William & Mary
CHARTERED 1693

PHYS 772 – The Standard Model of Particle Physics

Problem Set 2

Due: Tuesday, February 11 at 12:00pm

Term: Spring 2025

Instructor: Andrew W. Jackura

1. Consider a general *binary* reaction $ab \to cd$, where each particle has a mass m_j and four-momentum p_j, $j = a, b, c, d$. The *Mandelstam invariants* are defined to be

$$s = (p_a + p_b)^2\,, \qquad t = (p_a - p_c)^2\,, \qquad u = (p_a - p_d)^2\,.$$

   Show that $s + t + u = m_a^2 + m_b^2 + m_c^2 + m_d^2$.

2. Consider the (extremely rare) process $pp \to ppH$, where p is the proton and H the Higgs boson.
   - (a) Consider the reaction in a *fixed target experiment*, where one of the initial protons is at rest, while the other has an energy E. Determine the *minimum* value of E such that the Higgs production process can occur. Give your answer symbolically (in terms of m_p and m_H) as well as numerically.
   - (b) Consider the reaction performed in a *collider experiment*, where both proton have an energy E. Repeat the previous exercise for this scenario.
   - (c) Comment on the energy "reach" of the fixed target experiment vs. the collider experiment.

3. Consider a general two-body decay $a \to bc$, where each particle has a mass m_j, $j = a, b, c$ (with $m_a > m_b + m_c$ for the decay to be kinematically allowed).
   - (a) Show that in the rest frame of a, the three-momenta of b and c are equal in magnitude and opposite in direction, $\mathbf{p}_b = -\mathbf{p}_c$. Show that the magnitude is

$$p^\star \equiv |\mathbf{p}_b| = |\mathbf{p}_c| = \frac{1}{2m_a}\sqrt{[m_a^2 - (m_b + m_c)^2][m_a^2 - (m_b - m_c)^2]}\,.$$

   - (b) Use the previous result to numerically determine $p^\star$ for the process $\Delta^+ \to p + \pi^0$, taking the delta baryon mass to be $m_{\Delta^+} = 1232$ MeV.

4. The invariant *flux factor* of colliding particles a and b is defined as $\mathcal{F} = 4\sqrt{(p_a \cdot p_b)^2 - m_a^2 m_b^2}$.
   - (a) Show that $\mathcal{F} = 4E_aE_b(v_a + v_b)$ if the particles move towards each other with speeds v_a and v_b.
   - (b) Consider a Lorentz frame where $\mathbf{p}_a = -\mathbf{p}_b$, called the *center-of-momentum frame*. Show that $\mathcal{F} = 4p^\star\sqrt{s}$ in this frame, where $p^\star$ is the magnitude of the three momentum of both particles in this frame.
   - (c) Consider a Lorentz frame where particle b is at rest, called the *fixed-target frame* or *laboratory frame*. Show that $\mathcal{F} = 4m_b|\mathbf{p}_a|$ in this frame.

5. The two-body differential *Lorentz invariant phase space* for some initial total momentum P is defined as

$$\mathrm{d}\Phi_2(P \to p_1 + p_2) = (2\pi)^4 \delta^{(4)}(P - p_1 - p_2)\,\frac{1}{\mathcal{S}}\frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3\,2E_1}\frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^3\,2E_2}\,,$$

where $\mathcal{S}$ is a symmetry factor.

(a) Perform partial integrations in the center-of-momentum frame, where $P = (\sqrt{s}, \mathbf{0})$, to show that the differential phase space can be evaluated to

$$\mathrm{d}\Phi_2(P \to p_1 + p_2) = \frac{1}{\mathcal{S}}\frac{p^\star}{16\pi^2\sqrt{s}}\,\mathrm{d}\Omega^\star\,,$$

where $\mathrm{d}\Omega^\star = \mathrm{d}\cos\theta^\star\,\mathrm{d}\varphi^\star$ is the differential solid angle of particle 1 and $p^\star$ is the magnitude of momentum of the particles.

(b) Perform partial integrations in the laboratory frame, where $P = (E, \mathbf{P})$, to show that the differential phase space can be evaluated to

$$\mathrm{d}\Phi_2(P \to p_1 + p_2) = \frac{1}{\mathcal{S}}\frac{p_f^2}{16\pi^2|p_f E - p_i E_1\cos\theta|}\,\mathrm{d}\Omega\,,$$

where p_f is the magnitude of particle 1, p_i is the magnitude of the initial momentum, and $\mathrm{d}\Omega = \mathrm{d}\cos\theta\,\mathrm{d}\varphi$ is the differential solid angle of particle 1 with respect to the initial momentum.

6. The unpolarized differential decay rate for a two-body decay $a \to bc$ is defined by

$$\mathrm{d}\Gamma = \frac{1}{2E_a}\,\langle|\mathcal{M}|^2\rangle\,\mathrm{d}\Phi_2\,.$$

Show that in the rest frame of a, the decay rate can be written as

$$\Gamma = \frac{1}{32\pi^2}\frac{p^\star}{m_a^2}\frac{1}{\mathcal{S}}\int \mathrm{d}\Omega^\star\,\langle|\mathcal{M}|^2\rangle\,,$$

where $p^\star$ is the magnitude of the momentum of the decay products and $\mathrm{d}\Omega^\star = \mathrm{d}\cos\theta^\star\,\mathrm{d}\varphi^\star$ is the differential solid angle of particle b.

7. The unpolarized differential cross-section for a general binary reaction $ab \to cd$ is defined by

$$\mathrm{d}\sigma = \frac{1}{\mathcal{F}}\,\langle|\mathcal{M}|^2\rangle\,\mathrm{d}\Phi_2\,,$$

where $\langle|\mathcal{M}|^2\rangle$ is the spin-averaged matrix element.

(a) Show that the total cross section in the center-of-momentum (CM) frame is

$$\sigma_{\mathrm{CM}} = \frac{1}{64\pi^2 s}\frac{p_f^\star}{p_i^\star}\frac{1}{\mathcal{S}}\int \mathrm{d}\Omega^\star\,\langle|\mathcal{M}|^2\rangle\,,$$

where $p_i^\star$ and $p_f^\star$ are the initial and final state momenta.

(b) Show that the total cross section in the laboratory frame is

$$\sigma_{\mathrm{lab}} = \frac{1}{64\pi^2 m_b|\mathbf{p}_a|}\,\frac{1}{\mathcal{S}}\int \mathrm{d}\Omega\,\frac{|\mathbf{p}_c|^2}{||\mathbf{p}_c|(E_a + m_b) - |\mathbf{p}_a|E_c\cos\theta|}\,\langle|\mathcal{M}|^2\rangle\,.$$