WILLIAM & MARY
CHARTERED 1693

PHYS 772 – The Standard Model of Particle Physics

Problem Set 3 – Solution

Due: Tuesday, February 18 at 12:00pm

Term: Spring 2025

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1. The Dirac matrices $\gamma^\mu = (\gamma^0, \gamma^j)$ in the chiral (Weyl) representation are defined as

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \qquad \gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix},$$

where I is the 2×2 identity matrix and σ^j are the Pauli matrices.

(a) With this representation, confirm that $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$.

Solution: To clarify some of the manipulations in these problems, we introduce I_4 as the 4×4 identity, and let $I \to I_2$ be the 2×2 identity. Thus, what is to be shown is $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} I_4$, given the chiral representation

$$\gamma^0 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \qquad \gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}.$$

Recall the properties of the Pauli matrices, $\{\sigma^j, \sigma^k\} = 2\delta^{jk} I_2$.

$$\begin{aligned}
\{\gamma^0, \gamma^0\} &= 2(\gamma^0)^2, \\
&= 2 \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \\
&= 2 \begin{pmatrix} I_2 & 0 \\ 0 & I_2 \end{pmatrix} = 2g^{00}\, I_4, \\
\{\gamma^0, \gamma^j\} &= \gamma^0\gamma^j + \gamma^j\gamma^0, \\
&= \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \\
&= \begin{pmatrix} -\sigma^j & 0 \\ 0 & \sigma^j \end{pmatrix} + \begin{pmatrix} \sigma^j & 0 \\ 0 & -\sigma^j \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 2g^{0j}\, I_4,
\end{aligned}$$

where we note that $g^{00} = +1$ and $g^{0j} = g^{j0} = 0$. Continuing,

$$\begin{aligned}
\{\gamma^j, \gamma^0\} &= \gamma^j\gamma^0 + \gamma^0\gamma^j = \{\gamma^0, \gamma^j\} = 2g^{0j}\, I_4 = 2g^{j0}\, I_4\,, \\
\{\gamma^j, \gamma^k\} &= \gamma^j\gamma^k + \gamma^k\gamma^j\,, \\
&= \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}\,, \\
&= \begin{pmatrix} -\sigma^j\sigma^k & 0 \\ 0 & -\sigma^j\sigma^k \end{pmatrix} + \begin{pmatrix} -\sigma^k\sigma^j & 0 \\ 0 & -\sigma^k\sigma^j \end{pmatrix}\,, \\
&= -\begin{pmatrix} \sigma^j\sigma^k + \sigma^k\sigma^j & 0 \\ 0 & \sigma^j\sigma^k + \sigma^k\sigma^j \end{pmatrix}\,, \\
&= -\begin{pmatrix} 2\delta^{jk} I_2 & 0 \\ 0 & 2\delta^{jk} I_2 \end{pmatrix}\,, \\
&= -2\delta^{jk}\, I_4 = 2g^{jk}\, I_4
\end{aligned}$$

Therefore, we have shown $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\, I_4$

(b) Using the result in (a), show that $\gamma_\mu\gamma^\mu = 4$.

Solution: Contract $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\, I_4$ with $g_{\mu\nu}$,

$$\begin{aligned}
g_{\mu\nu}\{\gamma^\mu, \gamma^\nu\} &= 2g_{\mu\nu}g^{\mu\nu}\, I_4\,, \\
\{\gamma_\mu, \gamma^\mu\} &= 2g^\mu{}_\mu\, I_4\,, \\
2\gamma_\mu\gamma^\mu &= 2 \cdot 4\, I_4\,.
\end{aligned}$$

So, we conclude $\gamma_\mu\gamma^\mu = 4\, I_4$.

(c) Prove that $\gamma_\mu\gamma^\nu\gamma^\mu = -2\gamma^\nu$ without using an explicit matrix representation.

Solution: Using the anticommutator relation, as well as the result from part (b), we find

$$\begin{aligned}
\gamma_\mu\gamma^\nu\gamma^\mu &= \gamma_\mu(2g^{\mu\nu}\, I_4 - \gamma^\mu\gamma^\nu)\,, \\
&= 2\gamma_\nu - \gamma_\mu\gamma^\mu\gamma^\nu\,, \\
&= 2\gamma_\nu - 4\gamma^\nu\,, \\
&= -2\gamma_\nu\,.
\end{aligned}$$

(d) Similarly, prove that $\gamma_\mu\gamma^\nu\gamma^\rho\gamma^\mu = 4g^{\nu\rho}$.

