1. Can the following hadrons, in principle, exist within QCD? (a) qq, (b) $qq\bar{q}$, (c) $qq\bar{q}\bar{q}$, (d) gg, (e) qqg, (f) $q\bar{q}g$, (g) $qqqq\bar{q}$. **Hint:** Consider $\mathrm{SU}(3)_c$ symmetry transformations of observable hadrons. Gluons transform under the adjoint representation of $\mathrm{SU}(3)_c$.

Solution: Hadrons within QCD must be color neutral, that is a hadron h *must* belong to the $\mathbf{1}$ representation of $\mathrm{SU}(3)_c$. So, all we need to find is if the given combinations of quarks and gluons admit a singlet representation. Recall that quarks lie in the $\mathbf{3}$ of $\mathrm{SU}(3)_c$, antiquarks lie in the $\mathbf{3}^*$ of $\mathrm{SU}(3)_c$, and gluons lie in the $\mathbf{8}$ of $\mathrm{SU}(3)_c$.

So, for (a)

$$qq \to \mathbf{3}\times\mathbf{3} = \mathbf{3}^* + \mathbf{6} \not\supset \mathbf{1}\,,$$

therefore qq is **not** a valid hadron.

For (b), we have (recalling that $\mathbf{3}\times\mathbf{3}^* = \mathbf{1}+\mathbf{8}$),

$$qq\bar{q} \to \mathbf{3}\times\mathbf{3}\times\mathbf{3}^* = \mathbf{3}\times(\mathbf{1}\times\mathbf{8}) \not\supset \mathbf{1}\,,$$

since the $\mathbf{3}\times\mathbf{8} = \mathbf{3}+\mathbf{6}^*+\mathbf{15}$ which was found in Problem Set 7. Therefore, $qq\bar{q}$ is **not** a valid hadron.

For (c), $qq\bar{q}\bar{q}$ is

$$\begin{aligned} qq\bar{q}\bar{q} \to \mathbf{3}\times\mathbf{3}\times\mathbf{3}^*\times\mathbf{3}^* &= (\mathbf{3}\times\mathbf{3}^*)\times(\mathbf{3}\times\mathbf{3}^*)\,, \\ &= (\mathbf{1}+\mathbf{8})\times(\mathbf{1}+\mathbf{8}) \supset \mathbf{1}\,. \end{aligned}$$

So, $qq\bar{q}\bar{q}$ **is** a valid hadron. These are *tetraquarks*, which candidates have been observed in the heavy quark sector, e.g., the $Z_c(3900)$.

For (d), gg, we need the product $\mathbf{8}\times\mathbf{8}$. From lecture, we worked out this product, and found it contains a singlet representation. Therefore,

$$gg \to \mathbf{8}\times\mathbf{8} \supset \mathbf{1}\,,$$

and thus is a valid hadron. These are *glueballs*, bound states of gluons. There is suspicion that higher mass states in the $J^{PC} = 0^{+}+$ and 2^{++} sectors contain strong mixing into these glueball states.

For (e), qqg, we have $\mathbf{3}\times\mathbf{3}\times\mathbf{8} = \mathbf{3}^*+\mathbf{3}^*+\mathbf{6}+\mathbf{6}+\mathbf{15}^*+\mathbf{15}^*+\mathbf{24}$ from Problem Set 7. So,

$$qqg \to \mathbf{3}\times\mathbf{3}\times\mathbf{8} \not\supset \mathbf{1}\,,$$

and thus is **not** a valid hadron.

For (f), $q\bar{q}g$, we have from Problem Set 7, $\mathbf{3}\times\mathbf{3}^*\times\mathbf{8} = \mathbf{1}+\mathbf{8}+\mathbf{8}+\mathbf{8}+\mathbf{10}+\mathbf{10}^*+\mathbf{27}$. So,

$$q\bar{q}g \to \mathbf{3}\times\mathbf{3}^*\times\mathbf{8} \supset \mathbf{1}\,.$$

Therefore, $q\bar{q}g$ **is** a valid hadron. These are *hybrid* mesons, which had a substantial component from excited glue. The $\pi_1(1600)$ is an observed hybrid candidate.

