William & Mary
CHARTERED 1693

PHYS 772 – The Standard Model of Particle Physics

Problem Set 2 – Solution

Due: Tuesday, February 11 at 12:00pm

Term: Spring 2025

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1. Consider a general *binary* reaction $ab \to cd$, where each particle has a mass m_j and four-momentum p_j, $j = a, b, c, d$. The *Mandelstam invariants* are defined to be

$$s = (p_a + p_b)^2 , \qquad t = (p_a - p_c)^2 , \qquad u = (p_a - p_d)^2 .$$

Show that $s + t + u = m_a^2 + m_b^2 + m_c^2 + m_d^2$.

Solution: Taking the sum $s + t + u$, we find

$$\begin{aligned} s + t + u &= (p_a + p_b)^2 + (p_a - p_c)^2 + (p_a - p_d)^2 , \\ &= \sum_j p_j^2 + 2p_a^2 + 2p_a \cdot p_b - 2p_a \cdot p_c - 2p_a \cdot p_d , \\ &= \sum_j m_j^2 + 2p_a \cdot (p_a + p_b - p_c - p_d) , \\ &= \sum_j m_j^2 , \end{aligned}$$

where in the third line we used conservation of four-momentum states $p_a + p_b = p_c + p_d$.

2. Consider the (extremely rare) process $pp \to ppH$, where p is the proton and H the Higgs boson.

(a) Consider the reaction in a *fixed target experiment*, where one of the initial protons is at rest, while the other has an energy E. Determine the *minimum* value of E such that the Higgs production process can occur. Give your answer symbolically (in terms of m_p and m_H) as well as numerically.

Solution: For fixed target experiments, we have a proton beam with $p_a = (E, \mathbf{p})$, while the other is at rest, $p_b = (m_p, \mathbf{0})$. Therefore, the Mandelstam s is $s = (p_a + p_b)^2 = (E + m_p)^2 - |\mathbf{p}|^2$. Since the proton is on-shell, $|\mathbf{p}|^2 = E^2 - m_p^2$, and thus $s = E^2 + m_p^2 + 2m_pE - (E^2 - m_p^2) = 2m_p^2 + 2m_pE$. The minimum energy E for Higgs production to occur must coincide with the threshold energy of the production, that is where $s = (2m_p + m_H)^2$. Since s is a Lorentz invariant, we can equate the two expressions and solve,

$$s = 2m_p^2 + 2m_pE_{\text{min}} = (2m_p + m_H)^2 ,$$

$$\implies E_{\text{min}} = \frac{(2m_p + m_H)^2}{2m_p} - m_p .$$

Numerically, we find $E_{\text{min}} \approx 8,560\,\text{GeV}$ for a Higgs mass $m_H \approx 125\,\text{GeV}$ and proton mass

$m_p \approx 0.94\,\text{GeV}$.

(b) Consider the reaction performed in a *collider experiment*, where both proton have an energy E. Repeat the previous exercise for this scenario.

Solution: In a collider experiment, both initial protons are moving at equal but opposite momentum, $p_a = (E, \mathbf{p})$ and $p_b = (E, -\mathbf{p}_))$. Thus, the total s is $s = (p_a + p_b)^2 = 4E^2$. Again, the minimum energy coincides with threshold production, $s = (2m_p + m_H)^2$, therefore we can find the minimum energy

$$s = 4E_{\min}^2 = (2m_p + m_H)^2\,,$$

$$\implies E_{\min} = m_p + \frac{1}{2}m_H\,.$$

Numerically, $E_{\min} \approx 63\,\text{GeV}$.

(c) Comment on the energy "reach" of the fixed target experiment vs. the collider experiment.

Solution: The minimum energy for the fixed target experiment is about 140 times larger than that of the collider experiment, which is due to having to boost the beam to extremely large energies.

3. Consider a general two-body decay $a \to bc$, where each particle has a mass m_j, $j = a, b, c$ (with $m_a > m_b + m_c$ for the decay to be kinematically allowed).

