WILLIAM & MARY
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PHYS 772 – The Standard Model of Particle Physics

Problem Set 5 – Solution

Due: Tuesday, March 18 at 4:00pm

Term: Spring 2025

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1. Derive the classical equations of motion for spinor electrodynamics given the Lagrange density

$$\mathcal{L} = \frac{1}{2} i\bar{\psi}\not{D}\psi + \text{h.c.} - m\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,,$$

with $D_\mu = \partial_\mu + iqA_\mu$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and the Euler-Lagrange equations

$$\partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\psi)}\right) = \frac{\delta\mathcal{L}}{\delta\psi}\,, \qquad \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\bar{\psi})}\right) = \frac{\delta\mathcal{L}}{\delta\bar{\psi}}\,, \qquad \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu A_\nu)}\right) = \frac{\delta\mathcal{L}}{\delta A_\nu}\,.$$

Solution: Rewriting the Lagrange density as

$$\mathcal{L} = \frac{i}{2}\bar{\psi}\gamma^\mu(\partial_\mu\psi) - \frac{i}{2}(\partial_\mu\bar{\psi})\gamma^\mu\psi - m\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - qA_\mu\bar{\psi}\gamma^\mu\psi\,,$$

we can find the classical equations of motion by direct evaluation. Let us first obtains the equations for the ψ field, which come from the Euler-Lagrange equations as

$$\partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\bar{\psi})}\right) = \partial_\mu\left(-\frac{i}{2}\gamma^\mu\psi\right) = -\frac{i}{2}\gamma^\mu\partial_\mu\psi\,,$$

$$\frac{\delta\mathcal{L}}{\delta\bar{\psi}} = \frac{i}{2}\gamma^\mu(\partial_\mu\psi) - m\psi - qA_\mu\gamma^\mu\psi\,.$$

Combining together, we find the equations

$$-\frac{i}{2}\gamma^\mu\partial_\mu\psi = \frac{i}{2}\gamma^\mu(\partial_\mu\psi) - m\psi - qA_\mu\gamma^\mu\psi\,,$$

$$\implies (i\not{\partial} - m - q\not{A})\psi = 0\,.$$

For the $\bar{\psi}$ field,

$$\partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\psi)}\right) = \partial_\mu\left(\frac{i}{2}\bar{\psi}\gamma^\mu\right) = \frac{i}{2}\partial_\mu\bar{\psi}\gamma^\mu\,,$$

$$\frac{\delta\mathcal{L}}{\delta\psi} = -\frac{i}{2}(\partial_\mu\bar{\psi})\gamma^\mu - m\bar{\psi} - qA_\mu\bar{\psi}\gamma^\mu\,.$$

Combining, we find

$$\frac{i}{2}\partial_\mu\bar{\psi}\gamma^\mu = -\frac{i}{2}(\partial_\mu\bar{\psi})\gamma^\mu - m\bar{\psi} - qA_\mu\bar{\psi}\gamma^\mu\,,$$

$$\implies \bar{\psi}(i\overleftarrow{\not{\partial}} + q\not{A} + m) = 0\,.$$

Finally, for the electromagnetic field, we find

$$\begin{aligned}\partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu A_\nu)}\right) &= -\frac{1}{4}\partial_\mu\left(2F^{\alpha\beta}\frac{\delta F_{\alpha\beta}}{\delta(\partial_\mu A_\nu)}\right)\,,\\ &= -\frac{1}{2}\partial_\mu\left(F^{\alpha\beta}\frac{\delta}{\delta(\partial_\mu A_\nu)}(\partial_\alpha A_\beta - \partial_\beta A_\alpha)\right)\,,\\ &= -\frac{1}{2}\partial_\mu\left(F^{\alpha\beta}\left(\delta^\mu_\alpha\delta^\nu_\beta - \delta^\mu_\beta\delta^\nu_\alpha\right)\right)\,,\\ &= -\frac{1}{2}\partial_\mu\left(F^{\mu\nu} - F^{\nu\mu}\right)\,,\\ &= -\partial_\mu F^{\mu\nu}\,,\end{aligned}$$

where in the fourth line we used $F^{\nu\mu} = -F^{\mu\nu}$. For the potential term,

$$\frac{\delta\mathcal{L}}{\delta A_\nu} = -q\bar{\psi}\gamma^\nu\psi\,,$$

from which we arrive at

$$\partial_\mu F^{\mu\nu} = q\bar{\psi}\gamma^\nu\psi\,.$$

Therefore, the classical equations of motion are

$$(i\partial\!\!\!/ - qA\!\!\!/ - m)\psi = 0\,,\qquad \bar{\psi}(i\overleftarrow{\partial\!\!\!/} + qA\!\!\!/ + m) = 0\,,\qquad \partial_\mu F^{\mu\nu} = q\bar{\psi}\gamma^\mu\psi\,.$$

2. An alternative Lagrange density for the classical free electromagnetic field is

$$\mathcal{L}' = -\frac{1}{2}\partial_\mu A_\nu\,\partial^\mu A^\nu\,.$$

(a) Under what assumption does $\mathcal{L}'$ yield the free inhomogeneous Maxwell equations?

Solution: The free inhomogeneous Maxwell equations are $\partial_\mu F^{\mu\nu} = 0$. Since $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, we have

$$\begin{aligned}0 = \partial_\mu F^{\mu\nu} &= \partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu)\,,\\ &= \partial^\mu\partial_\mu A^\nu - \partial^\nu(\partial_\mu A^\mu)\,,\\ &= \partial^2 A^\nu - \partial^\nu(\partial_\mu A^\mu) = 0\,.\end{aligned}$$

So, the free inhomogeneous Maxwell equations, in terms of A_μ, are $\partial^2 A^\nu - \partial^\nu(\partial_\mu A^\mu) = 0$.