2. Consider a non-abelian gauge field $A_\mu \equiv A^j_\mu T_j$, where $T_j \in \mathfrak{su}(N)$ are generators satisfying the Lie algebra $[T_j, T_k] = ic_{jkl} T_l$ with c_{jkl} being structure constants and $j, k, l = 1, 2, \ldots, N^2 - 1$. Under a local gauge transformation, $U = \exp(i\alpha^j(x) T_j)$ where $\alpha_j(x) \in \mathbb{R}$ for every j, the gauge fields transform as

$$A_\mu \to U A_\mu U^{-1} + \frac{i}{g} (\partial_\mu U)\, U^{-1} .$$

Show that under infinitesimal transformations, $\alpha^a(x) \ll 1$, the gauge fields transform as

$$A^j_\mu \to A^j_\mu - \frac{1}{g} \partial_\mu \alpha^j(x) - c_{jkl}\, \alpha^k A^l_\mu + \mathcal{O}(\alpha^2) .$$

Solution: Taking $\alpha^j(x) \ll 1$ for all $j = 1, 2, \ldots, N^2 - 1$, w can Taylor expand the exponential

$$U = \exp(i\alpha^j(x) T_j) = 1 + i\alpha^j(x) T_j + \mathcal{O}(\alpha^2) .$$

So, the gauge transformation is

$$\begin{aligned}
A^j_\mu T_j &\to U A^j_\mu T_j U^{-1} + \frac{i}{g} (\partial_\mu U) U^{-1} , \\
&= (1 + i\alpha^j T_j + \mathcal{O}(\alpha^2)) A^k_\mu T_k (1 - i\alpha^l T_l + \mathcal{O}(\alpha^2)) \\
&\qquad + \frac{i}{g} \partial_\mu (1 + i\alpha^j T_j + \mathcal{O}(\alpha^2))(1 + i\alpha^k T_k + \mathcal{O}(\alpha^2)) , \\
&= A^j_\mu T_j + i\alpha^k A^l_\mu (T_k T_l - T_l T_k) - \frac{1}{g} \partial_\mu \alpha^j T_j + \mathcal{O}(\alpha^2) , \\
&= A^j_\mu T_j + i\alpha^k A^l_\mu (ic_{klj} T_j) - \frac{1}{g} \partial_\mu \alpha^j T_j + \mathcal{O}(\alpha^2) , \\
&= \left(A^j_\mu - c_{jkl} \alpha^k A^l_\mu - \frac{1}{g} \partial_\mu \alpha^j + \mathcal{O}(\alpha^2) \right) T_j .
\end{aligned}$$

Therefore, the infinitesimal transformation gives

$$A^j_\mu \to A^j_\mu - \frac{1}{g} \partial_\mu \alpha^j - c_{jkl} \alpha^k A^l_\mu + \mathcal{O}(\alpha^2) .$$

3. The $SU(3)_c$ Yang-Mills Lagrange density for interacting gluon fields is given by $\mathcal{L}_{\rm YM} = -\frac{1}{2} \operatorname{tr} (G_{\mu\nu} G^{\mu\nu})$, where the field-strength tensor is defined as $G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig_s [A_\mu, A_\nu]$ with $A_\mu = A^a_\mu \lambda_a / 2$ are the gluon gauge fields and λ_a are the Gell-Mann matrices. Write the Lagrange density as a free part $\mathcal{L}^{(\rm free)}_{\rm YM}$ and an interacting part $\mathcal{L}^{(\rm int)}_{\rm YM}$ which depends on the strong coupling g_s.

Solution: Contracting the field strength tensors,

$$G_{\mu\nu}G^{\mu\nu} = \Big(\partial_\mu A_\nu - \partial_\nu A_\mu + ig_s[A_\mu, A_\nu]\Big)\Big(\partial^\mu A^\nu - \partial^\nu A^\mu + ig_s[A^\mu, A^\nu]\Big)\,,$$
$$= (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)$$
$$+ ig_s(\partial_\mu A_\nu - \partial_\nu A_\mu)[A^\mu, A^\nu] + ig_s[A_\mu, A_\nu](\partial_\mu A_\nu - \partial_\nu A_\mu)$$
$$- g_s^2[A_\mu, A_\nu][A^\mu, A^\nu]\,.$$

