
> where the spin-averaged amplitude, $\left\langle|\mathcal{M}|^2\right\rangle$, is defined by
>
> $$\left\langle|\mathcal{M}|^2\right\rangle \equiv \frac{1}{2}\sum_s \frac{1}{2}\sum_r \sum_{s'} \sum_{r'} \left|\mathcal{M}(e_s^- e_r^+ \to e_{s'}^- e_{r'}^+)\right|^2.$$

At leading order in the QED coupling, there are two diagrams which contribute to $e^-e^+ \to e^-e^+$. The amplitude is then

$i\mathcal{M} =$



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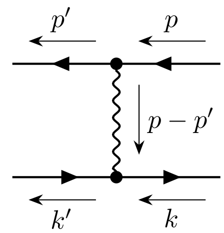

$+\,\mathcal{O}(\alpha^2),$

$$
\begin{aligned}
&= \bar{v}_r(k)(-ie\gamma^\mu)u_s(p)\,\frac{-ig_{\mu\nu}}{(p+k)^2+i\epsilon}\,\bar{u}_{s'}(p')(-ie\gamma^\nu)v_{r'}(k') \\
&\qquad - \bar{v}_r(k)(-ie\gamma^\mu)v_{r'}(k')\,\frac{-ig_{\mu\nu}}{(p-p')^2+i\epsilon}\,\bar{u}_{s'}(p')(-ie\gamma^\nu)u_s(p) + \mathcal{O}(\alpha^2)\,, \\
&= \frac{4\pi i\alpha}{s}[\bar{v}_r(k)\gamma^\mu u_s(p)][\bar{u}_{s'}(p')\gamma_\mu v_{r'}(k')] \\
&\qquad - \frac{4\pi i\alpha}{t}[\bar{v}_r(k)\gamma^\nu v_{r'}(k')][\bar{u}_{s'}(p')\gamma_\nu u_s(p)] + \mathcal{O}(\alpha^2)\,, \\
&\equiv i\mathcal{M}_s - i\mathcal{M}_t + \mathcal{O}(\alpha^2)\,,
\end{aligned}
$$

where we have defined the *annihilation* amplitude $\mathcal{M}_s$ (associated with the first diagram) and the *exchange* amplitude $\mathcal{M}_t$ (associated with the second diagram), and used that $e^2 = 4\pi\alpha$, $s = (p+k)^2$, and $t = (p-p')^2$. Note the relative minus sign on the exchange diagram is due to the exchange of an incoming positron and outgoing electron from the annihilation diagram. From here forward we implicitly assume an $i\epsilon$ shift in the propagators.

For the cross-section, we need the spin-averaged amplitude,

$$
\begin{aligned}
\langle|\mathcal{M}|^2\rangle &\equiv \frac{1}{4}\sum_{s,s'}\sum_{r,r'}\mathcal{M}^*\mathcal{M}\,, \\
&= \frac{1}{4}\sum_{s,s'}\sum_{r,r'}(|\mathcal{M}_s|^2 + |\mathcal{M}_t|^2 - \mathcal{M}_s^*\mathcal{M}_t - \mathcal{M}_t^*\mathcal{M}_s) + \mathcal{O}(\alpha^3)\,, \\
&= \frac{1}{4}\sum_{s,s'}\sum_{r,r'}\left[|\mathcal{M}_s|^2 + |\mathcal{M}_t|^2 - 2\mathrm{Re}\left(\mathcal{M}_s^*\mathcal{M}_t\right)\right] + \mathcal{O}(\alpha^3)\,.
\end{aligned}
$$