Now, for the Lagrange density $\mathcal{L}' = -\frac{1}{2}\partial_\mu A_\nu\partial^\mu A^\nu$, we can derive the equations of motion through the Euler-Lagrange equations,

$$\partial_\mu \left(\frac{\delta \mathcal{L}'}{\delta(\partial_\mu A_\nu)} \right) = \frac{\delta \mathcal{L}'}{\delta A_\nu}$$

$$\implies \partial_\mu \left[\frac{\delta}{\delta(\partial_\mu A_\nu)} \left(-\frac{1}{2} \partial_\alpha A_\beta \partial^\alpha A^\beta \right) \right] = \frac{\delta}{\delta A_\nu} \left(-\frac{1}{2} \partial_\alpha A_\beta \partial^\alpha A^\beta \right) ,$$

$$-\frac{1}{2} \partial_\mu \left(\frac{\delta(\partial_\alpha A_\beta)}{\delta(\partial_\mu A_\nu)} \partial^\alpha A^\beta + \partial^\alpha A^\beta \frac{\delta(\partial_\alpha A_\beta)}{\delta(\partial_\mu A_\nu)} \right) = 0 ,$$

$$-\partial_\mu \left(\delta^\mu_\alpha \delta^\nu_\beta \, \partial^\alpha A^\beta \right) = 0 ,$$

$$\implies \partial_\mu \partial^\mu A^\nu = 0 .$$

Therefore, we find that the equations of motion are $\partial^2 A^\nu = 0$, which differs from $\partial^2 A^\nu - \partial^\nu(\partial_\mu A^\mu) = 0$ by a four-divergence $\partial^\nu(\partial_\mu A^\mu)$. Therefore, the assumption that $\mathcal{L}'$ yields the free inhomogeneous Maxwell equations is the Lorentz gauge condition, $\partial_\mu A^\mu = 0$.

(b) With this assumption, show that $\mathcal{L}'$ differs from $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ by a four-divergence.

Solution: Starting from the definition,

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} , \\
&= -\frac{1}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) , \\
&= -\frac{1}{2} (\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\nu A_\mu \partial^\mu A^\nu) , \\
&= -\frac{1}{2} \partial_\mu A_\nu \partial^\mu A^\nu + \frac{1}{2} \partial_\mu A_\nu \partial^\nu A^\mu , \\
&= \mathcal{L}' + \frac{1}{2} \partial_\mu A_\nu \partial^\nu A^\mu .
\end{aligned}$$

Note that $\partial_\mu(A_\nu \partial^\nu A^\mu) = \partial_\mu A_\nu \partial^\nu A^\mu + A_\nu \partial_\mu \partial^\nu A^\mu = \partial_\mu A_\nu \partial^\nu A^\mu + A_\nu \partial^\nu \partial_\mu A^\mu$, where in the last equality we used the fact that the derivatives are symmetric on the second term. Moreover, we can rewrite the second term with $\partial_\nu(A_\nu \partial_\mu A^\mu) = (\partial_\mu A^\mu)^2 + A_\nu \partial^\nu \partial_\mu A^\mu$. Relabeling the summed indices on the second term, $\mu \leftrightarrow \nu$, and combining with the first relation we obtain

$$\partial_\mu (A^\nu \partial^\nu A^\mu - A^\mu \partial_\nu A^\nu) = \partial_\mu A_\nu \partial^\nu A^\mu - (\partial_\mu A^\mu)^2 .$$

So, substituting this into $\mathcal{L} = \mathcal{L}' + \frac{1}{2} \partial_\mu A_\nu \partial^\nu A^\mu$, we have

$$\mathcal{L} = \mathcal{L}' + \frac{1}{2} \partial_\mu (A^\nu \partial^\nu A^\mu - A^\mu \partial_\nu A^\nu) + \frac{1}{2} (\partial_\mu A^\mu)^2 .$$

So, $\mathcal{L}$ differs from $\mathcal{L}'$ by a four-divergence so long as we restrict $\mathcal{L}$ to the Lorentz gauge, $\partial_\mu A^\mu = 0$.

3. Consider the Lagrange density for *scalar electrodynamics*,

$$\mathcal{L} = (D_\mu\varphi)^\dagger\,(D^\mu\varphi) - m^2\varphi^\dagger\varphi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - V(\varphi^\dagger\varphi)\,,$$

where m is the mass of the scalar field, $D_\mu = \partial_\mu + iqA_\mu$ where q is the charge of the scalar field, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and $V(\varphi^*\varphi)$ is a U(1) invariant self-interaction term, e.g., $V(\varphi^\dagger\varphi) = \lambda(\varphi^\dagger\varphi)^2$. This theory is invariant under local U(1) gauge transformations. Split the Lagrange density as follows: $\mathcal{L} = \mathcal{L}_{\text{KG}} + \mathcal{L}_{\text{EM}} + \mathcal{L}_{\text{int.}}$, where $\mathcal{L}_{\text{KG}}$ is the usual free complex Klein-Gordon field theory,

$$\mathcal{L}_{\text{KG}} = \partial_\mu\varphi^\dagger\,\partial^\mu\varphi - m^2\varphi^\dagger\varphi\,,$$

and $\mathcal{L}_{\text{EM}}$ is the Lagrange density for the free electromagnetic field,

$$\mathcal{L}_{\text{EM}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,.$$

Determine the interacting Lagrange density $\mathcal{L}_{\text{int.}}$ for scalar electrodynamics.

