

## Generating vertex Feynman rules

Given an interaction Lagrange density  $\mathcal{L}_{\text{int.}}$  which is a functional of some generic field  $\varphi_{i_a}^a$  where  $a = 1, \dots, n$  denotes the type of field and  $i_a$  is its a generic index for the representation of the field under Poincaré and internal symmetry transformations. The momentum space interaction vertex  $i\Gamma(p_1, \dots, p_n)$  is then given by

$$(2\pi)^4 \delta^{(4)}(p_1 + p_2 + \dots + p_n) i\Gamma_{i_1 \dots i_n}(p_1, \dots, p_n) = \prod_{a=1}^n \frac{\delta}{\delta \tilde{\varphi}_{i_a}^a(p_a)} \left( i \int d^4x \mathcal{L}_{\text{int.}} \right), \quad (1)$$

where  $\tilde{\varphi}_{i_a}^a(p)$  is the Fourier transform of the field,

$$\varphi_{i_a}^a(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ik \cdot x} \tilde{\varphi}_{i_a}^a(p). \quad (2)$$

We assume that all momenta are flowing into the vertex. We will give two examples of how to generate the vertex function  $i\Gamma$ .

### Self-Interacting Scalar Field Theory

Fir consider  $\varphi^4$  theory with interacting Lagrange density

$$\mathcal{L}_{\text{int.}} = -\frac{\lambda}{4!} \varphi(x)^4. \quad (3)$$

The vertex function is defined by

$$(2\pi)^4 \delta^{(4)}(p_1 + p_2 + p_3 + p_4) i\Gamma = \frac{\delta}{\delta \tilde{\varphi}(p_1)} \frac{\delta}{\delta \tilde{\varphi}(p_2)} \frac{\delta}{\delta \tilde{\varphi}(p_3)} \frac{\delta}{\delta \tilde{\varphi}(p_4)} \left( -i \frac{\lambda}{4!} \int d^4x \varphi(x)^4 \right), \quad (4)$$

where the Fourier transform of the field is given by

$$\varphi(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ik \cdot x} \tilde{\varphi}(p). \quad (5)$$

Inserting this into the vertex function, we find

$$\begin{aligned} & (2\pi)^4 \delta^{(4)}(p_1 + p_2 + p_3 + p_4) i\Gamma \\ &= -i \frac{\lambda}{4!} \frac{\delta}{\delta \tilde{\varphi}(p_1)} \frac{\delta}{\delta \tilde{\varphi}(p_2)} \frac{\delta}{\delta \tilde{\varphi}(p_3)} \frac{\delta}{\delta \tilde{\varphi}(p_4)} \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \int \frac{d^4k_3}{(2\pi)^4} \int \frac{d^4k_4}{(2\pi)^4} \\ & \quad \times \int d^4x e^{-i(k_1+k_2+k_3+k_4) \cdot x} \tilde{\varphi}(k_1) \tilde{\varphi}(k_2) \tilde{\varphi}(k_3) \tilde{\varphi}(k_4). \end{aligned} \quad (6)$$

Note that the integration over  $x$  can be performed to give a Dirac delta function,

$$(2\pi)^4 \delta^{(4)}(k_1 + k_2 + k_3 + k_4) = \int d^4x e^{-i(k_1+k_2+k_3+k_4) \cdot x}. \quad (7)$$

One of the momentum integrals can then be performed (we choose  $k_4$ ), so that  $k_4 = -k_1 - k_2 - k_3$ ,

$$\begin{aligned} (2\pi)^4 \delta^{(4)}(p_1 + p_2 + p_3 + p_4) i\Gamma &= -i \frac{\lambda}{4!} \frac{\delta}{\delta \tilde{\varphi}(p_1)} \frac{\delta}{\delta \tilde{\varphi}(p_2)} \frac{\delta}{\delta \tilde{\varphi}(p_3)} \frac{\delta}{\delta \tilde{\varphi}(p_4)} \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \int \frac{d^4k_3}{(2\pi)^4} \\ & \quad \times \tilde{\varphi}(k_1) \tilde{\varphi}(k_2) \tilde{\varphi}(k_3) \tilde{\varphi}(-k_1 - k_2 - k_3), \end{aligned} \quad (8)$$

