



## PHYS 772 – The Standard Model of Particle Physics

### Problem Set 4

**Due:** Tuesday, March 04 at 4:00pm

**Term:** Spring 2025

**Instructor:** Andrew W. Jackura

1. Show that the Lie algebra structure constants  $c_{jkl}$ , defined by the Lie bracket  $[X^j, X^k] = c_{jkl}X^l$ , satisfy the relation  $c_{jkm}c_{mln} + c_{klm}c_{mjn} + c_{ljm}c_{mkn} = 0$ .
2. Consider a general Lie algebra  $[X^j, X^k] = c_{jkl}X^l$ , where  $c_{jkl} = -c_{kjl}$ . From the structure constants, we may form matrices  $M^j$  with matrix elements  $(M^j)_{lk} = c_{jkl}$ . Note the order of the indices. Show that these matrices furnish a representation of the algebra, i.e., show that  $[M^j, M^k] = c_{jkl}M^l$ . This representation is called the *adjoint representation*. **Hint:** The Jacobi identity may be helpful.
3. Suppose  $X^j$  is a generator for the Lie algebra  $[X^j, X^k] = c_{jkl}X^l$ . Show that  $X^2 = \sum_j X^j X^j$  commutes with the group generators, and therefore we may write  $(X^2)_{ab} = C_2(r) \delta_{ab}$  where  $C_2(r)$  is a constant called the *quadratic Casimir* of the representation  $r$ .
4. Let  $X^j$  be a generator for a generic  $\mathfrak{su}(N)$  Lie algebra,  $[X^j, X^k] = c_{jkl}X^l$ , and  $U(\alpha^j)$  is an element of the corresponding Lie group  $SU(N)$ , with  $U(\alpha^j) = \exp(\alpha^j X_j)$  with  $\alpha^j \in \mathbb{R}$ . Show that  $X^j$  are traceless, antihermitian  $N \times N$  matrices.
5. Consider the set of all complex  $2 \times 2$  matrices  $M$  with  $\det(M) = i$ . Does this set form a group under the usual matrix multiplication? Explain your reasoning.
6. Consider  $X_j = -\frac{1}{2}i\sigma_j$  as a bases element of the  $\mathfrak{su}(2)$  algebra,  $[X_j, X_k] = \epsilon_{jkl}X_l$ , where  $\sigma_j$  are the Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Verify the following:

- (a)  $[\sigma_j, \sigma_k] \equiv \sigma_j \sigma_k - \sigma_k \sigma_j = 2i\epsilon_{jkl}\sigma_l$ .
- (b)  $\{\sigma_j, \sigma_k\} \equiv \sigma_j \sigma_k + \sigma_k \sigma_j = 2\delta_{jk}I_2$ .
- (c)  $\sigma_j \sigma_k = \delta_{jk} + i\epsilon_{jkl}\sigma_l$ .
- (d) Show that a group element  $U(\alpha^j) \in SU(2)$  can be written as

$$U(\alpha^j) = \exp\left(-\frac{1}{2}i\alpha^j \sigma_j\right) = I_2 \cos\left(\frac{1}{2}\alpha\right) - i\frac{\alpha^j \sigma_j}{\alpha} \sin\left(\frac{1}{2}\alpha\right),$$

$$\text{where } \alpha^2 = \sum_j (\alpha_j)^2.$$

7. Consider  $X_j = L_j$  as a bases element of the  $\mathfrak{so}(3)$  algebra,  $[X_j, X_k] = \epsilon_{jkl}X_l$ , where  $L_j$  are the matrices,

$$L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad L_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Verify the following:

- (a)  $[L_j, L_k] = \epsilon_{jkl}L_l$ .
- (b)  $\{L_j, L_k\} \neq N\delta_{jk}$  for any  $j, k$ , and  $N$ .
- (c) Show that a group element  $O(\alpha^j) \in \text{SO}(3)$  can be written as

$$O(\alpha^j) = \exp(\alpha^j L_j) = I_3 + \frac{\alpha^j L_j}{\alpha} \sin \alpha + \left(\frac{\alpha^j L_j}{\alpha}\right)^2 (1 - \cos \alpha),$$

where  $\alpha^2 = \sum_j (\alpha_j)^2$ .