1. **Reading & Research:** It can be shown that the beta function for the weak interaction coupling is negative, similar to QCD. However, the weak interaction does not exhibit any "confinement" phenomena. Give an explanation of why the weak interaction does not have similar physics to QCD.

> **Solution:**
> While the beta function is negative for electroweak interactions, the running stops at the electroweak scale due to symmetry breaking, giving a vacuum expectation value $\sim 250\,\mathrm{GeV}$. The SU(2) confinement scale is much lower than the electroweak scale, see "Strongly coupled standard model" by M. Claudson, E. Farhi, and R. L. Jaffe, Phys. Rev. D 34, 873 (1986).

2. The leading order beta function for QCD gives a running coupling

$$\alpha_s(Q^2) = \frac{4\pi}{\beta_0}\frac{1}{\log(Q^2/\Lambda^2_{\mathrm{QCD}})}\,,$$

where $\beta_0 = 11 - 2N_f/3$, where N_f is the number of active quark flavors and Q^2 is the energy scale of the system. Active flavors mean quarks that contribute to the system, that is those with $Q^2 > m_q^2$. The scale Λ_{QCD} is defined where the coupling diverges. Qualitatively, $m_s < \Lambda_{\mathrm{QCD}} < m_c$, where m_s and m_c are the strange and charm quark masses.

(a) If at a given Q_0^2, $\alpha_s(Q_0^2)$ is measured, write an expression for Λ^2_{QCD} in terms of β_0, Q_0^2, and $\alpha_s(Q_0^2)$.

> **Solution:** At Q_0^2, the running coupling is
>
> $$\alpha_s(Q_0^2) = \frac{4\pi}{\beta_0}\frac{1}{\log\left(Q_0^2/\Lambda^2_{\mathrm{QCD}}\right)}\,.$$
>
> Solving for Λ^2_{QCD}, we have
>
> $$\frac{\beta_0\alpha_s(Q_0^2)}{4\pi} = \frac{1}{\log\left(Q_0^2/\Lambda^2_{\mathrm{QCD}}\right)} \implies \log\left(Q_0^2/\Lambda^2_{\mathrm{QCD}}\right) = \frac{4\pi}{\beta_0\alpha_s(Q_0^2)}\,,$$
>
> $$\implies \frac{Q_0^2}{\Lambda^2_{\mathrm{QCD}}} = \exp\left(\frac{4\pi}{\beta_0\alpha_s(Q_0^2)}\right)\,,$$
>
> $$\implies \Lambda^2_{\mathrm{QCD}} = Q_0^2\,\exp\left(-\frac{4\pi}{\beta_0\alpha_s(Q_0^2)}\right)\,.$$

(b) Fixing α_s at the Z^0 boson mass, $Q_0 = m_Z$, determine a numerical value for Λ_{QCD} in MeV. **Hint:** Since Λ_{QCD} is between the strange and charm masses, one needs to run the coupling from the Z boson scale to through the heavy quark thresholds. At each quark mass, one must match the coupling above and below this scale, where the number of active quark flavors differ. Perform this matching and running until you find a scale at which the coupling diverges. Use the Review of Particle Physics to find the masses and $\alpha_s(m_Z)$.

Solution: Since the QCD scale is $m_s < \Lambda_{\rm QCD} < m_c$, we need to run the coupling from the Z^0 boson mass through the b and c quarks. We then simply match the coupling at each threshold,

$$\Lambda_{\rm QCD} = m_c \exp\left(\frac{2\pi}{\beta_0(N_f=3)\alpha_s(m_c^2)}\right) ,$$

where

$$\alpha_s(m_c^2) = \frac{\alpha_s(m_b^2)}{1+\frac{\alpha_s(m_b^2)}{4\pi}\beta_0(N_f=4)\log\left(m_c^2/m_b^2\right)} ,$$

$$\alpha_s(m_b^2) = \frac{\alpha_s(m_Z^2)}{1+\frac{\alpha_s(m_Z^2)}{4\pi}\beta_0(N_f=5)\log\left(m_b^2/m_Z^2\right)} .$$

Using the RPP values, $m_Z = 91.1876(21)\,\text{GeV}$, $\alpha(m_Z^2) = 0.1185(6)$, $m_c = 1.275(25)\,\text{GeV}$, $m_b = 4.18(3)\,\text{GeV}$. This gives the following, $\alpha_s(m_b^2) \simeq 0.214$, $\alpha_s(m_c^2) \simeq 0.322$, and $\Lambda_{\rm QCD} \simeq 146\,\text{MeV}$.