Solution: Using the anticommutator relation, as well as the result from part (c), we find

$$\begin{aligned}\gamma_\mu\gamma^\nu\gamma^\rho\gamma^\mu &= \gamma_\mu\gamma^\nu(2g^{\rho\mu}\,I_4 - \gamma^\mu\gamma^\rho)\,,\\ &= 2\gamma^\rho\gamma^\nu - \gamma_\mu\gamma^\nu\gamma^\mu\gamma^\rho\,,\\ &= 2\gamma^\rho\gamma^\nu + 2\gamma^\nu\gamma^\rho\,,\\ &= 2\{\gamma^\rho,\gamma^\nu\}\,,\\ &= 4g^{\nu\rho}\,I_4\,.\end{aligned}$$

2. Given $\gamma^5 = \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, prove the following trace identities:

(a) $\mathrm{tr}\,(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}$,

Solution: Taking the trace, we use the cyclic properties of the trace and the anticommutation relations, we have

$$\begin{aligned}\mathrm{tr}\,(\gamma^\mu\gamma^\nu) &= \frac{1}{2}\,\mathrm{tr}(\gamma^\mu\gamma^\nu + \gamma^\mu\gamma^\nu)\,,\\ &= \frac{1}{2}\Big[\,\mathrm{tr}(\gamma^\mu\gamma^\nu) + \mathrm{tr}(\gamma^\mu\gamma^\nu)\Big]\,,\\ &= \frac{1}{2}\,\mathrm{tr}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu)\,,\\ &= \frac{1}{2}\,\mathrm{tr}(\{\gamma^\mu\gamma^\nu\})\,,\\ &= \frac{1}{2}\cdot 2g^{\mu\nu}\,\mathrm{tr}(I_4)\,,\\ &= 4g^{\mu\nu}\,,\end{aligned}$$

where $\mathrm{tr}(I_4) = 4$.

(b) $\mathrm{tr}\,(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})$,

Solution: Here we use the anticommutation relation inside the trace,

$$\begin{aligned}
\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) &= \mathrm{tr}[\gamma^\mu\gamma^\nu(2g^{\rho\sigma}\, I_4 - \gamma^\sigma\gamma^\rho)]\,,\\
&= 2g^{\rho\sigma}\,\mathrm{tr}(\gamma^\mu\gamma^\nu) - \mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\rho)\,,\\
&= 8g^{\mu\nu}g^{\rho\sigma} - \mathrm{tr}[\gamma^\mu(2g^{\nu\sigma}\, I_4 - \gamma^\sigma\gamma^\nu)\gamma^\rho]\,,\\
&= 8g^{\mu\nu}g^{\rho\sigma} - 2g^{\nu\sigma}\,\mathrm{tr}(\gamma^\mu\gamma^\rho) + \mathrm{tr}(\gamma^\mu\gamma^\sigma\gamma^\nu\gamma^\rho)\,,\\
&= 8g^{\mu\nu}g^{\rho\sigma} - 8g^{\nu\sigma}g^{\mu\rho} + \mathrm{tr}[2g^{\mu\sigma}\, I_4 - \gamma^\sigma\gamma^\mu)\gamma^\nu\gamma^\rho]\,,\\
&= 8g^{\mu\nu}g^{\rho\sigma} - 8g^{\nu\sigma}g^{\mu\rho} + 2g^{\mu\sigma}\,\mathrm{tr}(\gamma^\nu\gamma^\rho) - \mathrm{tr}(\gamma^\sigma\gamma^\mu\gamma^\nu\gamma^\rho)\,,\\
&= 8g^{\mu\nu}g^{\rho\sigma} - 8g^{\nu\sigma}g^{\mu\rho} + 8g^{\mu\sigma}g^{\nu\rho} - \mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma)\,,
\end{aligned}$$

where in the last line we used the cyclic property of the trace. Then, adding this final trace to the left-hand side, we find

$$\begin{aligned}
2\,\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) &= 8g^{\mu\nu}g^{\rho\sigma} - 8g^{\nu\sigma}g^{\mu\rho} + 8g^{\mu\sigma}g^{\nu\rho}\\
&\implies \mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4\,(g^{\mu\nu}g^{\rho\sigma} - g^{\nu\sigma}g^{\mu\rho} + g^{\mu\sigma}g^{\nu\rho})
\end{aligned}$$

(c) The trace of *any* odd number of gamma matrices is zero.

