1. Consider the Abelian Higgs model, with the symmetric Lagrange density given by

$$\mathcal{L} = (D_\mu \varphi)^*(D^\mu \varphi) + \mu^2 \varphi^* \varphi - \frac{\lambda}{3!}(\varphi^*\varphi)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,,$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $D_\mu = \partial_\mu + iqA_\mu$. This theory has an unstable extremum at $\varphi = 0$, and is invariant under global U(1) transformations, $\varphi \to e^{i\alpha(x)}\varphi$ and $A_\mu \to A_\mu - q^{-1}\partial_\mu \alpha$.

(a) Let $\varphi(x) = \frac{1}{\sqrt{2}} r(x)\, e^{i\theta(x)}$ where $r(x)$, $\theta(x)$ are real scalar fields. Show that the Lagrange density in terms of these fields is

$$\mathcal{L} = \frac{1}{2}\partial_\mu r\, \partial^\mu r + \frac{1}{2} r^2 (\partial_\mu \theta + qA_\mu)^2 - \frac{\lambda}{4!}(r^2 - a^2)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \text{const.}\,,$$

where we have ignored non-dynamical constants. What is a in terms of the theory parameters μ, λ, and q?

Solution: Taking the covariant derivative on φ,

$$\begin{aligned} D_\mu \varphi &= \frac{1}{\sqrt{2}}(\partial_\mu + iqA_\mu) r(x) e^{i\theta(x)}\,, \\ &= \frac{1}{\sqrt{2}}\partial_\mu \left(r(x) e^{i\theta(x)}\right) + \frac{1}{\sqrt{2}} iqA_\mu r(x) e^{i\theta(x)}\,, \\ &= \frac{1}{\sqrt{2}}\partial_\mu r(x) e^{i\theta(x)} + \frac{i}{\sqrt{2}} r(x) \partial_\mu \theta(x)\, e^{i\theta(x)} + \frac{1}{\sqrt{2}} iqA_\mu r(x) e^{i\theta(x)}\,, \\ &= \frac{1}{\sqrt{2}}\left[\partial_\mu r(x) + i r(x)\left(\partial_\mu \theta(x) + qA_\mu\right)\right] e^{i\theta(x)}\,. \end{aligned}$$

Similarly, we also find

$$(D_\mu \varphi)^* = \frac{1}{\sqrt{2}}\left[\partial_\mu r(x) - i r(x)\left(\partial_\mu \theta(x) + qA_\mu\right)\right] e^{-i\theta(x)}\,,$$

thus the kinetic term is

$$\begin{aligned} (D_\mu \varphi)^*(D^\mu \varphi) &= \frac{1}{2}\left[\partial_\mu r(x) - i r(x)\left(\partial_\mu \theta(x) + qA_\mu\right)\right]\left[\partial^\mu r(x) + i r(x)\left(\partial^\mu \theta(x) + qA^\mu\right)\right]\,, \\ &= \frac{1}{2}\partial_\mu r \partial^\mu r + \frac{1}{2} r^2 (\partial_\mu \theta + qA_\mu)^2\,. \end{aligned}$$

Trivially, $\varphi^* \varphi = r^2/2$, so the Lagrange density is

$$\mathcal{L} = \frac{1}{2}\partial_\mu r \partial^\mu r + \frac{1}{2} r^2 (\partial_\mu \theta + qA_\mu)^2 + \frac{1}{2}\mu^2 r^2 - \frac{\lambda}{4!} r^4 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,.$$

Completing the square on the potential terms, we find

$$\begin{aligned}\frac{1}{2}\mu^2 r^2 - \frac{\lambda}{4!}r^4 &= -\frac{\lambda}{4!}\left(r^4 - 2\frac{6\mu^2}{\lambda}r^2\right)\,,\\ &= -\frac{\lambda}{4!}\left(r^4 - 2\frac{6\mu^2}{\lambda}r^2 + \left(\frac{6\mu^2}{\lambda}\right)^2 - \left(\frac{6\mu^2}{\lambda}\right)^2\right)\,,\\ &= -\frac{\lambda}{4!}\left(r^2 - \left(\frac{6\mu^2}{\lambda}\right)^2\right)^2 - \frac{\lambda}{4!}\left(\frac{6\mu^2}{\lambda}\right)^2\,,\\ &\equiv -\frac{\lambda}{4!}\left(r^2 - a^2\right)^2 + \text{const.}\,,\end{aligned}$$

where in the last equality we identified $a = \sqrt{6\mu^2/\lambda}$, and discarded the irrelevant non-dynamical constant.
Combining this with before, we find the Lagrange density

$$\mathcal{L} = \frac{1}{2}\partial_\mu r\,\partial^\mu r + \frac{1}{2}r^2(\partial_\mu\theta + qA_\mu)^2 - \frac{\lambda}{4!}(r^2 - a^2)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \text{const.}\,,$$

as desired.