(a) Show that in the rest frame of a, the three-momenta of b and c are equal in magnitude and opposite in direction, $\mathbf{p}_b = -\mathbf{p}_c$. Show that the magnitude is

$$p^\star \equiv |\mathbf{p}_b| = |\mathbf{p}_c| = \frac{1}{2m_a}\sqrt{[m_a^2 - (m_b + m_c)^2][m_a^2 - (m_b - m_c)^2]}\,.$$

Solution: In the CM frame, $m_a = E_b + E_c$. Take the square of $(m_a - E_b)^2 = E_c^2$ to find

$$\begin{aligned} E_c^2 &= (m_a - E_b)^2\,, \\ &= m_a^2 + E_b^2 - 2m_a E_b\,. \end{aligned}$$

Now, the on-shell condition gives $E_b^2 = m_b^2 + \mathbf{p}_b^2$ and $E_c^2 = m_c^2 + \mathbf{p}_c^2$, with $\mathbf{p}_b^2 = \mathbf{p}_c^2$ in the CM frame. So, solving for E_b,

$$E_b = \frac{m_a^2 + m_b^2 - m_c^2}{2\sqrt{s}}\,.$$

To find the momentum, use

$$
\begin{aligned}
|\mathbf{p}_b|^2 &= E_b^2 - m_b^2 \,, \\
&= \left(\frac{m_a^2 + m_b^2 - m_c^2}{2m_a}\right)^2 - m_b^2 \,, \\
&= \frac{1}{4m_a^2}\Big((m_a^2 + m_b^2 - m_c^2)^2 - 4m_a^2 m_b^2\Big) \,, \\
&= \frac{1}{4m_a^2}\Big([m_a^2 - (m_b + m_c)^2][m_a^2 - (m_b - m_c)^2]\Big) \,.
\end{aligned}
$$

In the CM frame, $|\mathbf{p}_b| = |\mathbf{p}_c| = p^\star$, therefore

$$
p^\star = |\mathbf{p}_b| = |\mathbf{p}_c| = \frac{1}{2m_a^2}\sqrt{[m_a^2 - (m_b + m_c)^2][m_a^2 - (m_b - m_c)^2]} \,.
$$

(b) Use the previous result to numerically determine $p^\star$ for the process $\Delta^+ \to p + \pi^0$, taking the delta baryon mass to be $m_{\Delta^+} = 1232$ MeV.

Solution: Substituting the mass values, we find $p^\star \approx 230$ MeV.

4. The invariant *flux factor* of colliding particles a and b is defined as $\mathcal{F} = 4\sqrt{(p_a \cdot p_b)^2 - m_a^2 m_b^2}$.

(a) Show that $\mathcal{F} = 4E_a E_b (v_a + v_b)$ if the particles move towards each other with speeds v_a and v_b.

Solution: Take the system along the z direction, with $p_a = (E_a, 0, 0, E_a v_a)$ and $p_b = (E_b, 0, 0, -E_b v_b)$. So,

$$
\begin{aligned}
\mathcal{F} &= 4\sqrt{(p_a \cdot p_b)^2 - m_a^2 m_b^2} \,, \\
&= 4\sqrt{(E_a E_b + E_a E_b v_a v_b)^2 - E_a^2 E_b^2 (1 - v_a^2)(1 - v_b^2)} \,, \\
&= 4E_a E_b \sqrt{(1 + v_a v_b)^2 - (1 - v_a^2)(1 - v_b^2)} \,, \\
&= 4E_a E_b (v_a + v_b) \,.
\end{aligned}
$$

(b) Consider a Lorentz frame where $\mathbf{p}_a = -\mathbf{p}_b$, called the *center-of-momentum frame*. Show that $\mathcal{F} = 4p^\star\sqrt{s}$ in this frame, where $p^\star$ is the magnitude of the three momentum of both particles in this frame.

Solution: Note that $s = (p_a + p_b)^2 = m_a^2 + m_b^2 + 2p_a \cdot p_b$. So, $(p_a \cdot p_b)^2 = (s - m_a^2 - m_b^2)^2/4$, therefore

$$\begin{aligned}\mathcal{F} &= 4\sqrt{(p_a \cdot p_b)^2 - m_a^2 m_b^2}\,,\\ &= 4\sqrt{\frac{(s - m_a^2 - m_b^2)^2}{4} - m_a^2 m_b^2}\,,\\ &= 4\sqrt{s}p^\star\,,\end{aligned}$$

where we have used the result from problem 3.