We then evaluate the spin sums on the squared amplitudes using Casimir's trick. Focusing first on $|\mathcal{M}_s|^2$,

$$|\mathcal{M}_s|^2 = \left(\frac{4\pi\alpha}{s}\right)^2 [\bar{v}_r(k)\gamma^\mu u_s(p)]^*[\bar{u}_{s'}(p')\gamma_\mu v_{r'}(k')]^*[\bar{v}_r(k)\gamma^\nu u_s(p)][\bar{u}_{s'}(p')\gamma_\nu v_{r'}(k')]\,,$$

$$= \left(\frac{4\pi\alpha}{s}\right)^2 [\bar{v}_r(k)\gamma^\mu u_s(p)]^\dagger[\bar{u}_{s'}(p')\gamma_\mu v_{r'}(k')]^\dagger[\bar{v}_r(k)\gamma^\nu u_s(p)][\bar{u}_{s'}(p')\gamma_\nu v_{r'}(k')]\,,$$

$$= \left(\frac{4\pi\alpha}{s}\right)^2 [\bar{u}_s(p)\gamma^\mu v_r(k)][\bar{v}_{r'}(k')\gamma_\mu u_{s'}(p')][\bar{v}_r(k)\gamma^\nu u_s(p)][\bar{u}_{s'}(p')\gamma_\nu v_{r'}(k')]\,,$$

where we used $(\bar{\xi}\gamma^\mu\zeta)^\dagger = \bar{\zeta}\gamma^\mu\xi$ using $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$ for ξ and ζ being either a u or v spinor. Summing over spins,

$$\sum_{s,s'}\sum_{r,r'}|\mathcal{M}_s|^2 \propto \sum_{s,s'}\sum_{r,r'}[\bar{u}_s(p)\gamma^\mu v_r(k)][\bar{v}_{r'}(k')\gamma_\mu u_{s'}(p')][\bar{v}_r(k)\gamma^\nu u_s(p)][\bar{u}_{s'}(p')\gamma_\nu v_{r'}(k')]\,,$$

$$= \sum_{s,r}[\bar{u}_s(p)\gamma^\mu v_r(k)\bar{v}_r(k)\gamma^\nu u_s(p)]\sum_{s',r'}[\bar{v}_{r'}(k')\gamma_\mu u_{s'}(p')\bar{u}_{s'}(p')\gamma_\nu v_{r'}(k')]\,,$$

$$= \sum_{s,r}\mathrm{tr}[\bar{u}_s(p)\gamma^\mu v_r(k)\bar{v}_r(k)\gamma^\nu u_s(p)]\sum_{s',r'}\mathrm{tr}[\bar{v}_{r'}(k')\gamma_\mu u_{s'}(p')\bar{u}_{s'}(p')\gamma_\nu v_{r'}(k')]\,,$$

$$= \sum_{s,r}\mathrm{tr}[\gamma^\mu v_r(k)\bar{v}_r(k)\gamma^\nu u_s(p)\bar{u}_s(p)]\sum_{s',r'}\mathrm{tr}[\gamma_\mu u_{s'}(p')\bar{u}_{s'}(p')\gamma_\nu v_{r'}(k')\bar{v}_{r'}(k')]\,,$$

$$= \mathrm{tr}[\gamma^\mu \not{k}\gamma^\nu \not{p}]\,\mathrm{tr}[\gamma_\mu \not{p}'\gamma_\nu \not{k}']$$

where we used the completeness of the spinors in the high-energy limit,

$$\sum_s u_s(p)\bar{u}_s(p) = \not{p} = \sum_r v_r(p)\bar{v}_r(p)\,.$$

Evaluating the traces, using $\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})$, we find

$$\mathrm{tr}[\gamma^\mu \not{k}\gamma^\nu \not{p}] = 4\left(p^\nu k^\mu + p^\mu k^\nu - g^{\mu\nu}p\cdot k\right)\,,$$

$$\mathrm{tr}[\gamma_\mu \not{p}'\gamma_\nu \not{k}'] = 4\left(p'_\nu k'_\mu + p'_\mu k'_\nu - g_{\mu\nu}p'\cdot k'\right)\,.$$