Solution: We first prove that $\mathrm{tr}(\gamma^\mu) = 0$, which is obvious in the Weyl basis but is true in general. Recall that $(\gamma_5)^2 = \gamma_5\gamma^5 = 4I_4$. Therefore, the trace can be written as

$$\mathrm{tr}(\gamma^\mu) = \mathrm{tr}(\gamma^\mu I_4) = \mathrm{tr}(\gamma^\mu\gamma^5\gamma^5) = -\,\mathrm{tr}(\gamma^5\gamma^\mu\gamma^5) = -\,\mathrm{tr}(\gamma^\mu\gamma^5\gamma^5) = -\,\mathrm{tr}(\gamma^\mu)\,,$$

where in the fourth equality we used the anticommutation relation $\gamma^\mu\gamma^5 = -\gamma^5\gamma^\mu$ and in the fifth equality results from the cyclic properties of the trace. Therefore, we conclude

$$\mathrm{tr}(\gamma^\mu) = 0\,.$$

A generic trace over an odd number of gamma matrices can be written as a trace over $2n+1$ gamma matrices where $n \in \mathbb{N}$, $\mathrm{tr}(\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_{2n}}\gamma^{\mu_{2n+1}})$. So, inserting $I_4 = (\gamma^5)^2$ at the end gives,

$$\begin{aligned}
\mathrm{tr}(\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_{2n}}\gamma^{\mu_{2n+1}}) &= \mathrm{tr}(\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_{2n}}\gamma^{\mu_{2n+1}}I_4)\,,\\
&= \mathrm{tr}(\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_{2n}}\gamma^{\mu_{2n+1}}\gamma^5\gamma^5)\,,\\
&= (-1)^{2n+1}\,\mathrm{tr}(\gamma^5\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_{2n}}\gamma^{\mu_{2n+1}}\gamma^5)\,,\\
&= -\,\mathrm{tr}(\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_{2n}}\gamma^{\mu_{2n+1}}\gamma^5\gamma^5)\,,\\
&= -\,\mathrm{tr}(\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_{2n}}\gamma^{\mu_{2n+1}})\,,
\end{aligned}$$

where the factor $(-1)^{2n+1} = (-1)$ comes from anticommuting γ^5 to the left through all $2n+1$ gamma matrices. We conclude that $\mathrm{tr}(\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_{2n}}\gamma^{\mu_{2n+1}}) = 0$

(d) $\mathrm{tr}\,(\gamma^5) = \mathrm{tr}\,(\gamma^5\gamma^\mu) = \mathrm{tr}\,(\gamma^5\gamma^\mu\gamma^\nu) = \mathrm{tr}\,(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho) = 0,$

Solution: We begin by first proving $\mathrm{tr}(\gamma^5) = 0$. By definition, $\gamma^5\gamma^\mu = -\gamma^\mu\gamma^5$. Now, taking the trace, and inserting the identity in the form $I_4 = (\gamma^0)^2$,

$$\begin{aligned}\mathrm{tr}(\gamma^5) &= \mathrm{tr}(I_4\gamma^5)\,,\\ &= \mathrm{tr}(\gamma^0\gamma^0\gamma^5)\,,\\ &= -\,\mathrm{tr}(\gamma^0\gamma^5\gamma^0)\,,\\ &= -\,\mathrm{tr}(\gamma^0\gamma^0\gamma^5)\,,\\ &= -\,\mathrm{tr}(\gamma^5)\,,\end{aligned}$$

where we anticommuted γ^0 to the right going to line 3, and the used the cyclic property of the trace in line 4. Therefore, we conclude $\mathrm{tr}(\gamma^5) = 0$.

We note that since γ^5 is defined as $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, that $\mathrm{tr}(\gamma^5\gamma^\mu) = \mathrm{tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho) = 0$ since this is the trace of an odd number of gamma matrices.