(c) Consider a Lorentz frame where particle b is at rest, called the *fixed-target frame* or *laboratory frame*. Show that $\mathcal{F} = 4m_b|\mathbf{p}_a|$ in this frame.

Solution: In the lab frame, $p_b = (m_b, \mathbf{0})$, therefore,

$$\begin{aligned}\mathcal{F} &= 4\sqrt{(p_a \cdot p_b)^2 - m_a^2 m_b^2}\,,\\ &= 4\sqrt{E_a^2 m_b^2 - m_a^2 m_b^2} = 4m_b\sqrt{E_a^2 - m_a^2}\,,\\ &= 4m_b|\mathbf{p}_a|\,.\end{aligned}$$

5. The two-body differential *Lorentz invariant phase space* for some initial total momentum P is defined as

$$\mathrm{d}\Phi_2(P \to p_1 + p_2) = (2\pi)^4 \delta^{(4)}(P - p_1 - p_2)\,\frac{1}{\mathcal{S}}\frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3\,2E_1}\frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^3\,2E_2}\,,$$

where $\mathcal{S}$ is a symmetry factor.

(a) Perform partial integrations in the center-of-momentum frame, where $P = (\sqrt{s}, \mathbf{0})$, to show that the differential phase space can be evaluated to

$$\mathrm{d}\Phi_2(P \to p_1 + p_2) = \frac{1}{\mathcal{S}}\frac{p^\star}{16\pi^2\sqrt{s}}\,\mathrm{d}\Omega^\star\,,$$

where $\mathrm{d}\Omega^\star = \mathrm{d}\cos\theta^\star\,\mathrm{d}\varphi^\star$ is the differential solid angle of particle 1 and $p^\star$ is the magnitude of momentum of the particles.

Assume we are integrating against a test function $f(\mathbf{p}_1, \mathbf{p}_2)$. Since the phase space is Lorentz invariant, we can evaluate in any reference frame. We choose the CM frame. The four-dimensional Dirac delta can be written as

$$\delta^{(4)}(P - p_1 - p_2) = \delta^{(4)}(E - E_1 - E_2)\,\delta^{(3)}(\mathbf{p}_1 + \mathbf{p}_2)\,,$$

where we used $\mathbf{P} = \mathbf{0}$.
So, we can integrate over the measure $\mathrm{d}^3\mathbf{p}_2$, eliminating the spatial momentum Dirac delta

functions,

$$\mathrm{d}\Phi_2(P \to p_1 + p_2) = \frac{1}{(4\pi)^2}\frac{1}{\mathcal{S}}\frac{\mathrm{d}^3\mathbf{p}_1}{E_1E_2}\,\delta(E - E_1 - E_2)\,.$$

Note that since $\mathbf{p}_1 = -\mathbf{p}_2$, $E_1 = \sqrt{m_1^2 + \mathbf{p}_1^2}$ and $E_2 = \sqrt{m_2^2 + \mathbf{p}_1^2}$. The remaining delta function can be evaluated by a change of variables to $|\mathbf{p}_1|$,

$$\begin{aligned}\delta(E - E_1 - E_2) &= \left|\frac{\partial(E - E_1 - E_2)}{\partial|\mathbf{p}_1|}\right|^{-1}\delta(|\mathbf{p}_1| - |\mathbf{p}_1^\star|)\,,\\ &= \frac{E_1E_2}{|\mathbf{p}_1|\sqrt{s}}\,\delta(|\mathbf{p}_1| - |\mathbf{p}_1^\star|)\end{aligned}$$

where $|\mathbf{p}_1^\star|$ is the solution to $E - E_1 - E_2 = 0$. So, converting the measure to spherical coordinates, we find

$$\begin{aligned}\mathrm{d}\Phi_2(P \to p_1 + p_2) &= \frac{1}{(4\pi)^2}\frac{1}{\mathcal{S}}\frac{\mathrm{d}^3\mathbf{p}_1}{E_1E_2}\frac{E_1E_2}{|\mathbf{p}_1|\sqrt{s}}\,\delta(|\mathbf{p}_1| - |\mathbf{p}_1^\star|)\,,\\ &= \frac{1}{(4\pi)^2}\frac{1}{\mathcal{S}}\frac{\mathrm{d}\Omega\mathrm{d}|\mathbf{p}_1|\,|\mathbf{p}_1|^2}{E_1E_2}\frac{E_1E_2}{|\mathbf{p}_1|\sqrt{s}}\,\delta(|\mathbf{p}_1| - |\mathbf{p}_1^\star|)\,,\\ &= \frac{1}{\mathcal{S}}\frac{p^\star}{4\pi\sqrt{s}}\frac{\mathrm{d}\Omega^\star}{4\pi}\,,\end{aligned}$$

where $p^\star = |\mathbf{p}_1^\star|$.

