WILLIAM & MARY
CHARTERED 1693

PHYS 772 – The Standard Model of Particle Physics

Problem Set 8

Due: Tuesday, April 08 at 4:00pm

Term: Spring 2025

Instructor: Andrew W. Jackura

1. Can the following hadrons, in principle, exist within QCD? **(a)** qq, **(b)** $qq\bar{q}$, **(c)** $qq\bar{q}\bar{q}$, **(d)** gg, **(e)** qqg, **(f)** $q\bar{q}g$, **(g)** $qqqq\bar{q}$. **Hint:** Consider $\mathrm{SU}(3)_c$ symmetry transformations of observable hadrons. Gluons transform under the adjoint representation of $\mathrm{SU}(3)_c$.

2. Consider a non-abelian gauge field $A_\mu \equiv A_\mu^j T_j$, where $T_j \in \mathfrak{su}(N)$ are generators satisfying the Lie algebra $[T_j, T_k] = i c_{jkl} T_l$ with c_{jkl} being structure constants and $j, k, l = 1, 2, \ldots, N^2 - 1$. Under a local gauge transformation, $U = \exp(i\alpha^j(x) T_j)$ where $\alpha_j(x) \in \mathbb{R}$ for every j, the gauge fields transform as

$$A_\mu \to U A_\mu U^{-1} + \frac{i}{g} \left(\partial_\mu U\right) U^{-1} \, .$$

Show that under infinitesimal transformations, $\alpha^a(x) \ll 1$, the gauge fields transform as

$$A_\mu^j \to A_\mu^j - \frac{1}{g} \partial_\mu \alpha^j(x) - c_{jkl}\, \alpha^k A_\mu^l + \mathcal{O}(\alpha^2) \, .$$

3. The $SU(3)_c$ Yang-Mills Lagrange density for interacting gluon fields is given by $\mathcal{L}_{\mathrm{YM}} = -\frac{1}{2} \operatorname{tr}\left(G_{\mu\nu} G^{\mu\nu}\right)$, where the field-strength tensor is defined as $G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i g_s [A_\mu, A_\nu]$ with $A_\mu = A_\mu^a \, \lambda_a / 2$ are the gluon gauge fields and λ_a are the Gell-Mann matrices. Write the Lagrange density as a free part $\mathcal{L}_{\mathrm{YM}}^{(\mathrm{free})}$ and an interacting part $\mathcal{L}_{\mathrm{YM}}^{(\mathrm{int})}$ which depends on the strong coupling g_s.

4. We can learn about the structure of hadrons through interactions with electromagnetic probes. Consider elastic $e^- p \to e^- p$ for incident electron energies $E_e \gg m_p$. At leading order in the QED coupling, the process is dominated by one-photon exchange. The QED vertex for the proton can in general be written as

$$\Gamma_p^\mu(P', P) = \gamma^\mu F_1(Q^2) + \frac{i}{2 m_p} \sigma^{\mu\nu} q_\nu F_2(Q^2) \, ,$$

where P and P' are the initial and final momentum of the proton, respectively, and $q = P' - P$ is the momentum transfer by the photon with virtuality $Q^2 \equiv -q^2$. The form-factors F_1 and F_2 encode all the non-perturbative QCD interactions with the photon.

   (a) Show that, in the initial proton rest frame, that the ratio of the final to initial electron energy is

$$\frac{E_e'}{E_e} = \left(1 + \frac{2 E_e}{m_p} \sin^2 \frac{\theta}{2}\right)^{-1} \, ,$$

   where E_e and E_e' are the initial and final electron energies, respectively, and θ is the scattering angle defined with respect to the incident electron momentum.

(b) Show that, in the initial proton rest frame, that the differential cross-section is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2}{4E_e^2 \sin^4 \frac{\theta}{2}} \frac{E_e'}{E_e} \left[\left(F_1^2 + \frac{Q^2}{4m_p^2} F_2^2 \right) \cos^2 \frac{\theta}{2} + \frac{Q^2}{4m_p^2} \left(F_1 + F_2 \right)^2 \sin^2 \frac{\theta}{2} \right] ,$$

where Ω is the solid angle defined in the initial proton rest frame. **Hint:** Use the Gordon identity to rewrite the proton-photon vertex as

$$\Gamma^\mu = \gamma^\mu \left(F_1 + F_2 \right) - \frac{(P' + P)^\mu}{2m_p} F_2 ,$$

for simpler trace relations.

(c) Simplify the differential cross section for following limits: (i) the static source limit $m_p \to \infty$, and the (ii) structureless proton limit.

(d) The Mott cross section (modified for proton recoil) is that of an electron on a spinless target,

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \right)_{\text{Mott}} = \frac{\alpha^2}{4E_e^2 \sin^4 \frac{\theta}{2}} \left(\frac{E_e'}{E_e} \right) \cos^2 \frac{\theta}{2} .$$

Show that the electron-proton scattering cross section can bet written in the Rosenbluth form as

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \right)_{\text{Mott}} \left[\frac{G_E^2 + \tau G_M^2}{1 + \tau} + 2\tau G_M^2 \tan^2 \frac{\theta}{2} \right] ,$$

where $\tau = Q^2/4m_p^2$ and we have introduced the Sachs electric and magnetic form factors,

$$G_E = F_1 - \tau F_2 , \qquad G_M = F_1 + F_2 .$$

The Sachs form factors are often easier to measure, and offer interpretations for Fourier transforms of electromagnetic charge distributions. Show that $G_E(0) = 1$ (unit proton charge) and $G_M(0) = \mu_p$ (proton magnetic moment).