Solution: The boson's interaction with the electromagnetic field is due to the covariant derivatives,

$$\begin{aligned}
(D_\mu\varphi)^\dagger\,(D^\mu\varphi) &= [(\partial_\mu + iqA_\mu)\varphi]^\dagger\,[(\partial^\mu + iqA^\mu)\varphi]\,,\\
&= (\partial_\mu\varphi^\dagger - iqA_\mu\varphi^\dagger)(\partial^\mu\varphi + iqA^\mu\varphi)\,,\\
&= \partial_\mu\varphi^\dagger\,\partial^\mu\varphi - iqA_\mu\varphi^\dagger\,(\partial^\mu\varphi) + iqA^\mu(\partial_\mu\varphi^\dagger)\varphi + q^2A_\mu A^\mu\,\varphi^\dagger\varphi\,,\\
&= \partial_\mu\varphi^\dagger\,\partial^\mu\varphi - iqA^\mu\left[\varphi^\dagger(\partial_\mu\varphi) - (\partial_\mu\varphi^\dagger)\varphi\right] + q^2A_\mu A^\mu\,\varphi^\dagger\varphi\,.
\end{aligned}$$

So, the interaction Lagrange density is

$$\mathcal{L}_{\text{int.}} = -V(\varphi^\dagger\varphi) - iqA^\mu\left[\varphi^\dagger(\partial_\mu\varphi) - (\partial_\mu\varphi^\dagger)\varphi\right] + q^2A_\mu A^\mu\,\varphi^\dagger\varphi\,.$$

4. Verify that the field strength tensor $F_{\mu\nu}$ can be computed through the commutator $iqF_{\mu\nu} = [D_\mu, D_\nu]$.

Solution: Evaluating the commutator against some test function φ,

$$\begin{aligned}
[D_\mu, D_\nu]\varphi &= [\partial_\mu + iqA_\mu, \partial_\nu + iqA_\nu]\varphi\,,\\
&= [\partial_\mu, \partial_\nu]\varphi + iq[\partial_\mu, A_\nu]\varphi + iq[A_\mu, \partial_\nu]\varphi - q^2[A_\mu, A_\nu]\varphi\,,\\
&= iq[\partial_\mu, A_\nu]\varphi + iq[A_\mu, \partial_\nu]\varphi\,,\\
&= iq\left(\partial_\mu(A_\nu\varphi) - A_\nu\partial_\mu\varphi + A_\mu\partial_\nu\varphi - \partial_\nu(A_\mu\varphi)\right)\,,
\end{aligned}$$

where in going to the third line we used that $\partial_\mu\partial_\nu\varphi = \partial_\nu\partial_\mu\varphi$, and $[A_\mu, A_\nu] = 0$. Now, $\partial_\mu(A_\nu\varphi) =$

$(\partial_\mu A_\nu)\varphi + A_\nu \partial_\mu \varphi$ and $\partial_\nu(A_\mu \varphi) = (\partial_\nu A_\mu)\varphi + A_\mu \partial_\nu \varphi$. So,

$$
\begin{aligned}
\frac{1}{iq}[D_\mu, D_\nu]\varphi &= \partial_\mu(A_\nu\varphi) - A_\nu\partial_\mu\varphi + A_\mu\partial_\nu\varphi - \partial_\nu(A_\mu\varphi)\,, \\
&= (\partial_\mu A_\nu)\varphi + A_\nu\partial_\mu\varphi - A_\nu\partial_\mu\varphi + A_\mu\partial_\nu\varphi - (\partial_\nu A_\mu)\varphi - A_\mu\partial_\nu\varphi\,, \\
&= (\partial_\mu A_\nu - \partial_\nu A_\mu)\varphi\,, \\
&= F_{\mu\nu}\varphi\,,
\end{aligned}
$$

so we conclude that $iqF_{\mu\nu} = [D_\mu, D_\nu]$.

5. Show that the radiative transition, $e^- \to e^- + \gamma$, is forbidden in vacuum.

Solution: Let us defined the following kinematics,

$$e^-(p) \to e^-(p') + \gamma(k)\,,$$

where $p = (E, \mathbf{p})$, $p' = (E', \mathbf{p}')$, and $k = (\omega, \mathbf{k})$ are the four-momenta of the incoming electron, outgoing electron, and outgoing photon, respectively. In vacuum, each of these particles are on their mass-shell, $p^2 = p'^2 = m_e^2$, and $k^2 = 0$. The S matrix element is given by

$$S(e^- \to e^-\gamma) = (2\pi)^4 \delta^{(4)}(p - p' - k)\, i\mathcal{M}(e^- \to e^-\gamma)\,,$$

where the delta function enforces conservation of four-momentum, $p = p' + k$ and $\mathcal{M}$ is the amplitude. The leading order amplitude is non-zero, given by $i\mathcal{M} = -ie\bar{u}(p')\not{\epsilon}u(p) + \mathcal{O}(e^2)$.

Let us examine conservation of four-momentum, which in terms of its components are $E = E' + \omega$ and $\mathbf{p} = \mathbf{p}' + \mathbf{k}$. Let us choose to evaluate the amplitude in the rest frame of the initial electron, so $\mathbf{p} = \mathbf{0}$, and $E = m_e$. Therefore, by conservation of energy and momentum, we have $m_e = E' + \omega$ and $\mathbf{p}' = -\mathbf{k}$, respectively. Since the particles are on-shell, we further have $E' = \sqrt{m_e^2 + \mathbf{p}'^2}$ and $\omega = |\mathbf{k}|$. Combining these results, conservation of energy imposes the condition

$$m_e = \sqrt{m_e^2 + \mathbf{k}^2} + |\mathbf{k}|\,,$$

This condition is only true if $\mathbf{k} = \mathbf{0}$, that is there is no photon emitted. We conclude that conservation of momentum forbids this reaction, giving $S(e^- \to e^-\gamma) = 0$.