Now, we use that $A_\mu = A^a_\mu T_a$ where $T_a = \lambda_a/2$, so

$$G_{\mu\nu}G^{\mu\nu} = \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)\left(\partial^\mu A^{\nu\, b} - \partial^\nu A^{\mu\, b}\right) T_a T_b$$
$$+ ig_s\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right) A^{\mu\, b} A^{\nu\, c}\, T_a[T_b, T_c] + ig_s A^a_\mu A^b_\nu \left(\partial_\mu A^{\nu\, c} - \partial_\nu A^{\mu\, c}\right)[T_a, T_b]T_c$$
$$- g_s^2 A^a_\mu A^b_\nu A^{\mu\, c} A^{\nu\, d}\,[T_a, T_b][T_c, T_d]\,.$$

Furthermore, $[T_a, T_b] = if_{abc}T_c$, so

$$G_{\mu\nu}G^{\mu\nu} = \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)\left(\partial^\mu A^{\nu\, b} - \partial^\nu A^{\mu\, b}\right) T_a T_b$$
$$+ ig_s\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right) A^{\mu\, b} A^{\nu\, c}\, T_a(if_{bcd}T_d) + ig_s A^a_\mu A^b_\nu \left(\partial_\mu A^{\nu\, c} - \partial_\nu A^{\mu\, c}\right)(if_{abd}T_d)T_c$$
$$- g_s^2 A^a_\mu A^b_\nu A^{\mu\, c} A^{\nu\, d}\,(if_{abe}T_e)(if_{cdf}T_f)\,.$$

Now, taking the trace, we use $\mathrm{tr}(T_a T_b) = \mathrm{tr}(\lambda_a\lambda_b)/4 = \delta_{ab}/2$, so the Yang-Mills Lagrange density is

$$\mathcal{L}_{\mathrm{YM}} = -\frac{1}{4}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)\left(\partial^\mu A^{\nu\, a} - \partial^\nu A^{\mu\, a}\right)$$
$$- \frac{1}{4}g_s f_{bca}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right) A^{\mu\, b} A^{\nu\, c} - \frac{1}{4}g_s f_{abc} A^a_\mu A^b_\nu \left(\partial_\mu A^{\nu\, c} - \partial_\nu A^{\mu\, c}\right)$$
$$+ \frac{1}{4}g_s^2 f_{abe}f_{cde}\, A^a_\mu A^b_\nu A^{\mu\, c} A^{\nu\, d}\,,$$
$$= -\frac{1}{4}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)\left(\partial^\mu A^{\nu\, a} - \partial^\nu A^{\mu\, a}\right)$$
$$- \frac{1}{2}g_s f_{abc}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right) A^{\mu\, b} A^{\nu\, c} + \frac{1}{4}g_s^2 f_{abe}f_{cde}\, A^a_\mu A^b_\nu A^{\mu\, c} A^{\nu\, d}\,,$$
$$\equiv \mathcal{L}^{(\mathrm{free})}_{\mathrm{YM}} + \mathcal{L}^{(\mathrm{int})}_{\mathrm{YM}}\,,$$

where we used $f_{bca} = f_{abc}$ from the antisymmetry properties of the structure constants. So, the free and interacting Lagrange densities are

$$\mathcal{L}^{(\mathrm{free})}_{\mathrm{YM}} = -\frac{1}{4}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)\left(\partial^\mu A^{\nu\, a} - \partial^\nu A^{\mu\, a}\right)\,,$$
$$\mathcal{L}^{(\mathrm{int})}_{\mathrm{YM}} = -\frac{1}{2}g_s f_{abc}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right) A^{\mu\, b} A^{\nu\, c} + \frac{1}{4}g_s^2\, f_{abe}f_{cde}\, A^a_\mu A^b_\nu A^{\mu\, c} A^{\nu\, d}\,.$$

