WILLIAM & MARY

CHARTERED 1693

PHYS 772 – The Standard Model of Particle Physics

Problem Set 3

Due: Tuesday, February 18 at 12:00pm

Term: Spring 2025

Instructor: Andrew W. Jackura

1. The Dirac matrices $\gamma^\mu = (\gamma^0, \gamma^j)$ in the chiral (Weyl) representation are defined as

$$\gamma^0 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \qquad \gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix},$$

where I_2 is the 2×2 identity matrix and σ^j are the Pauli matrices.

(a) With this representation, confirm that $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} I_4$, where I_4 is the 4×4 identity matrix.
(b) Using the result in (a), show that $\gamma_\mu \gamma^\mu = 4\, I_4$.
(c) Prove that $\gamma_\mu \gamma^\nu \gamma^\mu = -2\gamma^\nu$ without using an explicit matrix representation.
(d) Similarly, prove that $\gamma_\mu \gamma^\nu \gamma^\rho \gamma^\mu = 4g^{\nu\rho} I_4$.

2. Given $\gamma^5 = \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, prove the following trace identities:

(a) $\mathrm{tr}\,(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}$,
(b) $\mathrm{tr}\,(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})$,
(c) The trace of *any* odd number of gamma matrices is zero.
(d) $\mathrm{tr}\,(\gamma^5) = \mathrm{tr}\,(\gamma^5\gamma^\mu) = \mathrm{tr}\,(\gamma^5\gamma^\mu\gamma^\nu) = \mathrm{tr}\,(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho) = 0$,
(e) $\mathrm{tr}\,(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = -4i\epsilon^{\mu\nu\rho\sigma}$.

3. The chiral projectors are defined as

$$P_R = \frac{1}{2}(I_4 + \gamma^5), \qquad P_L = \frac{1}{2}(I_4 - \gamma^5),$$

where I_4 is the 4×4 identity matrix. Prove the following properties:

(a) $\gamma^5 P_L = -P_L$, and $\gamma^5 P_R = P_R$,
(b) $(P_{L/R})^2 = P_{L/R}$,
(c) $P_L P_R = P_R P_L = 0$,
(d) $P_L + P_R = I_4$.

4. Suppose the charge conjugation operator is defined as $C = i\gamma^2\gamma^0$. Confirm that in the Weyl representation,

(a) $C^{-1} = C^\top = C^\dagger = -C$.
(b) $C\gamma^\mu C^{-1} = -(\gamma^\mu)^\top$,
(c) $C\gamma^5 C^{-1} = (\gamma^5)^\top$,

5. A Dirac spinor ψ is called a Majorana spinor if it satisfies the condition $\psi = C\bar{\psi}^\top$, and is called a Weyl spinor if it satisfies either $\psi = P_R\psi$ or $\psi = P_L\psi$. Determine whether or not a spinor can be both Majorana and Weyl.