6. Consider the pair production of pions in electron-positron annihilation, $e^-e^+ \to \pi^-\pi^+$. Assume the reaction occurs at a center-of-momentum (CM) energy $\sqrt{s} \gg m_e$, but is comparable to the mass of the produced pions, $\sqrt{s} \sim m_\pi$. For simplicity, describe the charged pions by quantum scalar electrodynamics (for the Feynman rules, see the notes on `Feynman Rules - SQED`).

(a) Show that the unpolarized differential cross-section to leading order in α is given by

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{8}\frac{\alpha^2\beta_\pi^3}{s}(1 - \cos^2\theta) + \mathcal{O}(\alpha^3)\,,$$

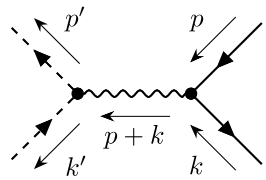
where θ is the CM frame scattering angle and β_π is the speed of the pion (recall that $|\mathbf{p}_\pi| = E_\pi\beta_\pi$).

Solution: Let us consider the reaction with the following kinematics

$$e^-(p,s) + e^+(k,r) \to \pi^-(p') + \pi^+(k')\,,$$

where in the CM frame $p = (E_e, \mathbf{p})$, $k = (E_e, -\mathbf{p})$, $p' = (E_\pi, \mathbf{p}')$, and $k' = (E_\pi, -\mathbf{p}')$. For the electron we have $E_e = \sqrt{s}/2$ and $|\mathbf{p}| = \sqrt{s - 4m_e^2}/2$, while for the pions $E_\pi = \sqrt{s}/2$ and $|\mathbf{p}'| = \sqrt{s - 4m_\pi^2}/2$. The scattering amplitude at leading order is given by

$$i\mathcal{M} =$$



$$+\,\mathcal{O}(\alpha^2)\,,$$

$$
\begin{aligned}
&= (-iq)(k'-p')^\mu \,\frac{-ig_{\mu\nu}}{(p+k)^2}\,\bar{v}_r(k)(-iq\gamma^\nu)u_s(p) + \mathcal{O}(\alpha^2)\,,\\
&= -i\frac{4\pi\alpha}{s}\,\bar{v}_r(k)(\not{k}' - \not{p}')u_s(p) + \mathcal{O}(\alpha^2)\,,\\
&= -i\frac{4\pi\alpha}{s}\,\bar{v}_r(k)(\not{k}' + \not{p}' - 2\not{p}')u_s(p) + \mathcal{O}(\alpha^2)\,,\\
&= -i\frac{4\pi\alpha}{s}\,\bar{v}_r(k)(\not{k} + \not{p} - 2\not{p}')u_s(p) + \mathcal{O}(\alpha^2)\,,\\
&= i\frac{8\pi\alpha}{s}\,\bar{v}_r(k)\not{p}'u_s(p) + \mathcal{O}(\alpha^2)\,,
\end{aligned}
$$

where in the third line we added zero in the form of $0 = \not{p}' - \not{p}'$, then in the fourth line used $k' + p' = k + p$, and finally in the fifth line used the on-shell Dirac conditions $(\not{p} - m_e)u_s(p) = 0$ and $\bar{v}_r(k)(\not{p} + m_e) = 0$, which subsequently gives $\bar{v}_r(k)(\not{k} + \not{p})u_s(p) = 0$.

The spin-averaged amplitude is then

$$\begin{aligned}
\langle|\mathcal{M}|^2\rangle &= \frac{1}{4}\sum_{s,r}|\mathcal{M}(e^-_s e^+_r \to \pi^-\pi^+)|^2\,,\\
&= \frac{1}{4}\left(\frac{8\pi\alpha}{s}\right)^2\sum_{s,r}(\bar{v}_r(k)\not{p}' u_s(p))^*(\bar{v}_r(k)\not{p}' u_s(p)) + \mathcal{O}(\alpha^3)\,,\\
&= \frac{1}{4}\left(\frac{8\pi\alpha}{s}\right)^2\sum_{s,r}\mathrm{tr}\left[\bar{u}_s(p)\not{p}' v_r(k)\bar{v}_r(k)\not{p}' u_s(p)\right] + \mathcal{O}(\alpha^3)\,,\\
&= \frac{1}{4}\left(\frac{8\pi\alpha}{s}\right)^2\mathrm{tr}\left[\not{p}'(\not{k} - m_e)\not{p}'(\not{p} + m_e)\right] + \mathcal{O}(\alpha^3)\,,\\
&= \frac{1}{4}\left(\frac{8\pi\alpha}{s}\right)^2\left\{\mathrm{tr}\left[\not{p}'\not{k}\not{p}'\not{p}\right] - m_e^2\,\mathrm{tr}\left[\not{p}'\not{p}'\right]\right\} + \mathcal{O}(\alpha^3)\,,\\
&= \left(\frac{8\pi\alpha}{s}\right)^2\left[2(p'\cdot k)\,(p'\cdot p) - (p'\cdot p')\,(k\cdot p) - m_e^2\,(p'\cdot p')\right] + \mathcal{O}(\alpha^3)\,,\\
&= \frac{1}{2}\left(\frac{8\pi\alpha}{s}\right)^2\left[4(p'\cdot k)\,(p'\cdot p) - m_\pi^2 s\right] + \mathcal{O}(\alpha^3)\,,
\end{aligned}$$