4. Consider a $q\bar{q}$ meson within an exact flavor SU(3) quark model, i.e., $q = u, d, s$. Assume the meson is flavor neutral. A generic wave function for this meson is given by

$$|n\,^{2S+1}L_J, m_J\rangle_{q\bar{q}} = \sum_{m_L, m_S} \langle L m_L; S m_S | J m_J \rangle \sum_{s,\bar{s}} \langle \frac{1}{2} s; \frac{1}{2} \bar{s} | S m_S \rangle$$
$$\times \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\, \varphi_{n,L}(p)\, Y_{L m_L}(\hat{\mathbf{p}})\, |q_s(\mathbf{p}) \bar{q}_{\bar{s}}(-\mathbf{p})\rangle \ ,$$

where n is the radial quantum number, S is the total intrinsic spin, L is the orbital angular momentum, J is the total angular momentum, m_J is the total angular momentum projection on some fixed z-axis, m_L is the orbital angular momentum projection, m_S is the total intrinsic spin projection, $\varphi_{n,L}$ is the momentum-space radial wave function, and $Y_{L m_L}$ are spherical harmonics. The quarks are spin-1/2 fermions with spin s and $\bar{s}$ for the q and $\bar{q}$, respectively. The two-quark state is defined in the center-of-momentum frame as the usual direct product $|q_s(\mathbf{p}) \bar{q}_{\bar{s}}(-\mathbf{p})\rangle \equiv |q_s(\mathbf{p})\rangle \otimes |\bar{q}_{\bar{s}}(-\mathbf{p})\rangle$.

(a) Determine the allowed values of S.

> **Solution:** Since we have two spin-1/2 objects, the total spin is either $S = 0$ or 1. This can be seen from the Clebsch-Gordan decomposition. If $\mathbf{2}$ is the fundamental representation of $\mathfrak{su}(2)$, then $\mathbf{2} \times \mathbf{2} = \mathbf{1} + \mathbf{3}$. Therefore, we have either a singlet ($S = 0$) or a triplet ($S = 1$) state.

(b) Show that under parity $\mathcal{P}$, the $q\bar{q}$ meson has an eigenvalue

$$\mathcal{P}\, |n\,^{2S+1}L_J, m_J\rangle_{q\bar{q}} = (-1)^{L+1}\, |n\,^{2S+1}L_J, m_J\rangle_{q\bar{q}} \ .$$

Hint: Recall that $\mathcal{P}\, |q_s(\mathbf{p})\rangle = \eta_q\, |q_s(-\mathbf{p})\rangle$ and $\eta_{\bar{q}} \equiv -\eta_q$.

Solution: By direct evaluation,

$$\begin{aligned}
\mathcal{P}\left|n\,^{2S+1}L_J, m_J\right\rangle_{q\bar{q}} &= \sum_{m_L,m_S} \langle Lm_L; Sm_S | Jm_J\rangle \sum_{s,\bar{s}} \left\langle \frac{1}{2}s; \frac{1}{2}\bar{s} | Sm_S \right\rangle \\
&\qquad \times \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\, \varphi_{n,L}(p)\, Y_{Lm_L}(\hat{\mathbf{p}}) \mathcal{P} \left| q_s(\mathbf{p}) \bar{q}_{\bar{s}}(-\mathbf{p}) \right\rangle \,, \\
&= \sum_{m_L,m_S} \langle Lm_L; Sm_S | Jm_J\rangle \sum_{s,\bar{s}} \left\langle \frac{1}{2}s; \frac{1}{2}\bar{s} | Sm_S \right\rangle \\
&\qquad \times \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\, \varphi_{n,L}(p)\, Y_{Lm_L}(\hat{\mathbf{p}}) \eta_q \eta_{\bar{q}} \left| q_s(-\mathbf{p}) \bar{q}_{\bar{s}}(\mathbf{p}) \right\rangle \,, \\
&= \eta_q \eta_{\bar{q}} \sum_{m_L,m_S} \langle Lm_L; Sm_S | Jm_J\rangle \sum_{s,\bar{s}} \left\langle \frac{1}{2}s; \frac{1}{2}\bar{s} | Sm_S \right\rangle \\
&\qquad \times \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\, \varphi_{n,L}(p)\, Y_{Lm_L}(-\hat{\mathbf{p}}) \left| q_s(\mathbf{p}) \bar{q}_{\bar{s}}(-\mathbf{p}) \right\rangle \,, \\
&= \eta_q \eta_{\bar{q}} (-1)^L \sum_{m_L,m_S} \langle Lm_L; Sm_S | Jm_J\rangle \sum_{s,\bar{s}} \left\langle \frac{1}{2}s; \frac{1}{2}\bar{s} | Sm_S \right\rangle \\
&\qquad \times \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\, \varphi_{n,L}(p)\, Y_{Lm_L}(\hat{\mathbf{p}}) \left| q_s(\mathbf{p}) \bar{q}_{\bar{s}}(-\mathbf{p}) \right\rangle \,, \\
&= \eta_q \eta_{\bar{q}} (-1)^L \left|n\,^{2S+1}L_J, m_J\right\rangle_{q\bar{q}} \,,
\end{aligned}$$