Therefore, the remaining identity to show is $\mathrm{tr}(\gamma^5\gamma^\mu\gamma^\nu) = 0$. Note that if $\mu = \nu$, then $(\gamma^\mu)^2 = \pm I_4$ where the '+' is for $\mu = 0$, and '−' otherwise. So, if $\mu = \nu$, then $\mathrm{tr}(\gamma^5\gamma^\mu\gamma^\nu) \to \pm\,\mathrm{tr}(\gamma^5 I_4) = 0$ by the first identity proved in this solution. What remains is the case where $\mu \neq \nu$. We insert an identity of the form $I_4 = \pm(\gamma^\rho)^2$, where we are free to choose $\rho \neq \mu$ and $\rho \neq \nu$, so that $\{\gamma^\rho, \gamma^\mu\} = \{\gamma^\rho, \gamma^\nu\} = 0$. Taking the trace,

$$\begin{aligned}\mathrm{tr}(\gamma^5\gamma^\mu\gamma^\nu) &= \mathrm{tr}(I_4\gamma^5\gamma^\mu\gamma^\nu)\,,\\ &= \pm\,\mathrm{tr}(\gamma^\rho\gamma^\rho\gamma^5\gamma^\mu\gamma^\nu)\,,\\ &= (\pm 1)(-1)^3\,\mathrm{tr}(\gamma^\rho\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho)\,,\\ &= (\pm 1)(-1)^3\,\mathrm{tr}(\gamma^\rho\gamma^\rho\gamma^5\gamma^\mu\gamma^\nu)\,,\\ &= (-1)^3\,\mathrm{tr}(\gamma^5\gamma^\mu\gamma^\nu) = -\,\mathrm{tr}(\gamma^5\gamma^\mu\gamma^\nu)\,,\end{aligned}$$

where in the third line we anticommuted γ^ρ to the right three times, and in the fourth used the cyclic property of the trace. We conclude that $\mathrm{tr}(\gamma^5\gamma^\mu\gamma^\nu) = 0$ for all μ, ν.

(e) $\mathrm{tr}\,(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = -4i\epsilon^{\mu\nu\rho\sigma}.$

Solution: Using the anticommutation relation on the last two gamma matrices, we find

$$\begin{aligned}\mathrm{tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) &= \mathrm{tr}[\gamma^5\gamma^\mu\gamma^\nu(2g^{\rho\sigma}\,I_4 - \gamma^\sigma\gamma^\rho)]\,,\\ &= 2g^{\rho\sigma}\,\mathrm{tr}(\gamma^5\gamma^\mu\gamma^\nu) - \mathrm{tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\rho)\,,\\ &= -\,\mathrm{tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\rho)\,,\end{aligned}$$

where we used the result from part d that $\mathrm{tr}(\gamma^5\gamma^\mu\gamma^\nu) = 0$. If $\rho = \sigma$, then $(\gamma^\rho)^2 = \pm I_4$

where the '+' is for $\rho = 0$ and '−' otherwise. Thus, if $\rho = \sigma$, we have $\text{tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) \to \mp\,\text{tr}(\gamma^5\gamma^\mu\gamma^\nu) = 0$. So, we conclude that $\text{tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma)$ is antisymmetric in ρ and σ. We can repeat this argument for any pair of indices, ultimately concluding that $\text{tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma)$ is completely antisymmetric in all μ, ν, ρ, σ indices. In 4D spacetime, the only Lorentz tensor that is completely antisymmetric is the Levi-Civita, therefore we conclude

$$\text{tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = A\epsilon^{\mu\nu\rho\sigma}\,,$$

where A is an undetermined constant and $\epsilon^{\mu\nu\rho\sigma}$ is defined such that $\epsilon^{0123} = +1$.

To determine the constant, we can take any particular combination of Lorentz indices. Let us take $(\mu, \nu, \rho, \sigma) = (0, 1, 2, 3)$, so that

$$\text{tr}(\gamma^5\gamma^0\gamma^1\gamma^2\gamma^3) = A\epsilon^{0123} = A\,.$$