where in going to the last line we used $s = 2m_e^2 + 2k\cdot p$ and $p'^2 = m_\pi^2$. Now, we evaluate the remaining scalar products as $(p'\cdot k)\,(p'\cdot p) = (E_\pi E_e + \mathbf{p}'\cdot\mathbf{p})(E_\pi E_e - \mathbf{p}'\cdot\mathbf{p}) = (E_\pi E_e)^2 - (\mathbf{p}'\cdot\mathbf{p})^2$, where we used $\mathbf{p} = -\mathbf{k}$ in the CM frame. Since $\mathbf{p}'\cdot\mathbf{p} = |\mathbf{p}'||\mathbf{p}|\cos\theta$, and $\beta_\pi = |\mathbf{p}'|/E_\pi$ and $\beta_e = |\mathbf{p}|/E_e$, then $(p'\cdot k)\,(p'\cdot p) = (E_\pi E_e)^2(1 - (\beta_\pi\beta_e)^2\cos^2\theta)$. Furthermore, $E_\pi = E_e = \sqrt{s}/2$, so $(E_\pi E_e)^2 = (s/4)^2$ So, the spin-averaged matrix element is

$$\begin{aligned}
\langle|\mathcal{M}|^2\rangle &= \frac{1}{2}\left(\frac{8\pi\alpha}{s}\right)^2\frac{s^2}{4}\left[\left(1 - \frac{4m_\pi^2}{s}\right) - (\beta_\pi\beta_e)^2\cos^2\theta\right] + \mathcal{O}(\alpha^3)\,,\\
&= 8\pi^2\alpha^2\beta_\pi^2\left[1 - \beta_e^2\cos^2\theta\right] + \mathcal{O}(\alpha^3)\,,
\end{aligned}$$

where $\beta_\pi = |\mathbf{p}'|/E_\pi = \sqrt{1 - 4m_\pi^2/s}$. The differential cross-section is

$$\begin{aligned}
\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} &= \frac{1}{64\pi^2 s}\frac{|\mathbf{p}'|}{|\mathbf{p}|}\langle|\mathcal{M}|^2\rangle\,,\\
&= \frac{1}{64\pi^2 s}\sqrt{\frac{1 - 4m_\pi^2/s}{1 - 4m_e^2/s}}\,8\pi^2\alpha^2\beta_\pi^2\left[1 - \beta_e^2\cos^2\theta\right] + \mathcal{O}(\alpha^3)\,,\\
&= \frac{\alpha^2}{8s}\frac{\beta_\pi^3}{\beta_e}\left(1 - \beta_e^2\cos^2\theta\right) + \mathcal{O}(\alpha^3)\,.
\end{aligned}$$

For energies near $\sqrt{s} \sim 2m_\pi$, $\beta_e = 1 + \mathcal{O}(m_e^2/s)$, so

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{8}\frac{\alpha^2\beta_\pi^3}{s}\,(1 - \cos^2\theta) + \mathcal{O}(\alpha^3, m_e^2/s)\,,$$

(**b**) Compute the total cross-section, and compute ratio, $\sigma(e^-e^+ \to \pi^-\pi^+)/\sigma(e^-e^+ \to \mu^-\mu^+)$ where $\sigma(e^-e^+ \to \mu^-\mu^+) = 4\pi\alpha^2/3s$. Compute the theoretical value at $\sqrt{s} = 0.40\,\text{GeV}$ and $.77\,\text{GeV}$, and compare to the experimental R ratio, $R(\sqrt{s} = 0.40\,\text{GeV}) = 0.18 \pm 0.02$ and $R(\sqrt{s} = 0.77\,\text{GeV}) = 9.99 \pm 0.09$. Comment on the comparison. **Hint:** Examining the plots of the R ratio may be helpful, see Fig. 53.2 of https://pdg.lbl.gov/2022/reviews/rpp2022-rev-cross-section-plots.pdf.

Solution: The cross-section is

$$\sigma = \frac{\pi}{3}\frac{\alpha^2\beta_\pi^3}{s} + \mathcal{O}(\alpha^3, m_e^2/s)\,,$$

and the R ratio for this process is defined as

$$\begin{aligned} R &= \frac{\sigma(e^-e^+ \to \pi^-\pi^+)}{\sigma(e^-e^+ \to \mu^-\mu^+)}\,, \\ &= \frac{\pi}{3}\frac{\alpha^2\beta_\pi^3}{s}\cdot\frac{3s}{4\pi\alpha^2}\,, \\ &= \frac{\beta_\pi^3}{4} = \frac{1}{4}\left(1 - \frac{4m_\pi^2}{s}\right)^{3/2} \xrightarrow[s\to\infty]{} \frac{1}{4} \end{aligned}$$

At $\sqrt{s} = 0.4\,\text{GeV}$, the R ratio is $R = 0.18 \pm 0.02$, which our theoretical result, $R_{\text{th.}} \approx 0.091$, which is within a factor of 2 of the experimental result. The significance of the deviation is 4.5σ. At $\sqrt{s} = 0.770\,\text{GeV}$, the experimental value is $R = 9.99 \pm 0.09$, but the theoretical value is $R_{\text{th.}} \approx 0.202$, which is a large discrepancy of nearly a factor of 50, which corresponds to a $\sim 100\sigma$ deviation. So, the low-energy region is in qualitative agreement, while the larger energy region categorically disagrees with experiment. Examining the plot of the R ratio, there is a large resonance around $\sqrt{s} \sim 0.770\,\text{GeV}$. This is the isovector $J^{PC} = 1^{--}$ ρ^0 resonance with a mass $m_\rho \approx 0.770\,\text{GeV}$, which gives a large dynamical enhancement in the $e^-e^+ \to \pi^-\pi^+$ cross-section. Strongly interacting resonances physics must be captured non-perturbatively, as divergences in amplitudes can only be found by summing the entire series, and are not found at any given order. So, perturbation theory (even more sophisticated theories like chiral effective theory) will always fail to capture the structure of the cross-sections at energies away from threshold.