3. To first approximation, the electric form factor of the proton is measured to be

$$G_E(Q^2) = \left(1+\frac{Q^2}{\Lambda^2}\right)^{-2} ,$$

where Q^2 is the momentum transfer to the proton and $\Lambda^2 \approx 0.71\,\text{GeV}^2$.

(a) Determine the root-mean-square charge radius of the proton, $\sqrt{\langle r^2\rangle}$, in fm. Compare the result to the experimentally measured value in the Review of Particle Physics.

Solution: The rms charge radius is defined via

$$G_E(Q^2) = 1 + \frac{1}{6}\langle r^2\rangle\, Q^2 ,$$

so that we can get access to the charge radius via

$$\langle r^2\rangle = -6\frac{\partial}{\partial Q^2}G_E(Q^2)\bigg|_{Q^2=0} .$$

Evaluating the derivative, we find

$$\begin{aligned}\langle r^2\rangle &= -6\frac{\partial}{\partial Q^2}\left(1+\frac{Q^2}{\Lambda^2}\right)^{-2}\bigg|_{Q^2=0} ,\\ &= 12\frac{1}{\Lambda^2}\left(1+\frac{Q^2}{\Lambda^2}\right)^{-3}\bigg|_{Q^2=0} ,\\ &= \frac{12}{\Lambda^2} .\end{aligned}$$

So, the rms radius is

$$\sqrt{\langle r^2\rangle} = \sqrt{\frac{12}{\Lambda^2}} = \sqrt{\frac{12}{0.71\,\mathrm{GeV}^2}} \cdot 0.197\,\mathrm{GeV}\cdot\mathrm{fm} \approx 0.81\,\mathrm{fm}.$$

From the RPP, the rms charge radius of the proton is $0.8409(4)\,\mathrm{fm}$, which qualitatively agrees with this model.

(b) Compute the distribution of electric charge $\rho(\mathbf{x})$ associated with this form-factor. **Hint:** Assume a static distribution, such that the form-factor is defined as

$$G_E(Q^2) = \int \mathrm{d}^3\mathbf{x}\, e^{i\mathbf{q}\cdot\mathbf{x}}\, \rho(\mathbf{x})\,,$$

where $Q^2 = -q^2$ and $q^2 = -|\mathbf{q}|^2$.

Solution: Given the assumptions, the inverse Fourier transform is

$$\begin{aligned}\rho(\mathbf{x}) &= \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3}\, e^{-i\mathbf{q}\cdot\mathbf{x}}\, G_E(|\mathbf{q}|^2)\,,\\ &= \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3}\, e^{-i\mathbf{q}\cdot\mathbf{x}} \left(1 + \frac{|\mathbf{q}|^2}{\Lambda^2}\right)^{-2}.\end{aligned}$$

Recall the plane wave expansion for the exponential

$$e^{-i\mathbf{q}\cdot\mathbf{x}} = 4\pi \sum_{\ell,m_\ell} (-i)^\ell j_\ell(|\mathbf{q}||\mathbf{x}|)\, Y^*_{\ell m_\ell}(\hat{\mathbf{q}}) Y_{\ell m_\ell}(\hat{\mathbf{x}})\,,$$

where j_ℓ are the spherical Bessel functions and $Y_{\ell m_\ell}$ are the spherical harmonics. Transforming the integral to spherical coordinates, the charge density becomes

$$\rho(\mathbf{x}) = \frac{4\pi}{(2\pi)^3} \sum_{\ell,m_\ell} (-i)^\ell Y_{\ell m_\ell}(\hat{\mathbf{x}}) \int_0^\infty \mathrm{d}|\mathbf{q}|\, |\mathbf{q}|^2\, j_\ell(|\mathbf{q}||\mathbf{x}|) \left(1 + \frac{|\mathbf{q}|^2}{\Lambda^2}\right)^{-2} \int \mathrm{d}\hat{\mathbf{q}}\, Y^*_{\ell m_\ell}(\hat{\mathbf{q}})\,.$$