Using the definition $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, we evaluate the trace,

$$\begin{aligned}
\text{tr}(\gamma^5\gamma^0\gamma^1\gamma^2\gamma^3) &= i\,\text{tr}(\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3)\,,\\
&= (-)^3 i\,\text{tr}(\gamma^0\gamma^0\gamma^1\gamma^2\gamma^3\gamma^1\gamma^2\gamma^3)\,,\\
&= (-1)^3(-1)^2 i\,\text{tr}(\gamma^0\gamma^0\gamma^1\gamma^1\gamma^2\gamma^3\gamma^2\gamma^3)\,,\\
&= (-1)^3(-1)^2(-1) i\,\text{tr}(\gamma^0\gamma^0\gamma^1\gamma^1\gamma^2\gamma^2\gamma^3\gamma^3)\,,\\
&= (-1)^3(-1)^2(-1) i\,\text{tr}((+I_4)(-I_4)(-I_4)(-I_4))\,,\\
&= -i\,\text{tr}(I_4)\,,\\
&= -i4\,,
\end{aligned}$$

where in the second, third, and fourth lines we anticommutation relations to arrange identical gamma matrices into pairs, and in the fifth line we used that $(\gamma^0)^2 = +I_4$ while $(\gamma^j)^2 = -I_4$. We conclude that $A = -4i$, so that

$$\text{tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = -4i\epsilon^{\mu\nu\rho\sigma}$$

3. The chiral projectors are defined as

$$P_R = \frac{1}{2}(I_4 + \gamma^5)\,, \qquad P_L = \frac{1}{2}(I_4 - \gamma^5)\,,$$

where I_4 is the 4×4 identity matrix. Prove the following properties:

(a) $\gamma^5 P_L = -P_L$, and $\gamma^5 P_R = P_R$,

Solution: Note that $(\gamma^5)^2 = I_4$, thus,

$$\begin{aligned}\gamma^5 P_{L/R} &= \frac{1}{2}\gamma^5(I_4 \mp \gamma^5)\,, \\ &= \frac{1}{2}(\gamma^5 \mp (\gamma^5)^2)\,, \\ &= \frac{1}{2}(\gamma^5 \mp I_4)\,, \\ &= \mp\frac{1}{2}(I_4 \mp \gamma^5)\,, \\ &= \mp P_{L/R}\,.\end{aligned}$$

(b) $(P_{L/R})^2 = P_{L/R}$,

Solution: Taking the square of the projectors,

$$\begin{aligned}(P_{L/R})^2 &= \left(\frac{1}{2}(I_4 \mp \gamma^5)\right)^2\,, \\ &= \frac{1}{4}(I_4 \mp \gamma^5)(I_4 \mp \gamma^5)\,, \\ &= \frac{1}{4}(I_4 \mp \gamma^5 \mp \gamma^5 + (\gamma^5)^2)\,, \\ &= \frac{1}{2}(I_4 \mp \gamma^5) = P_{L/R}\end{aligned}$$

(c) $P_L P_R = P_R P_L = 0$,

Solution: Taking product

$$\begin{aligned}P_{L/R}P_{R/L} &= \frac{1}{2}(I_4 \mp \gamma^5)\cdot\frac{1}{2}(I_4 \pm \gamma^5)\,, \\ &= \frac{1}{2}(I_4 \mp \gamma^5 \pm \gamma^5 - (\gamma^5)^2)\,, \\ &= \frac{1}{2}(I_4 - I_4) = 0\,.\end{aligned}$$

Therefore, we conclude $P_L P_R = P_R P_L = 0$.

(d) $P_L + P_R = I_4$.

Solution: Taking $P_L + P_R$, we find

$$\begin{aligned} P_L + P_R &= \frac{1}{2}(I_4 - \gamma^5) + \frac{1}{2}(I_4 + \gamma^5)\,, \\ &= \frac{1}{2}\left(2I_4 - \gamma^5 + \gamma^5\right)\,, \\ &= I_4\,. \end{aligned}$$

4. Suppose the charge conjugation operator is defined as $C = i\gamma^2\gamma^0$. Confirm that in the Weyl representation,

   (a) $C^{-1} = C^\top = C^\dagger = -C$.