7. Consider lepton pair production in electron-positron annihilation within QED, $e^-e^+ \to \ell^-\ell^+$, where $\ell = \mu$ or τ. Assume the reaction occurs at a center-of-momentum (CM) energy $\sqrt{s} \gg m_e$, but is comparable to the mass of the produced leptons, $\sqrt{s} \sim m_\ell$.

(**a**) Show that the unpolarized differential cross-section $\mathrm{d}\sigma/\mathrm{d}\Omega$, to order α^2, is given by

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2}{4s}\,\beta_\ell\left[1 + \cos^2\theta + (1 - \beta_\ell^2)\sin^2\theta\right] + \mathcal{O}(\alpha^3)\,,$$

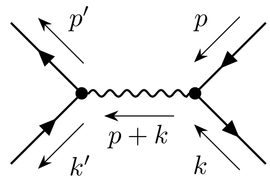
where β_ℓ is the speed of the produced lepton in the CM frame, and θ is the scattering angle.

Solution: Let us consider the reaction with the following kinematics

$$e^-(p,s) + e^+(k,r) \to \ell^-(p',s') + \ell^+(k',r')\,,$$

where in the CM frame $p = (E_e, \mathbf{p})$, $k = (E_e, -\mathbf{p})$, $p' = (E_\ell, \mathbf{p}')$, and $k' = (E_\ell, -\mathbf{p}')$. Since the electron mass is negligible, we have $E_e = |\mathbf{p}| = \sqrt{s}/2$, while for the leptons $E_\ell = \sqrt{s}/2$ and $|\mathbf{p}'| = \sqrt{s - 4m_\ell^2}/2$. The scattering amplitude at leading order is given by

$$i\mathcal{M} =$$



$$+ \mathcal{O}(\alpha^2)\,,$$

$$= \bar{u}_{s'}(p')(-iq\gamma^\mu)v_{r'}(k')\,\frac{-ig_{\mu\nu}}{(p+k)^2}\,\bar{v}_r(k)(-iq\gamma^\nu)u_s(p) + \mathcal{O}(\alpha^2)\,,$$

$$= -i\frac{4\pi\alpha}{s}\,[\bar{u}_{s'}(p')\gamma^\mu v_{r'}(k')]\,[\bar{v}_r(k)\gamma_\mu u_s(p)] + \mathcal{O}(\alpha^2)\,.$$

For the unpolarized differential cross section, we require the spin-averaged matrix element,

$$\begin{aligned}
\langle|\mathcal{M}|^2\rangle &\equiv \frac{1}{4}\sum_{s,r}\sum_{s',r'}\mathcal{M}^\dagger\mathcal{M}\,,\\
&= \frac{1}{4}\left(\frac{4\pi\alpha}{s}\right)^2\sum_{s,r}[\bar{v}_r(k)\gamma^\mu u_s(p)]^\dagger\,[\bar{v}_r(k)\gamma^\nu u_s(p)]\\
&\qquad\times\sum_{s',r'}[\bar{u}_{s'}(p')\gamma_\mu v_{r'}(k')]^\dagger\,[\bar{u}_{s'}(p')\gamma_\nu v_{r'}(k')] + \mathcal{O}(\alpha^3)\,,\\
&= \frac{1}{4}\left(\frac{4\pi\alpha}{s}\right)^2\sum_{s,r}\mathrm{tr}\,[\bar{u}_s(p)\gamma^\mu v_r(k)\bar{v}_r(k)\gamma^\nu u_s(p)]\\
&\qquad\times\sum_{s',r'}\mathrm{tr}\,[\bar{v}_{r'}(k')\gamma_\mu\bar{u}_{s'}(p')\bar{u}_{s'}(p')\gamma_\nu v_{r'}(k')] + \mathcal{O}(\alpha^3)\,,\\
&= \frac{1}{4}\left(\frac{4\pi\alpha}{s}\right)^2\mathrm{tr}\,[\not{p}\gamma^\mu\not{k}\gamma^\nu]\,\mathrm{tr}\left[(\not{k}'-m_\ell)\gamma_\mu(\not{p}'+m_\ell)\gamma_\nu\right] + \mathcal{O}(\alpha^3)\,,
\end{aligned}$$

where we used $\sum_s u_s(p)\bar{u}_s(p) = \sum_r v_s(p)\bar{v}_s(p) = \not{p}$ for the electron and positron, and $\sum_s u_s(p')\bar{u}_s(p') = \not{p}' + m_\ell$ and $\sum_s v_s(p')\bar{v}_s(p') = \not{p}' - m_\ell$ for the lepton and anti-lepton, respectively. Now, $\mathrm{tr}\,[\not{p}\gamma^\mu\not{k}\gamma^\nu] = 4(p^\mu k^\nu + p^\nu k^\mu - g^{\mu\nu}p\cdot k) = 4(p^\mu k^\nu + p^\nu k^\mu - g^{\mu\nu}s/2)$ where we used $s = (p+k)^2 = 2p\cdot k$ since $m_e^2/s \to 0$ in the high-energy limit. Also,

$$\begin{aligned}
\mathrm{tr}\left[(\not{k}'-m_\ell)\gamma_\mu(\not{p}'+m_\ell)\gamma_\nu\right] &= \mathrm{tr}\left[\not{k}'\gamma_\mu\not{p}'\gamma_\nu\right] - m_\ell^2\,\mathrm{tr}\,[\gamma_\mu\gamma_\nu]\,,\\
&= 4[k'_\mu p'_\nu + k'_\nu p'_\mu - g_{\mu\nu}(p'\cdot k' + m_\ell^2)]\,,\\
&= 4[k'_\mu p'_\nu + k'_\nu p'_\mu - g_{\mu\nu}s/2]\,,
\end{aligned}$$

where we used $s = (p'+k')^2 = 2m_\ell^2 + 2p'\cdot k'$.