(b) Perform partial integrations in the laboratory frame, where $P = (E, \mathbf{P})$, to show that the differential phase space can be evaluated to

$$\mathrm{d}\Phi_2(P \to p_1 + p_2) = \frac{1}{\mathcal{S}}\frac{p_f^2}{16\pi^2|p_fE - p_iE_1\cos\theta|}\,\mathrm{d}\Omega\,,$$

where p_f is the magnitude of particle 1, p_i is the magnitude of the initial momentum, and $\mathrm{d}\Omega = \mathrm{d}\cos\theta\,\mathrm{d}\varphi$ is the differential solid angle of particle 1 with respect to the initial momentum.

Solution: For the lab frame, we again first integrate $\mathbf{p}_2$ against $\delta^{(3)}(\mathbf{P} - \mathbf{p}_1 - \mathbf{p}_2)$ to fix $\mathbf{p}_2 = \mathbf{P} - \mathbf{p}_1$,

$$\mathrm{d}\Phi_2(P \to p_1 + p_2) = \frac{1}{(4\pi)^2}\frac{1}{\mathcal{S}}\frac{\mathrm{d}^3\mathbf{p}_1}{E_1E_2}\,\delta(E - \sqrt{m_1 + \mathbf{p}_1^2} - \sqrt{m_2^2 + (\mathbf{P} - \mathbf{p}_1)^2})\,,$$

where $E_1 = \sqrt{m_1 + \mathbf{p}_1^2}$ and $E_2 = \sqrt{m_2^2 + (\mathbf{P} - \mathbf{p}_1)^2}$. Now, $(\mathbf{P} - \mathbf{p}_1)^2 = \mathbf{P}^2 + \mathbf{p}_1^2 - 2|\mathbf{p}_1||\mathbf{P}|\cos\theta$, and we define $|\mathbf{P}| = p_i$. So, we evaluate the remaining integrals in spherical coordinates

$$\begin{aligned}
\mathrm{d}\Phi_2(P \to p_1 + p_2) &= \frac{1}{(4\pi)^2}\frac{1}{\mathcal{S}}\frac{\mathrm{d}^3\mathbf{p}_1}{E_1E_2}\,\delta(E - \sqrt{m_1 + \mathbf{p}_1^2} - \sqrt{m_2^2 + p_i^2 + \mathbf{p}_1^2 - 2p_i|\mathbf{p}_1|\cos\theta})\,, \\
&= \frac{1}{(4\pi)^2}\frac{1}{\mathcal{S}}\frac{\mathrm{d}|\mathbf{p}_1|\,\mathbf{p}_1^2\,\mathrm{d}\Omega}{E_1E_2}\left(\left|\frac{|\mathbf{p}_1|}{E_1} + \frac{|\mathbf{p}_1| + p_i\cos\theta}{E_2}\right|\right)^{-1}\delta(|\mathbf{p}_1| - p_f)\,, \\
&= \frac{1}{16\pi^2}\frac{1}{\mathcal{S}}\frac{p_f^2}{|p_f(E_1+E_2) + p_iE_1\cos\theta|}\,\mathrm{d}\Omega\,.
\end{aligned}$$

With $E = E_1 + E_2$, we find the desired result.

6. The unpolarized differential decay rate for a two-body decay $a \to bc$ is defined by

$$\mathrm{d}\Gamma = \frac{1}{2E_a}\,\langle|\mathcal{M}|^2\rangle\,\mathrm{d}\Phi_2\,.$$

Show that in the rest frame of a, the decay rate can be written as

$$\Gamma = \frac{1}{32\pi^2}\frac{p^\star}{m_a^2}\frac{1}{\mathcal{S}}\int \mathrm{d}\Omega^\star\,\langle|\mathcal{M}|^2\rangle\ ,$$

where $p^\star$ is the magnitude of the momentum of the decay products and $\mathrm{d}\Omega^\star = \mathrm{d}\cos\theta^\star\,\mathrm{d}\varphi^\star$ is the differential solid angle of particle b.