Solution: Given $C = i\gamma^2\gamma^0$, we first check if the matrix C is unitary, $C^\dagger C = CC^\dagger = I_4$. Note the following useful property,

$$\begin{aligned} (\gamma^\mu)^2 &= \frac{1}{2}\{\gamma^\mu, \gamma^\mu\} \text{ (no sum on } \mu)\,, \\ &= \frac{1}{2}\cdot 2g^{\mu\mu}\, I_4 \text{ (no sum on } \mu)\,, \\ &= g^{\mu\mu}\, I_4 \text{ (no sum on } \mu)\,, \end{aligned}$$

so $(\gamma^0)^2 = I_4$ and $(\gamma^j)^2 = -I_4$.

$$C^\dagger = (i\gamma^2\gamma^0)^\dagger = -i(\gamma^0)^\dagger(\gamma^2)^\dagger = -i\gamma^0(-\gamma^2) = i\gamma^0\gamma^2 = -i\gamma^2\gamma^0 = -C$$

So,

$$C^\dagger C = (-i\gamma^2\gamma^0)(i\gamma^2\gamma^0) = \gamma^2\gamma^0\gamma^2\gamma^0 = -\gamma^2\gamma^0\gamma^0\gamma^2 = -\gamma^2\gamma^2 = +I_4$$

Since C is unitary, we conclude that $C^\dagger = C^{-1}$.

$$C^\top = (C^\dagger)^* = -C^* = -(i\gamma^2\gamma^0)^* = -(-i)(\gamma^2)^*(\gamma^0)^*\,.$$

In the Weyl basis, γ^0 is real, so $(\gamma^0)^* = \gamma^0$, and $(\gamma^2)^*$ is

$$(\gamma^2)^* = \begin{pmatrix} 0 & (\sigma^2)^* \\ -(\sigma^2)^* & 0 \end{pmatrix} = -\begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix} = -\gamma^2\,,$$

where we used the fact that $(\sigma^2)^* = -\sigma^2$ since the non-zero entries of σ^2 are purely imaginary. Therefore,

$$C^\top = -(-i)(\gamma^2)^*(\gamma^0)^* = i(-\gamma^2)\gamma^0 = -i\gamma^2\gamma^0 = -C\,.$$

We thus conclude that C is a real matrix, $C^* = C$, and that

$$C^{-1} = C^\top = C^\dagger = -C\,,$$

as desired.

(b) $C\gamma^\mu C^{-1} = -(\gamma^\mu)^\top$,

Solution: To prove this, first recall that $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$. Since γ^0 is real in the Weyl basis, then we conclude $(\gamma^\mu)^\top = \gamma^0(\gamma^\mu)^*\gamma^0$, where the $*$ denote complex conjugation. Now, with $C = i\gamma^2\gamma^0 = -i\gamma^0\gamma^2$, with $C^{-1} = C^\dagger = -i\gamma^2\gamma^0 = -C$, multiply $(\gamma^\mu)^\top = \gamma^0(\gamma^\mu)^*\gamma^0$ on the left with C^{-1} and on the right with C,

$$\begin{aligned} C^{-1}(\gamma^\mu)^\top C &= C^\dagger \left(\gamma^0(\gamma^\mu)^*\gamma^0\right) C\,, \\ &= (-i\gamma^2\gamma^0)\left(\gamma^0(\gamma^\mu)^*\gamma^0\right)(-i\gamma^0\gamma^2)\,, \\ &= -\gamma^2(\gamma^\mu)^*\gamma^2\,, \end{aligned}$$

where in the last line we used $(\gamma^0)^2 = I_4$. For $\mu = 0, 1, 3$, then $(\gamma^\mu)^* = \gamma^\mu$ since they are real in the Weyl basis. Then, $\gamma^\mu\gamma^2 = -\gamma^2\gamma^\mu$ for $\mu \neq 2$. Since $(\gamma^2)^2 = -I_4$, we find $C^{-1}(\gamma^\mu)^\top C = -\gamma^2(-\gamma^2\gamma^\mu) = (-1)^2(-\gamma^\mu) = -\gamma^\mu$ for $\mu \neq 2$. When $\mu = 2$, then $(\gamma^2)^* = -\gamma^2$. So, $C^{-1}(\gamma^\mu)^\top C = -\gamma^2(-\gamma^2)\gamma^2 = -\gamma^2$. Therefore, we conclude for all μ, $C^{-1}(\gamma^\mu)^\top C = -\gamma^\mu$. Now, multiply on the left by C, and on the right by C^{-1},

$$CC^{-1}(\gamma^\mu)^\top CC^{-1} = -C\gamma^\mu C^{-1}\,, \implies C\gamma^\mu C^{-1} = -(\gamma^\mu)^\top\,,$$

which was to be proved.