We now perform the functional derivatives. The first derivative gives 4 terms,

$$\begin{aligned}
 & \frac{\delta}{\delta\tilde{\varphi}(p_1)} \frac{\delta}{\delta\tilde{\varphi}(p_2)} \frac{\delta}{\delta\tilde{\varphi}(p_3)} \frac{\delta}{\delta\tilde{\varphi}(p_4)} \tilde{\varphi}(k_1)\tilde{\varphi}(k_2)\tilde{\varphi}(k_3)\tilde{\varphi}(-k_1 - k_2 - k_3), \\
 &= (2\pi)^4 \frac{\delta}{\delta\tilde{\varphi}(p_1)} \frac{\delta}{\delta\tilde{\varphi}(p_2)} \frac{\delta}{\delta\tilde{\varphi}(p_3)} \left( \delta^{(4)}(k_1 - p_4)\tilde{\varphi}(k_2)\tilde{\varphi}(k_3)\tilde{\varphi}(-k_1 - k_2 - k_3) \right. \\
 &\quad + \tilde{\varphi}(k_1)\delta^{(4)}(k_2 - p_4)\tilde{\varphi}(k_3)\tilde{\varphi}(-k_1 - k_2 - k_3) \\
 &\quad + \tilde{\varphi}(k_1)\tilde{\varphi}(k_2)\delta^{(4)}(k_3 - p_4)\tilde{\varphi}(-k_1 - k_2 - k_3) \\
 &\quad \left. + \tilde{\varphi}(k_1)\tilde{\varphi}(k_2)\tilde{\varphi}(k_3)\delta^{(4)}(p_4 + k_1 + k_2 + k_3) \right). \tag{9}
 \end{aligned}$$

We can repeat taking derivatives. It is easy to see that taking the second derivative will give 3 terms for each of the 4 terms. Therefore, the total number of terms will be  $4 \times 3$ , leaving only two fields left on each term. The third derivative then has two options for the  $4 \times 3$  terms, which gives  $4 \times 3 \times 2$  terms with one remaining field. The final derivative will eliminate the last field. Each of the  $4 \times 3 \times 2 \times 1 = 4!$  terms contain 4 delta functions. The three momentum integrals will yield a single momentum conserving delta function,  $\delta(p_1 + p_2 + p_3 + p_4)$ . Therefore, since there is no momentum dependence in the interaction, all  $4!$  terms are identical, and cancel the  $4!$  in the denominator. This finally gives

$$i\Gamma = -i\lambda. \tag{10}$$

### Quantum Electrodynamics

The QED interaction Lagrange density is

$$\mathcal{L}_{\text{int.}} = -Q\bar{\psi}(x)\not{A}(x)\psi(x), \tag{11}$$

where  $Q$  is the coupling. The vertex is then

$$(2\pi)^4 \delta^{(4)}(q + p_1 + p_2) i\Gamma_{\alpha\beta}^{\mu} = \frac{\delta}{\delta\tilde{A}_{\mu}(q)} \frac{\delta}{\delta\tilde{\psi}_{\alpha}(p_1)} \frac{\delta}{\delta\tilde{\bar{\psi}}_{\beta}(p_2)} \left( -iQ \int d^4x \bar{\psi}_{\epsilon}(x) \gamma_{\epsilon}^{\nu} \psi_{\delta}(x) A_{\nu}(x) \right), \tag{12}$$

where the Fourier transforms of the fields are

$$\begin{aligned}
 A_{\nu}(x) &= \int \frac{d^4\bar{q}}{(2\pi)^4} e^{-i\bar{q}\cdot x} \tilde{A}_{\nu}(\bar{q}), \\
 \psi_{\delta}(x) &= \int \frac{d^4k_1}{(2\pi)^4} e^{-ik_1\cdot x} \tilde{\psi}_{\delta}(k_1), \\
 \bar{\psi}_{\epsilon}(x) &= \int \frac{d^4k_2}{(2\pi)^4} e^{-ik_2\cdot x} \tilde{\bar{\psi}}_{\epsilon}(k_2).
 \end{aligned} \tag{13}$$