where in the third line we let $\mathbf{p} \to -\mathbf{p}$ in the integrand, and then in the fourth line we used $Y_{Lm_L}(-\hat{\mathbf{p}}) = (-1)^L Y_{Lm_L}(\hat{\mathbf{p}})$. Therefore, since $\eta_{\bar{q}} = -\eta_q$, the parity of the quark model hadron is

$$\mathcal{P}\left|n\,^{2S+1}L_J, m_J\right\rangle_{q\bar{q}} = (-1)^{L+1} \left|n\,^{2S+1}L_J, m_J\right\rangle_{q\bar{q}} \,,$$

since $\eta_q^2 = 1$.

(c) Show that under charge conjugation $\mathcal{C}$, the $q\bar{q}$ meson has an eigenvalue

$$\mathcal{C}\left|n\,^{2S+1}L_J, m_J\right\rangle_{q\bar{q}} = (-1)^{L+S} \left|n\,^{2S+1}L_J, m_J\right\rangle_{q\bar{q}} \,.$$

Hint: Recall that $\mathcal{C}\left|q_s(\mathbf{p})\right\rangle = \left|\bar{q}_s(\mathbf{p})\right\rangle$, and under interchange $\mathrm{P}_{12}\left|q_1 q_2\right\rangle = -\left|q_2 q_1\right\rangle$.

Solution: By direct evaluation,

$$\begin{aligned}
\mathcal{C}\,|n\,^{2S+1}L_J, m_J\rangle_{q\bar{q}} &= \sum_{m_L,m_S} \langle Lm_L; Sm_S|Jm_J\rangle \sum_{s,\bar{s}} \Big\langle \frac{1}{2}s; \frac{1}{2}\bar{s}|Sm_S\Big\rangle \\
&\qquad \times \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\,\varphi_{n,L}(p)\,Y_{Lm_L}(\hat{\mathbf{p}})\mathcal{C}\,|q_s(\mathbf{p})\bar{q}_{\bar{s}}(-\mathbf{p})\rangle\ , \\
&= \sum_{m_L,m_S} \langle Lm_L; Sm_S|Jm_J\rangle \sum_{s,\bar{s}} \Big\langle \frac{1}{2}s; \frac{1}{2}\bar{s}|Sm_S\Big\rangle \\
&\qquad \times \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\,\varphi_{n,L}(p)\,Y_{Lm_L}(\hat{\mathbf{p}})\,|\bar{q}_s(\mathbf{p})q_{\bar{s}}(-\mathbf{p})\rangle\ , \\
&= -\sum_{m_L,m_S} \langle Lm_L; Sm_S|Jm_J\rangle \sum_{s,\bar{s}} \Big\langle \frac{1}{2}s; \frac{1}{2}\bar{s}|Sm_S\Big\rangle \\
&\qquad \times \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\,\varphi_{n,L}(p)\,Y_{Lm_L}(\hat{\mathbf{p}})\,|q_{\bar{s}}(-\mathbf{p})\bar{q}_s(\mathbf{p})\rangle\ , \\
&= -\sum_{m_L,m_S} \langle Lm_L; Sm_S|Jm_J\rangle \sum_{s,\bar{s}} (-1)^{S+1}\Big\langle \frac{1}{2}\bar{s}; \frac{1}{2}s|Sm_S\Big\rangle \\
&\qquad \times \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\,\varphi_{n,L}(p)\,Y_{Lm_L}(-\hat{\mathbf{p}})\,|q_{\bar{s}}(\mathbf{p})\bar{q}_s(-\mathbf{p})\rangle\ , \\
&= -(-1)^{S+1}(-1)^L \sum_{m_L,m_S} \langle Lm_L; Sm_S|Jm_J\rangle \sum_{s,\bar{s}} \Big\langle \frac{1}{2}s; \frac{1}{2}\bar{s}|Sm_S\Big\rangle \\
&\qquad \times \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\,\varphi_{n,L}(p)\,Y_{Lm_L}(\hat{\mathbf{p}})\,|q_s(\mathbf{p})\bar{q}_{\bar{s}}(-\mathbf{p})\rangle\ , \\
&= (-1)^{L+S}\,|n\,^{2S+1}L_J, m_J\rangle_{q\bar{q}}\ ,
\end{aligned}$$