(c) $C\gamma^5 C^{-1} = (\gamma^5)^\top$,

Solution: Here, let us take the commutator of C and γ^5,

$$\begin{aligned} [C, \gamma^5] &= C\gamma^5 - \gamma^5 C\,, \\ &= i\gamma^2\gamma^0\gamma^5 - i\gamma^5\gamma^2\gamma^0\,, \\ &= i\gamma^2\gamma^0\gamma^5 - i\gamma^2\gamma^0\gamma^5\,, \\ &= 0 \end{aligned}$$

since $\{\gamma^5, \gamma^\mu\} = 0$. Thus, $C\gamma^5 C^{-1} = \gamma^5$. Note that $(\gamma^5)^\dagger = \gamma^5$, and in the Weyl basis $\gamma^5 = (\gamma)^*$. Thus, we conclude $C\gamma^5 C^{-1} = (\gamma^5)^\top$.

5. A Dirac spinor ψ is called a Majorana spinor if it satisfies the condition $\psi = C\bar{\psi}^\top$, and is called a Weyl spinor if it satisfies either $\psi = P_R\psi$ or $\psi = P_L\psi$. Determine whether or not a spinor can be both Majorana and Weyl.

Solution: Let us define $\psi^c \equiv C\bar{\psi}^\top$, $\psi_L \equiv P_L\psi$, and $\psi_R \equiv P_R\psi$. We take the charge conjugation of a chiral fermion. For example, let us take $(\psi_L)^c = C\bar{\psi}_L^\top$. Since $\bar{\psi} = \psi^\dagger\gamma^0$, we have

$$\begin{aligned}(\psi_L)^c &= C\bar{\psi}_L^\top\,,\\ &= C(\psi_L^\dagger\gamma^0)^\top\,,\\ &= C((P_L\psi)^\dagger\gamma^0)^\top\,,\\ &= C(\psi^\dagger P_L\gamma^0)^\top\,,\\ &= C(\gamma^0)^\top(P_L)^\top\psi^*\,,\end{aligned}$$

where we used that $\psi^\dagger = (\psi^\top)^*$ and $P_L^\dagger = P_L$ since $(\gamma^5)^\dagger = \gamma^5$. Now, we note that

$$\begin{aligned}(P_L)^\top &= \frac{1}{2}(I_4 - \gamma^5)^\top\,,\\ &= \frac{1}{2}(I_4 - (\gamma^5)^\top)\,,\\ &= \frac{1}{2}(I_4 - C^{-1}\gamma^5 C)\,,\\ &= C^{-1}\left[\frac{1}{2}(I_4 - \gamma^5)\right]C\,,\\ &= C^{-1}P_L C\,,\end{aligned}$$

where we used $C^{-1}\gamma^5 C = (\gamma^5)^\top$ and $C^{-1}C = I_4$. So, we have

$$\begin{aligned}(\psi_L)^c &= C(\gamma^0)^\top C^{-1}P_L C\psi^*\,,\\ &= -\gamma^0 P_L C\psi^*\,,\end{aligned}$$

where we used $C(\gamma^0)^\top C^{-1} = -\gamma^0$. Recall that $\gamma^0\gamma^5 = -\gamma^5\gamma^0$, so $\gamma^0 P_L = P_R\gamma^0$. Thus, we have

$$\begin{aligned}(\psi_L)^c &= -\gamma^0 P_L C\psi^*\,,\\ &= -P_R\gamma^0 C\psi^*\,.\end{aligned}$$

Finally, we use again $C^{-1}\gamma^0 C = -(\gamma^0)^\top \implies \gamma^0 C = -C(\gamma^0)^\top$, so that

$$\begin{aligned}(\psi_L)^c &= -P_R\gamma^0 C\psi^*\,,\\ &= P_R C(\gamma^0)^\top\psi^*\,,\\ &= P_R C(\psi^\dagger\gamma^0)^\top\,,\\ &= P_R C\bar{\psi}^\top\,,\\ &= P_R\psi^c\,,\\ &= (\psi^c)_R\,.\end{aligned}$$

We find that $(\psi_L)^c = (\psi^c)_R$. Similar arguments show that $(\psi_R)^c = (\psi^c)_L$. We conclude under charge conjugation the chirality flips for a Weyl fermion. So, if we consider a four-component spinor, $\psi = (\psi_L, \psi_R)^\top$, then charge conjugation flips the chirality, but the spinor is simply a rotated version. The two-component spinors themselves are not eigenstates of both the Majorana and Weyl equation.