Contracting these traces,

$$\mathrm{tr}[\gamma^\mu \not{k}\gamma^\nu \not{p}]\,\mathrm{tr}[\gamma^\mu \not{p}'\gamma^\nu \not{k}'] = 16\left(p^\nu k^\mu + p^\mu k^\nu - g^{\mu\nu}p\cdot k\right)\left(p'_\nu k'_\mu + p'_\mu k'_\nu - g_{\mu\nu}p'\cdot k'\right)\,,$$

$$= 32\left(p\cdot p'\,k\cdot k' + p\cdot k'\,k\cdot p'\right)\,,$$

$$= 8\left(t^2 + u^2\right)\,,$$

where we used $s = 2p\cdot k = 2p'\cdot k'$, $t = 2p\cdot p' = 2k\cdot k'$, and $u = 2p\cdot k' = 2k\cdot p'$. So, we have for the annihilation term

$$\frac{1}{4}\sum_{s,s'}\sum_{r,r'}|\mathcal{M}_s|^2 = 32\pi^2\alpha^2\left(\frac{t^2+u^2}{s^2}\right)\,.$$

Next look at the exchange term $|\mathcal{M}_t|^2$,

$$
\begin{aligned}
|\mathcal{M}_t|^2 &= \left(\frac{4\pi\alpha}{t}\right)^2 [\bar{v}_r(k)\gamma^\nu v_{r'}(k')]^*[\bar{u}_{s'}(p')\gamma_\nu u_s(p)]^*[\bar{v}_r(k)\gamma^\mu v_{r'}(k')][\bar{u}_{s'}(p')\gamma_\mu u_s(p)]\,, \\
&= \left(\frac{4\pi\alpha}{t}\right)^2 [\bar{v}_{r'}(k')\gamma^\nu v_r(k)][\bar{u}_s(p)\gamma_\mu u_{s'}(p')][\bar{v}_r(k)\gamma^\nu v_{r'}(k')][\bar{u}_{s'}(p')\gamma_\mu u_s(p)]\,,
\end{aligned}
$$

so that the spin sums are

$$
\begin{aligned}
\sum_{s,s'}\sum_{r,r'}|\mathcal{M}_t|^2 &\propto \sum_{s,s'}\sum_{r,r'}[\bar{v}_{r'}(k')\gamma^\nu v_r(k)][\bar{u}_s(p)\gamma_\nu u_{s'}(p')][\bar{v}_r(k)\gamma^\mu v_{r'}(k')][\bar{u}_{s'}(p')\gamma_\mu u_s(p)]\,, \\
&= \sum_{s,s'}[\bar{u}_s(p)\gamma_\nu u_{s'}(p')\bar{u}_{s'}(p')\gamma_\mu u_s(p)]\sum_{r,r'}[\bar{v}_{r'}(k')\gamma^\nu v_r(k)\bar{v}_r(k)\gamma^\mu v_{r'}(k')]\,, \\
&= \sum_{s,s'}\mathrm{tr}[\gamma_\nu u_{s'}(p')\bar{u}_{s'}(p')\gamma_\mu u_s(p)\bar{u}_s(p)]\sum_{r,r'}\mathrm{tr}[\gamma^\nu v_r(k)\bar{v}_r(k)\gamma^\mu v_{r'}(k')\bar{v}_{r'}(k')]\,, \\
&= \mathrm{tr}[\gamma_\nu \not{p}'\gamma_\mu \not{p}]\,\mathrm{tr}[\gamma^\nu \not{k}\gamma^\mu \not{k}']\,.
\end{aligned}
$$

We now contract the traces,

$$
\begin{aligned}
\mathrm{tr}[\gamma_\nu \not{p}'\gamma_\mu \not{p}]\,\mathrm{tr}[\gamma^\nu \not{k}\gamma^\mu \not{k}'] &= 16\left(p^\nu p'^\mu + p^\mu p'^\nu - g^{\mu\nu} p\cdot p'\right)\left(k_\nu k'_\mu + k_\mu k'_\nu - g_{\mu\nu} k\cdot k'\right)\,, \\
&= 32\left(p'\cdot k'\, p\cdot k + p\cdot k'\, k\cdot p'\right)\,, \\
&= 8\left(s^2+u^2\right)\,.
\end{aligned}
$$