Therefore, the vertex function is given by

$$\begin{aligned}
 (2\pi)^4 \delta^{(4)}(q + p_1 + p_2) i\Gamma_{\alpha\beta}^{\mu} &= -iQ \frac{\delta}{\delta \tilde{A}_{\mu}(q)} \frac{\delta}{\delta \tilde{\psi}_{\alpha}(p_1)} \frac{\delta}{\delta \tilde{\psi}_{\beta}(p_2)} \int d^4x \bar{\psi}_{\epsilon}(x) \gamma_{\epsilon\delta}^{\nu} \psi_{\delta}(x) A_{\nu}(x), \\
 &= -iQ \frac{\delta}{\delta \tilde{A}_{\mu}(q)} \frac{\delta}{\delta \tilde{\psi}_{\alpha}(p_1)} \frac{\delta}{\delta \tilde{\psi}_{\beta}(p_2)} \int \frac{d^4\bar{q}}{(2\pi)^4} \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \\
 &\quad \times \int d^4x e^{-i(\bar{q}+k_1+k_2)\cdot x} \tilde{\psi}_{\epsilon}(k_2) \gamma_{\epsilon\delta}^{\nu} \tilde{\psi}_{\delta}(k_1) \tilde{A}_{\nu}(\bar{q}), \\
 &= -iQ \frac{\delta}{\delta \tilde{A}_{\mu}(q)} \frac{\delta}{\delta \tilde{\psi}_{\alpha}(p_1)} \frac{\delta}{\delta \tilde{\psi}_{\beta}(p_2)} \int \frac{d^4\bar{q}}{(2\pi)^4} \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \\
 &\quad \times (2\pi)^4 \delta^{(4)}(\bar{q} + k_1 + k_2) \tilde{\psi}_{\epsilon}(k_2) \gamma_{\epsilon\delta}^{\nu} \tilde{\psi}_{\delta}(k_1) \tilde{A}_{\nu}(\bar{q}), \\
 &= -iQ \frac{\delta}{\delta \tilde{A}_{\mu}(q)} \frac{\delta}{\delta \tilde{\psi}_{\alpha}(p_1)} \frac{\delta}{\delta \tilde{\psi}_{\beta}(p_2)} \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \tilde{\psi}_{\epsilon}(k_2) \gamma_{\epsilon\delta}^{\nu} \tilde{\psi}_{\delta}(k_1) \tilde{A}_{\nu}(-k_1 - k_2).
 \end{aligned}$$

Taking now the functional derivatives,

$$\frac{\delta}{\delta f(p)} f(k) = (2\pi)^4 \delta^{(4)}(p - k)$$

we find

$$\begin{aligned}
 (2\pi)^4 \delta^{(4)}(q + p_1 + p_2) i\Gamma_{\alpha\beta}^{\mu} &= -iQ \frac{\delta}{\delta \tilde{A}_{\mu}(q)} \frac{\delta}{\delta \tilde{\psi}_{\alpha}(p_1)} \frac{\delta}{\delta \tilde{\psi}_{\beta}(p_2)} \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \tilde{\psi}_{\epsilon}(k_2) \gamma_{\epsilon\delta}^{\nu} \tilde{\psi}_{\delta}(k_1) \tilde{A}_{\nu}(-k_1 - k_2), \\
 &= -iQ \frac{\delta}{\delta \tilde{A}_{\mu}(q)} \gamma_{\beta\alpha}^{\nu} \tilde{A}_{\nu}(-p_1 - p_2), \\
 &= (2\pi)^4 \delta^{(4)}(q + p_1 + p_2) \left( -iQ \gamma_{\beta\alpha}^{\mu} \right)
 \end{aligned} \tag{14}$$

Thus, we conclude that the vertex function for QED is

$$\Gamma_{\beta\alpha} = -iQ \gamma_{\beta\alpha}^{\mu}. \tag{15}$$