Since the form-factor is independent of the orientation, the orthonormality of the spherical harmonics restricts $\ell = m_\ell = 0$, giving the spherical Bessel function $j_0(qx) = \sin(qx)/qx$. Therefore, the integral reduces to

$$\rho(\mathbf{x}) = \frac{1}{2\pi^2 |\mathbf{x}|} \int_0^\infty \mathrm{d}|\mathbf{q}|\, |\mathbf{q}| \sin(|\mathbf{q}||\mathbf{x}|) \left(1 + \frac{|\mathbf{q}|^2}{\Lambda^2}\right)^{-2}.$$

Notice that the integrand is even under $|\mathbf{q}| \to -|\mathbf{q}|$. So we extend the integration bounds to $|\mathbf{q}| \in (-\infty, \infty)$, and divide by 2. Furthermore, since the integrand is real, let's use $\sin x = \mathrm{Im}\, e^{ix}$ to write the integral as

$$\rho(\mathbf{x}) = \frac{\Lambda^4}{(2\pi)^2 |\mathbf{x}|} \mathrm{Im} \int_{-\infty}^{\infty} \mathrm{d}|\mathbf{q}|\, \frac{|\mathbf{q}| e^{i|\mathbf{q}||\mathbf{x}|}}{(|\mathbf{q}|^2 + \Lambda^2)^2}\,.$$

Let's consider a contour integral in the complex $|\mathbf{q}|$-plane,

$$\int_C \mathrm{d}|\mathbf{q}|\, \frac{|\mathbf{q}| e^{i|\mathbf{q}||\mathbf{x}|}}{(|\mathbf{q}|^2 + \Lambda^2)^2}\,,$$

where the contour C is taken along the real line and then a semi-circle of radius R in the upper-half plane, $\mathrm{Im}\,|\mathbf{q}| > 0$. With this contour and integrand, the integral over the semi-circle vanishes in the limit $R \to \infty$, thus

$$\int_C \mathrm{d}|\mathbf{q}|\, \frac{|\mathbf{q}| e^{i|\mathbf{q}||\mathbf{x}|}}{(|\mathbf{q}|^2 + \Lambda^2)^2} = \int_{-\infty}^{\infty} \mathrm{d}|\mathbf{q}|\, \frac{|\mathbf{q}| e^{i|\mathbf{q}||\mathbf{x}|}}{(|\mathbf{q}|^2 + \Lambda^2)^2} \,.$$

The integrand contains two poles of order 2, $(|\mathbf{q}|^2 + \Lambda^2)^2 = (|\mathbf{q}| + i\Lambda)^2(|\mathbf{q}| - i\Lambda)^2$. Since we close the contour in the upper-half plane, we use the residue theorem to pick up only the pole $|\mathbf{q}| = i\Lambda$,

$$\int_C \mathrm{d}|\mathbf{q}|\, \frac{|\mathbf{q}| e^{i|\mathbf{q}||\mathbf{x}|}}{(|\mathbf{q}| + i\Lambda)^2(|\mathbf{q}| - i\Lambda)^2} = 2\pi i \lim_{|\mathbf{q}| \to i\Lambda} \frac{\mathrm{d}}{\mathrm{d}|\mathbf{q}|}\, \frac{|\mathbf{q}| e^{i|\mathbf{q}||\mathbf{x}|}}{(|\mathbf{q}| + i\Lambda)^2} \,.$$