So, the exchange term yields

$$
\frac{1}{4}\sum_{s,s'}\sum_{r,r'}|\mathcal{M}_t|^2 = 32\pi^2\alpha^2\left(\frac{s^2+u^2}{t^2}\right)\,.
$$

Finally, we evaluate the interference term

$$
\begin{aligned}
\mathrm{Re}\left(\mathcal{M}_s^*\mathcal{M}_t\right) &= \frac{(4\pi\alpha)^2}{st}[\bar{v}_r(k)\gamma^\mu u_s(p)]^*[\bar{u}_{s'}(p')\gamma_\mu v_{r'}(k')]^*[\bar{v}_r(k)\gamma^\nu v_{r'}(k')][\bar{u}_{s'}(p')\gamma_\nu u_s(p)]\,, \\
&= \frac{(4\pi\alpha)^2}{st}[\bar{u}_s(p)\gamma^\mu v_r(k)][\bar{v}_{r'}(k')\gamma_\mu u_{s'}(p')][\bar{v}_r(k)\gamma^\nu v_{r'}(k')][\bar{u}_{s'}(p')\gamma_\nu u_s(p)]\,, \\
&= \frac{(4\pi\alpha)^2}{st}[\bar{u}_s(p)\gamma^\mu v_r(k)\bar{v}_r(k)\gamma^\nu v_{r'}(k')\bar{v}_{r'}(k')\gamma_\mu u_{s'}(p')\bar{u}_{s'}(p')\gamma_\nu u_s(p)]\,, \\
&= \frac{(4\pi\alpha)^2}{st}\,\mathrm{tr}[\bar{u}_s(p)\gamma^\mu v_r(k)\bar{v}_r(k)\gamma^\nu v_{r'}(k')\bar{v}_{r'}(k')\gamma_\mu u_{s'}(p')\bar{u}_{s'}(p')\gamma_\nu u_s(p)]\,, \\
&= \frac{(4\pi\alpha)^2}{st}\,\mathrm{tr}[\gamma^\mu v_r(k)\bar{v}_r(k)\gamma^\nu v_{r'}(k')\bar{v}_{r'}(k')\gamma_\mu u_{s'}(p')\bar{u}_{s'}(p')\gamma_\nu u_s(p)\bar{u}_s(p)]\,.
\end{aligned}
$$

Evaluating the spin sums,

$$\sum_{s,s'}\sum_{r,r'}\mathrm{Re}\,(\mathcal{M}_s^*\mathcal{M}_t) \propto \sum_{s,s'}\sum_{r,r'}\mathrm{tr}[\gamma^\mu v_r(k)\bar{v}_r(k)\gamma^\nu v_{r'}(k')\bar{v}_{r'}(k')\gamma_\mu u_{s'}(p')\bar{u}_{s'}(p')\gamma_\nu u_s(p)\bar{u}_s(p)]\,,$$
$$= \mathrm{tr}(\gamma^\mu \not{k}\gamma^\nu \not{k}'\gamma_\mu \not{p}'\gamma_\nu \not{p})\,.$$

Note the identities $\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu = -2\gamma^\sigma\gamma^\rho\gamma^\nu$, $\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = 4g^{\nu\rho}\,I_4$, and $\mathrm{tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}$ can be used to simplify the traces,

$$\begin{aligned}\mathrm{tr}(\gamma^\mu \not{k}\gamma^\nu \not{k}'\gamma_\mu \not{p}'\gamma_\nu \not{p}) &= -2\,\mathrm{tr}(\gamma^\mu \not{k}\not{p}'\gamma_\mu \not{k}'\not{p})\,,\\ &= -8k\cdot p'\,\mathrm{tr}(\not{k}'\not{p})\,,\\ &= -32\,k\cdot p'\,k'\cdot p\,,\\ &= -8u^2\,.\end{aligned}$$