Contracting the traces, we find

$$\begin{aligned}\operatorname{tr}\left[\not{p}\gamma^\mu\not{k}\gamma^\nu\right]\operatorname{tr}\left[(\not{k}'-m_\ell)\gamma_\mu(\not{p}'+m_\ell)\gamma_\nu\right]\\ &= 16(p^\mu k^\nu + p^\nu k^\mu - g^{\mu\nu}s/2)(k'_\mu p'_\nu + k'_\nu p'_\mu - g_{\mu\nu}s/2)\,,\\ &= 16\left(2p\cdot k'\,k\cdot p' + 2p\cdot p'\,k\cdot k' - s\,(p\cdot k + p'\cdot k') + s^2\right)\,,\\ &= 32\left((p\cdot k')^2 + (p\cdot p')^2 + sm_\ell^2/2\right)\,,\end{aligned}$$

where we used the following relations for the Mandelstam variables, $s = (p+k)^2 = 2p\cdot k = (p'+k')^2 = 2m_\ell^2 + 2p'\cdot k'$, $t = (p-p')^2 = m_\ell^2 - 2p'\cdot p = (k-k')^2 = m_\ell^2 - 2k\cdot k'$, $u = (p-k')^2 = m_\ell^2 - 2p\cdot k' = (k-p')^2 = m_\ell^2 - 2k\cdot p'$. Therefore, the spin-averaged amplitude is

$$\begin{aligned}\langle|\mathcal{M}|^2\rangle &= 8\left(\frac{4\pi\alpha}{s}\right)^2\left((p\cdot k')^2 + (p\cdot p')^2 + \frac{sm_\ell^2}{2}\right) + \mathcal{O}(\alpha^3)\,,\\ &= 8\left(\frac{4\pi\alpha}{s}\right)^2\left(E_e^2E_\ell^2(1+\beta_\ell\cos\theta)^2 + E_e^2E_\ell^2(1-\beta_\ell\cos\theta)^2 + \frac{sm_\ell^2}{2}\right) + \mathcal{O}(\alpha^3)\,,\\ &= (4\pi\alpha)^2\left(1 + \beta_\ell^2\cos^2\theta + \frac{4m_\ell^2}{s}\right) + \mathcal{O}(\alpha^3)\end{aligned}$$

where in the second line we used $(p\cdot k') = E_eE_\ell + |\mathbf{p}||\mathbf{p}'|\cos\theta = E_eE_\ell(1+\beta_\ell\cos\theta)$ and $(p\cdot p') = E_eE_\ell - |\mathbf{p}||\mathbf{p}'|\cos\theta = E_eE_\ell(1-\beta_\ell\cos\theta)$, with $E_e = |\mathbf{p}|$ and $|\mathbf{p}'| = E_\ell\beta_\ell$. Further in the third line, we used $E_\ell = E_e = \sqrt{s}/2$. Recall that $\beta_\ell = |\mathbf{p}'|/E_\ell = 1 - 4m_\ell^2/s$, so we simplify the spin-averaged amplitude as

$$\begin{aligned}\langle|\mathcal{M}|^2\rangle &= (4\pi\alpha)^2\left(1 + \beta_\ell^2\cos^2\theta + 1 - \beta_\ell^2\right) + \mathcal{O}(\alpha^3)\,,\\ &= (4\pi\alpha)^2(\cos^2\theta + \sin^2\theta + \beta_\ell^2(1-\sin^2\theta) + 1 - \beta_\ell^2) + \mathcal{O}(\alpha^3)\,,\\ &= (4\pi\alpha)^2(1 + \cos^2\theta + (1-\beta_\ell^2)\sin^2\theta) + \mathcal{O}(\alpha^3)\,.\end{aligned}$$

Finally, the unpolarized differential cross section is

$$\begin{aligned}\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} &= \frac{1}{64\pi^2 s}\frac{|\mathbf{p}'|}{|\mathbf{p}|}\langle|\mathcal{M}|^2\rangle\,,\\ &= \frac{1}{64\pi^2 s}\beta_\ell(4\pi\alpha)^2[1 + \cos^2\theta + (1-\beta_\ell^2)\sin^2\theta] + \mathcal{O}(\alpha^3)\,,\\ &= \frac{\alpha^2}{4s}\beta_\ell\,[1 + \cos^2\theta + (1-\beta_\ell^2)\sin^2\theta] + \mathcal{O}(\alpha^3)\,.\end{aligned}$$

(b) Show that the total $e^-e^+ \to \ell^-\ell^+$ cross-section at leading order is

$$\sigma = \frac{4\pi\alpha^2}{3s}\sqrt{1 - \frac{4m_\ell^2}{s}}\left(1 + \frac{2m_\ell^2}{s}\right) + \mathcal{O}(\alpha^3)\,.$$

Solution: Directly integrating the unpolarized differential cross-section,

$$\sigma = \int \mathrm{d}\Omega \, \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = 2\pi \int_{-1}^{1} \mathrm{d}\cos\theta \, \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \,,$$

where we integrated over φ since the system has an azimuthal symmetry. Therefore, we have

$$\begin{aligned}
\sigma &= \frac{\pi\alpha^2}{2s}\beta_\ell \int_{-1}^{1} \mathrm{d}\cos\theta \, \left[1 + \cos^2\theta + (1-\beta_\ell^2)\sin^2\theta\right] + \mathcal{O}(\alpha^3)\,, \\
&= \frac{\pi\alpha^2}{2s}\beta_\ell \left[\int_{-1}^{1} \mathrm{d}\cos\theta \, (1+\cos^2\theta) + (1-\beta_\ell^2)\int_{-1}^{1} \mathrm{d}\cos\theta \, (1-\cos^2\theta)\right] + \mathcal{O}(\alpha^3)\,, \\
&= \frac{\pi\alpha^2}{2s}\beta_\ell \left[\left(\cos\theta + \frac{\cos^3\theta}{3}\right)_{-1}^{1} + (1-\beta_\ell^2)\left(\cos\theta - \frac{\cos^3\theta}{3}\right)_{-1}^{1}\right] + \mathcal{O}(\alpha^3)\,, \\
&= \frac{\pi\alpha^2}{2s}\beta_\ell \left[\left(2 + \frac{2}{3}\right) + (1-\beta_\ell^2)\left(2 - \frac{2}{3}\right)\right] + \mathcal{O}(\alpha^3)\,, \\
&= \frac{4\pi\alpha^2}{3s}\beta_\ell \left[1 + \frac{1}{2}(1-\beta_\ell^2)\right] + \mathcal{O}(\alpha^3)\,.
\end{aligned}$$