Evaluating the derivative,

$$\frac{\mathrm{d}}{\mathrm{d}|\mathbf{q}|}\, \frac{|\mathbf{q}| e^{i|\mathbf{q}||\mathbf{x}|}}{(|\mathbf{q}| + i\Lambda)^2} = \frac{e^{i|\mathbf{q}||\mathbf{x}|}}{(|\mathbf{q}| + i\Lambda)^2} + \frac{i|\mathbf{q}||\mathbf{x}| e^{i|\mathbf{q}||\mathbf{x}|}}{(|\mathbf{q}| + i\Lambda)^2} - \frac{2|\mathbf{q}| e^{i|\mathbf{q}||\mathbf{x}|}}{(|\mathbf{q}| + i\Lambda)^3} \,,$$

and taking the limit $|\mathbf{q}| \to i\Lambda$, we find

$$\begin{aligned}
\lim_{|\mathbf{q}| \to i\Lambda} \frac{\mathrm{d}}{\mathrm{d}|\mathbf{q}|}\, \frac{|\mathbf{q}| e^{i|\mathbf{q}||\mathbf{x}|}}{(|\mathbf{q}| + i\Lambda)^2} &= e^{-\Lambda|\mathbf{x}|} \left(-\frac{1}{4\Lambda^2} + \frac{\Lambda|\mathbf{x}|}{4\Lambda^2} + \frac{2\Lambda}{8\Lambda^3} \right) , \\
&= \frac{|\mathbf{x}|}{4\Lambda}\, e^{-\Lambda|\mathbf{x}|} \,,
\end{aligned}$$

thus our integral is

$$\int_{-\infty}^{\infty} \mathrm{d}|\mathbf{q}|\, \frac{|\mathbf{q}| e^{i|\mathbf{q}||\mathbf{x}|}}{(|\mathbf{q}|^2 + \Lambda^2)^2} = \frac{i\pi|\mathbf{x}|}{2\Lambda}\, e^{-\Lambda|\mathbf{x}|} \,,$$

and the charge distribution is

$$\begin{aligned}
\rho(\mathbf{x}) &= \frac{\Lambda^4}{(2\pi)^2|\mathbf{x}|}\, \mathrm{Im} \int_{-\infty}^{\infty} \mathrm{d}|\mathbf{q}|\, \frac{|\mathbf{q}| e^{i|\mathbf{q}||\mathbf{x}|}}{(|\mathbf{q}|^2 + \Lambda^2)^2} \,, \\
&= \frac{\Lambda^3}{8\pi}\, e^{-\Lambda|\mathbf{x}|} \,,
\end{aligned}$$

giving an exponential charge distribution for the proton.
As a check, we expect that if we integrate over all $\mathbf{x}$ that $\int \mathrm{d}^3\mathbf{x}\, \rho(\mathbf{x}) = 1$. We find

$$\int \mathrm{d}^3\mathbf{x}\, \rho(\mathbf{x}) = 4\pi \int_0^{\infty} \mathrm{d}|\mathbf{x}|\, |\mathbf{x}|^2\, \rho(\mathbf{x}) = \frac{\Lambda^3}{2} \int_0^{\infty} \mathrm{d}|\mathbf{x}|\, |\mathbf{x}|^2\, e^{-\Lambda|\mathbf{x}|} \,,$$

where the second integral follows from the angular independence of the charge density. To perform the integral, let us recognize that

$$|\mathbf{x}|^2\, e^{-\Lambda|\mathbf{x}|} = \frac{\partial^2}{\partial\Lambda^2} e^{-\Lambda|\mathbf{x}|} \,,$$

so that

$$\begin{aligned}\int \mathrm{d}^3\mathbf{x}\,\rho(\mathbf{x}) &= \frac{\Lambda^3}{2}\frac{\partial^2}{\partial\Lambda^2}\int_0^\infty \mathrm{d}|\mathbf{x}|\,e^{-\Lambda|\mathbf{x}|}\,,\\ &= \frac{\Lambda^3}{2}\frac{\partial^2}{\partial\Lambda^2}\left(-\frac{1}{\Lambda}e^{-\Lambda|\mathbf{x}|}\right)\Bigg|_0^\infty\,,\\ &= \frac{\Lambda^3}{2}\frac{\partial^2}{\partial\Lambda^2}\,\frac{1}{\Lambda}\,,\\ &= \frac{\Lambda^3}{2}\,\frac{2}{\Lambda^3} = 1\,,\end{aligned}$$

as expected.