So, the interference term yields

$$\frac{1}{4}\sum_{s,s'}\sum_{r,r'}2\mathrm{Re}\,(\mathcal{M}_s^*\mathcal{M}_t) = -32\pi^2\alpha^2\left(\frac{2u^2}{st}\right)\,.$$

Combining these terms, we find for the spin-averaged amplitude

$$\langle|\mathcal{M}|^2\rangle = 32\pi^2\alpha^2\left(\frac{t^2+u^2}{s^2}+\frac{s^2+u^2}{t^2}+\frac{2u^2}{st}\right)+\mathcal{O}(\alpha^3)\,,$$

which gives for the unpolarized differential cross-section

$$\begin{aligned}\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} &= \frac{1}{64\pi^2 s}\langle|\mathcal{M}|^2\rangle\,,\\ &= \frac{\alpha^2}{2s}\left(\frac{t^2+u^2}{s^2}+\frac{s^2+u^2}{t^2}+\frac{2u^2}{st}\right)+\mathcal{O}(\alpha^3)\,,\\ &= \frac{\alpha^2}{2s}\left(\left(\frac{t}{s}\right)^2+\left(\frac{s}{t}\right)^2+u^2\left(\frac{1}{s}+\frac{1}{t}\right)^2\right)+\mathcal{O}(\alpha^3)\,.\end{aligned}$$

Since $s+t+u=0$, then $u^2=(s+t)^2$, so we write the cross-section in terms of s and t as

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2}{2s}\left(\left(\frac{t}{s}\right)^2+\left(\frac{s}{t}\right)^2+\left(1+\frac{s}{t}\right)^2\left(1+\frac{t}{s}\right)^2\right)+\mathcal{O}(\alpha^3)\,.$$

(c) Make a Semi-log plot of the $\mathcal{O}(\alpha^2)$ theoretical $\mathrm{d}\sigma/\mathrm{d}\Omega$ vs. $\cos\theta \in [-.8,.8]$ for each CM energy $\sqrt{s}/\,\mathrm{GeV} = \{14, 22, 34.8, 38.3, 43.6\}$. Plot the cross-section in nb, and restrict the y-axis to $(\mathrm{d}\sigma/\mathrm{d}\Omega)/\,\mathrm{nb} \in [0.001, 10.000]$. Plot the experimental data, measured from the TASSO experiment at PETRA, for each of the CM energies over the theoretical curves. Compare and comment on the quality of the theoretical description of the experimental data. **Note:** The data file presents the cross-section as $s \cdot \mathrm{d}\sigma/\mathrm{d}\Omega$. The data file was obtained from HEPData at https://www.hepdata.net/record/ins249557. The article by the TASSO collaboration may be helpful, https://link.springer.com/article/10.1007/BF01579904.

Solution: To compare the cross-section to data, we note that $t = -s(1-\cos\theta)/2$, so the ratio $t/s = -(1-\cos\theta)/2$. So, we have

$$\begin{aligned}
\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} &= \frac{\alpha^2}{2s}\left(\left(\frac{t}{s}\right)^2 + \left(\frac{s}{t}\right)^2 + \left(1+\frac{s}{t}\right)^2\left(1+\frac{t}{s}\right)^2\right) + \mathcal{O}(\alpha^3)\,, \\
&= \frac{\alpha^2}{2s}\left(\frac{1}{4}\left(1-\cos\theta\right)^2 + \frac{4}{\left(1-\cos\theta\right)^2} + \frac{1}{4}\left(\frac{1+\cos\theta}{1-\cos\theta}\right)^2\left(1+\cos\theta\right)^2\right) + \mathcal{O}(\alpha^3)\,, \\
&= \frac{\alpha^2}{8s}\left(\frac{16 + \left(1-\cos\theta\right)^4 + \left(1+\cos\theta\right)^4}{\left(1-\cos\theta\right)^2}\right) + \mathcal{O}(\alpha^3)\,, \\
&= \frac{\alpha^2}{8s}\left(\frac{16 + 2(\cos^4\theta + 6\cos^2\theta + 1)}{\left(1-\cos\theta\right)^2}\right) + \mathcal{O}(\alpha^3)\,, \\
&= \frac{\alpha^2}{4s}\left(\frac{3+\cos^2\theta}{1-\cos\theta}\right)^2 + \mathcal{O}(\alpha^3)\,,
\end{aligned}$$