where in the third line we used the antisymmetry properties of fermions, in the fourth line used $\langle j_1m_1; j_2m_2|jm\rangle = (-1)^{j_1+j_2-s}\langle j_2m_2; j_1m_1|jm\rangle$ and the fact that j is integer, and in the fifth line we let $\mathbf{p} \to -\mathbf{p}$ in the integrand, and then used $Y_{Lm_L}(-\hat{\mathbf{p}}) = (-1)^L Y_{Lm_L}(\hat{\mathbf{p}})$. Therefore, the C-parity of the quark model hadron is

$$\mathcal{C}\,|n\,^{2S+1}L_J, m_J\rangle_{q\bar{q}} = (-1)^{L+S}\,|n\,^{2S+1}L_J, m_J\rangle_{q\bar{q}}\ .$$

(d) Determine *all* allowed J^{PC} quantum numbers for of the meson for $L \leq 3$. List all J^{PC} that are forbidden for $J \leq 3$ (observed mesons with these quantum numbers are called *exotic*, as they are not allowed in the quark model).

Solution: The angular momentum quantum numbers of the $q\bar{q}$ state are $S = 0$ or 1, $L = 0, 1, 2, 3, \ldots$, and $|L - S| \leq J \leq L + S$. The parity of a given state is $P = (-1)^{L+1}$, and the C-parity is $C = (-1)^{L+S}$. So, we can make a table of the allowed J^{PC} for all

$L \leq 3$.

Orbital Angular Momentum	Spin	J^{PC}
$L = 0$ (S)	$S = 0$	0^{-+}
	$S = 1$	1^{--}
$L = 1$ (P)	$S = 0$	1^{+-}
	$S = 1$	$(0, 1, 2)^{++}$
$L = 2$ (D)	$S = 0$	2^{-+}
	$S = 1$	$(1, 2, 3)^{--}$
$L = 3$ (F)	$S = 0$	3^{+-}
	$S = 1$	$(2, 3, 4)^{++}$

So, the allowed quantum numbers for a $q\bar{q}$ state in the quark model are

$$J^{PC} = (0, 2, \ldots)^{-+}, (1, 3, \ldots)^{+-}, (1, 2, 3, \ldots)^{--}, (0, 1, 2, \ldots)^{++} .$$

Notice that there is a set of states not allowed within this model, called exotic, are

$$J^{PC}_{\text{exotic}} = 0^{--}, (1, 3, \ldots)^{-+}, (0, 2, \ldots)^{+-} .$$

(e) List *one* example (if one exist) of an observed unflavored meson for each J^{PC} supermultiplet by searching the Particle Data Group database (https://pdglive.lbl.gov) for *light unflavored mesons*. Are there any examples of observed mesons with exotic quantum numbers?

Solution: The following hadrons correspond to the multiplets found in the previous part,

J^{PC}	hadron
$(0, 2)^{-+}$	$(\pi^0, \pi_2(1880))$
$(1, 3)^{+-}$	$(b_1(1235), ???)$
$(1, 2, 3)^{--}$	$(\rho(770), ???, \rho_3(1690))$
$(0, 1, 2, 3, 4)^{++}$	$(f_0(500), a_1(1260), f_2(1270), ???, a_4(1970))$

where the "???" indicate that no unflavored neutral hadron has been observed with these quantum numbers.

There has been some observations of exotic quantum numbers, one example being the $\pi_1(1600)$ which has $J^{PC} = 1^{-+}$.