(b) Expand the theory about the true vacuum of the theory, $r(x) \to a + \rho(x)$, $\theta(x) \to \theta(x)$, $A_\mu(x) \to A_\mu(x)$. Write each term of the new Lagrange density in terms of the fields ρ, A_μ, and θ. Determine the mass of each field in terms of the parameters of the theory μ, λ, and q, and the vacuum expectation value a.

Solution: Expanding the theory about the true vacuum, $r(x) \to a + \rho(x)$, we find for the kinetic term,

$$\frac{1}{2}\partial_\mu r\,\partial^\mu r \to \frac{1}{2}\partial_\mu(a+\rho)\,\partial^\mu(a+\rho) = \frac{1}{2}\partial_\mu\rho\,\partial^\mu\rho\,,$$

and

$$\begin{aligned}\frac{1}{2}r^2(\partial_\mu\theta + qA_\mu)^2 &\to \frac{1}{2}(a+\rho)^2(\partial_\mu\theta + qA_\mu)^2\,,\\ &= \frac{1}{2}a^2(\partial_\mu\theta + qA_\mu)^2 + a\rho(\partial_\mu\theta + qA_\mu)^2 + \frac{1}{2}\rho^2(\partial_\mu\theta + qA_\mu)^2\,,\end{aligned}$$

For the potential term, we find

$$
\begin{aligned}
-\frac{\lambda}{4!}(r^2-a^2)^2 \to -\frac{\lambda}{4!}((a+\rho)^2-a^2)^2\,, \\
&= -\frac{\lambda}{4!}(\rho^2+2a\rho)^2\,, \\
&= -\frac{\lambda}{4!}(\rho^4+4a\rho^3+4a^2\rho^2)\,, \\
&= -\frac{\lambda a^2}{6}\rho^2-\frac{\lambda a}{6}\rho^3-\frac{\lambda}{4!}\rho^4\,.
\end{aligned}
$$

Combining all these terms, we find after spontaneous symmetry breaking

$$
\begin{aligned}
\mathcal{L} = \frac{1}{2}\partial_\mu\rho\,\partial^\mu\rho + \frac{1}{2}a^2(\partial_\mu\theta+qA_\mu)^2 + a\rho(\partial_\mu\theta+qA_\mu)^2 + \frac{1}{2}\rho^2(\partial_\mu\theta+qA_\mu)^2 \\
- \frac{\lambda a^2}{6}\rho^2 - \frac{\lambda a}{6}\rho^3 - \frac{\lambda}{4!}\rho^4 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,.
\end{aligned}
$$

The mass of the θ field is zero, while the mass of A_μ field is $m_A = aq$, and for the ρ field it is $m_\rho = \sqrt{2}\mu$.

(c) Eliminate the mixing term, $\partial_\mu\theta A^\mu$, by choosing the *unitary gauge*, $A_\mu \to A_\mu - q^{-1}\partial_\mu\theta$. Write down each term of the Lagrange density in this gauge.

Solution: Under the unitary gauge transformation, $A_\mu \to A_\mu - q^{-1}\partial_\mu\theta$, we find that the $F_{\mu\nu}F^{\mu\nu}$ term is invariant, while all terms with $(\partial_\mu\theta+qA_\mu)^2 \to (qA_\mu)^2 = q^2A_\mu A^\mu$. Thus the Lagrange density becomes

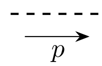
$$
\begin{aligned}
\mathcal{L} = \frac{1}{2}\partial_\mu\rho\,\partial^\mu\rho - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\
+ \frac{1}{2}a^2q^2A_\mu A^\mu + a\rho q^2A_\mu A^\mu + \frac{1}{2}\rho^2q^2A_\mu A^\mu \\
- \frac{\lambda a^2}{6}\rho^2 - \frac{\lambda a}{6}\rho^3 - \frac{\lambda}{4!}\rho^4\,.
\end{aligned}
$$

Let $m_A = qa$, $m_\rho = \sqrt{2}\mu$ where $\lambda a^2/6 = \mu^2$, we can write the theory in terms of the masses and the Higgs v.e.v. as