where we used $(1\pm z)^4 = z^4 \pm 4z^3 + 6z^2 \pm 4z + 1$ to obtain $(1-z)^4 + (1+z)^2 = 2(z^4 + 6z^2 + 1)$ in the fourth line.

We can now plot $\mathrm{d}\sigma/\mathrm{d}\Omega$ as a function of $\cos\theta$ for each s. Since the TASSO data are for $\sqrt{s} \geq 14.0\,\mathrm{GeV}$, we find $(m_e/\sqrt{s})^2 = (5.11\times 10^{-4}/14.0)^2 \approx 1\times 10^{-9}$, so we expect the ultrarelativistic approximation to hold with respect to the precision of the experimental data. Figure 1 shows the cross-section $\mathrm{d}\sigma/\mathrm{d}\Omega$ at $\sqrt{s}/\,\mathrm{GeV} = \{14, 22, 34.8, 38.3, 43.6\}$, as well as $s\cdot\mathrm{d}\sigma/\mathrm{d}\Omega$ in the second plot. Note that to obtain the physical cross-section in nb, we multiply the theoretical cross section by $(\hbar c)^2$. Overall the quality of the description is ver well.

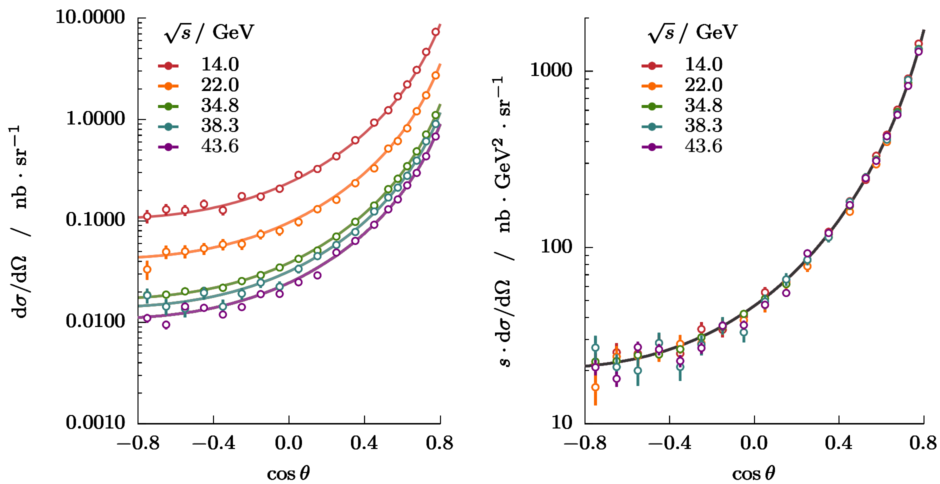

Figure 1: The unpolarized differential cross-section (left) and $s \cdot \mathrm{d}\sigma/\mathrm{d}\Omega$ (right) for each energy compared with the TASSO data.

(d) Make a plot of the ratio of the experimentally measured differential cross-section to the leading order QED prediction as a function of $\cos\theta \in [-.8, .8]$ for each CM energy $\sqrt{s}/\,\mathrm{GeV} = \{14, 22, 34.8, 38.3, 43.6\}$. Restrict the y axis between 0.5 and 1.5. Compare and comment on the quality of the theoretical description of the experimental data. **Hint:** Plot each energy on a separate plot to see if you notice any subtle trends.