Solution: For the particle at rest, $E_a = m_a$. Using the result from problem 5(a) with the invariant mass $\sqrt{s} = m_a$, we find

$$\mathrm{d}\Gamma = \frac{1}{2m_a}\langle|\mathcal{M}|^2\rangle\,\frac{1}{\mathcal{S}}\frac{p^\star}{4\pi m_a}\frac{\mathrm{d}\Omega^\star}{4\pi}\ ,$$

so

$$\Gamma = \frac{1}{32\pi^2}\frac{p^\star}{m_a^2}\frac{1}{\mathcal{S}}\int \mathrm{d}\Omega^\star\,\langle|\mathcal{M}|^2\rangle\,.$$

7. The unpolarized differential cross-section for a general binary reaction $ab \to cd$ is defined by

$$\mathrm{d}\sigma = \frac{1}{\mathcal{F}}\,\langle|\mathcal{M}|^2\rangle\,\mathrm{d}\Phi_2\,,$$

where $\langle|\mathcal{M}|^2\rangle$ is the spin-averaged matrix element.

(a) Show that the total cross section in the center-of-momentum (CM) frame is

$$\sigma_{\mathrm{CM}} = \frac{1}{64\pi^2 s}\frac{p_f^\star}{p_i^\star}\frac{1}{\mathcal{S}}\int \mathrm{d}\Omega^\star\,\langle|\mathcal{M}|^2\rangle\,,$$

where $p_i^\star$ and $p_f^\star$ are the initial and final state momenta.

Solution: Using the previous results from Problem 4 and 5, we find

$$\begin{aligned}\sigma &= \frac{1}{\mathcal{F}} \int \langle |\mathcal{M}|^2 \rangle \, \mathrm{d}\Phi_2 \,, \\ &= \frac{1}{4\sqrt{s} p_i^\star} \int \langle |\mathcal{M}|^2 \rangle \, \frac{1}{\mathcal{S}} \frac{p_f^\star}{16\pi^2 \sqrt{s}} \, \mathrm{d}\Omega^\star \,, \\ &= \frac{1}{64\pi^2 s} \frac{p_f^\star}{p_i^\star} \frac{1}{\mathcal{S}} \int \mathrm{d}\Omega^\star \, \langle |\mathcal{M}|^2 \rangle \,.\end{aligned}$$

(b) Show that the total cross section in the laboratory frame is

$$\sigma_{\text{lab}} = \frac{1}{64\pi^2 m_b |\mathbf{p}_a|} \frac{1}{\mathcal{S}} \int \mathrm{d}\Omega \, \frac{|\mathbf{p}_c|^2}{||\mathbf{p}_c|(E_a + m_b) - |\mathbf{p}_a| E_c \cos\theta|} \, \langle |\mathcal{M}|^2 \rangle \,.$$

Solution: Using the previous results from Problem 4 and 5, with $s = (E_a + m_b)$, we find

$$\begin{aligned}\sigma &= \frac{1}{\mathcal{F}} \int \langle |\mathcal{M}|^2 \rangle \, \mathrm{d}\Phi_2 \,, \\ &= \frac{1}{4 m_b |\mathbf{p}_a|} \int \langle |\mathcal{M}|^2 \rangle \, \frac{1}{16\pi^2} \frac{1}{\mathcal{S}} \frac{|\mathbf{p}_c|^2}{||\mathbf{p}_c|(E_a + m_b) + |\mathbf{p}_a| E_c \cos\theta|} \, \mathrm{d}\Omega \,, \\ &= \frac{1}{64\pi^2 m_b |\mathbf{p}_a|} \frac{1}{\mathcal{S}} \int \mathrm{d}\Omega \, \frac{|\mathbf{p}_c|^2}{||\mathbf{p}_c|(E_a + m_b) + |\mathbf{p}_a| E_c \cos\theta|} \, \langle |\mathcal{M}|^2 \rangle\end{aligned}$$