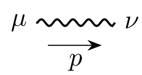
$$
\begin{aligned}
\mathcal{L} = \frac{1}{2}\partial_\mu\rho\partial^\mu\rho - \frac{1}{2}m_\rho^2\rho^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m_A^2A_\mu A^\mu \\
+ \frac{m_A^2}{a}\rho\,A_\mu A^\mu + \frac{m_A^2}{2a^2}\rho^2\,A_\mu A^\mu - \frac{m_\rho^2}{2a}\rho^3 - \frac{m_\rho^2}{8a^2}\rho^4\,.
\end{aligned}
$$

(d) Write the Feynman rules for the Abelian Higgs model in the unitary gauge.

Solution: We can read off the Feynman rules from the Lagrange density in part (c). The propagators are

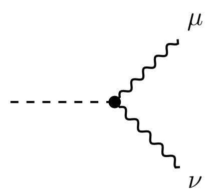


$$= \frac{i}{p^2 - m_\rho^2 + i\epsilon} \, ;$$

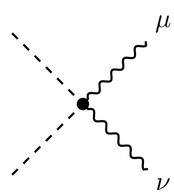


$$= \frac{-i}{p^2 - m_A^2 + i\epsilon} \left(g_{\mu\nu} - \frac{p_\mu p_\nu}{m_A^2} \right) \, ;$$

The Higgs-gauge boson interactions are

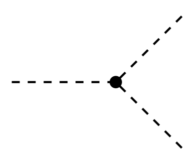


$$= i \frac{2m_A^2}{a} g^{\mu\nu} \, ;$$

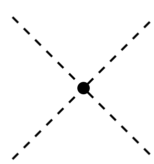


$$= -i \frac{2m_A^2}{a^2} g^{\mu\nu} \, ;$$

The Higgs self-interactions are



$$= -i \frac{3m_\rho^2}{a} \, ;$$



$$= -i \frac{3m_\rho^2}{a^2} \, ;$$

2. Recall that the decay rate for an $X \to n$ reaction in the X particle's rest frame is

$$\mathrm{d}\Gamma(X \to 1 + \cdots + n) = \frac{1}{2m_X} \langle |\mathcal{M}|^2 \rangle \, \mathrm{d}\Phi_n \left(p_X - \sum_{j=1}^{n} p_j \right) \, ,$$

where the n-body differential phase space is defined by

$$\mathrm{d}\Phi_n\left(p_X - \sum_{j=1}^{n} p_j\right) \equiv (2\pi)^4 \delta^{(4)}\left(p_X - \sum_{j=1}^{n} p_j\right) \frac{1}{\mathcal{S}} \prod_{k=1}^{n} \frac{\mathrm{d}^3\mathbf{p}_k}{(2\pi)^3 2E_k}\,,$$

with each momentum being $p_j = (E_j, \mathbf{p}_j)$ and $\mathcal{S}$ is a symmetry factor for identical particles. Show that for a two-body X decay, the decay rate is given by

$$\mathrm{d}\Gamma(X \to 1+2) = \frac{1}{32\pi^2}\langle|\mathcal{M}|^2\rangle \frac{1}{\mathcal{S}} \frac{|\mathbf{p}|}{m_X^2}\,\mathrm{d}\Omega\,,$$

where $|\mathbf{p}| = \lambda^{1/2}(m_X^2, m_1^2, m_2^2)/2m_X$ and Ω are the momentum and the solid angle of particle 1.

Solution: From Problem Set 2, we know that the two-body Lorentz invariant phase space in the CM frame is given by

$$\mathrm{d}\Phi_2 = \frac{1}{\mathcal{S}} \frac{|\mathbf{p}|}{4\pi\sqrt{s}} \frac{\mathrm{d}\Omega}{4\pi}\,.$$

For the decay of particle X in its rest frame, $\sqrt{s} \to m_X$, so that we may readily find

$$\begin{aligned}\mathrm{d}\Gamma(X \to 1+2) &= \frac{1}{2m_X}\langle|\mathcal{M}|^2\rangle \frac{1}{\mathcal{S}} \frac{|\mathbf{p}|}{4\pi m_X} \frac{\mathrm{d}\Omega}{4\pi}\,,\\ &= \frac{1}{32\pi^2}\langle|\mathcal{M}|^2\rangle \frac{1}{\mathcal{S}} \frac{|\mathbf{p}|}{m_X^2}\,\mathrm{d}\Omega\,,\end{aligned}$$

as desired.