Solution: Here we normalize the experimental cross-section by the leading order theory. Figure 2 shows the ratios for each energy. Here we see some discrepency with the leading order QED theory near the backward direction ($\cos\theta \sim -1$). We will see a greater discrepancy for $e^-e^+ \to \mu^-\mu^+$ and $e^-e^+ \to \tau^-\tau^+$ in the subsequent problem set, which is due to the presence of Z^0 boson intermediate states.

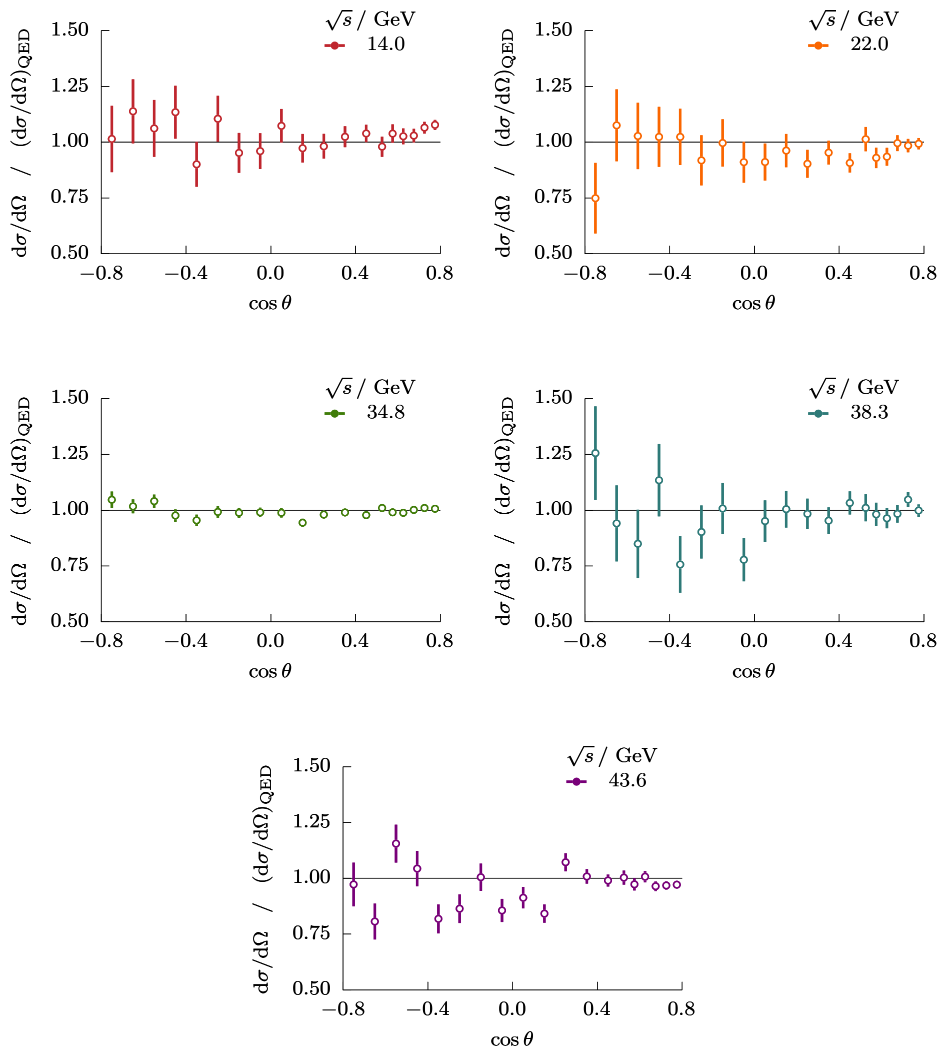

Figure 2: The ratios of the experimental TASSO data with respect to leading order QED theory at each energy.