Since $\beta_\ell^2 = 1 - 4m_\ell^2/s$, we find $1 - \beta_\ell^2 = +4m_\ell^2/s$. So, we conclude

$$\sigma = \frac{4\pi\alpha^2}{3s}\sqrt{1 - \frac{4m_\ell^2}{s}}\left(1 + \frac{2m_\ell^2}{s}\right) + \mathcal{O}(\alpha^3)\,.$$

(c) Take the ultrarelativistic limit of the results of **(a)** and **(b)**, that is $m_\ell^2/s \to 0$, to recover the following high-energy results,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2}{4s}\,(1+\cos^2\theta) + \mathcal{O}(\alpha^3)\,, \quad \text{and} \quad \sigma = \frac{4\pi\alpha^2}{3s} + \mathcal{O}(\alpha^3)\,.$$

Solution: Trivially, as $m_\ell^2/s \to \infty$, then $\beta_\ell = 1 + \mathcal{O}(m_\ell^2/s)$. So, for the differential cross-section,

$$\begin{aligned}
\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} &= \frac{\alpha^2}{4s}\,\beta_\ell\left[1 + \cos^2\theta + (1-\beta_\ell^2)\sin^2\theta\right] + \mathcal{O}(\alpha^3)\,, \\
&= \frac{\alpha^2}{4s}\left(1+\cos^2\theta\right) + \mathcal{O}(\alpha^3, m_\ell^2/s)\,,
\end{aligned}$$

while for the total cross-section

$$\begin{aligned}
\sigma &= \frac{4\pi\alpha^2}{3s}\sqrt{1 - \frac{4m_\ell^2}{s}}\left(1 + \frac{2m_\ell^2}{s}\right) + \mathcal{O}(\alpha^3)\,, \\
&= \frac{4\pi\alpha^2}{3s} + \mathcal{O}(\alpha^3, m_\ell^2/s)\,.
\end{aligned}$$

(d) Plot the $\mathcal{O}(\alpha^2)$ theoretical $s \cdot \mathrm{d}\sigma/\mathrm{d}\Omega$ vs. $\cos\theta \in [-1, 1]$ at a CM energy $\sqrt{s} = 35\,\mathrm{GeV}$ for both $e^-e^+ \to \mu^-\mu^+$ and $e^-e^+ \to \tau^-\tau^+$ (make a separate plot for each reaction). Plot the y-axis in $\mathrm{nb} \cdot \mathrm{GeV}^2$, restricted to $(s \cdot \mathrm{d}\sigma/\mathrm{d}\Omega)/(\mathrm{nb} \cdot \mathrm{GeV}^2) \in [0.0, 12.0]$. Plot the experimental data for each reaction, measured from the JADE experiment at PETRA, over the theoretical curves. Compare and comment on the quality of the theoretical description of the experimental data. **Note:** The data file presents the cross-section as $s \cdot \mathrm{d}\sigma/\mathrm{d}\Omega$. The data files were obtained from the article by the JADE collaboration, https://link.springer.com/article/10.1007/BF01560255.

Solution: Plotting the cross-sections, remembering to multiply by $(\hbar c)^2 = 1$ and $1\,\mathrm{b}/100\mathrm{fm}^2 = 1$, we find the following results in 1. Noticeably, there is a discrepancy in between the theoretical and experimental values. The theoretical cross-section is symmetric in $\cos\theta$, however the data is clearly asymmetric. In the next part, we examine the ratio of the experimental value to QED theory to show more clearly how large the asymmetry is.

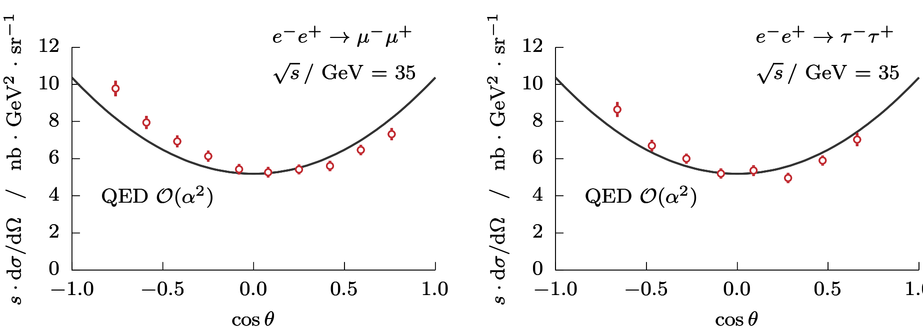


Figure 1: Plots of $s \cdot \mathrm{d}\sigma/\mathrm{d}\Omega$ vs. $\cos\theta$ for $\sqrt{s} = 35\,\mathrm{GeV}$ compared with the JADE data for the reactions $e^-e^+ \to \mu^-\mu^+$ (left) and $e^-e^+ \to \tau^-\tau^+$ (right).

(e) Make a plot of the ratio of the experimentally measured differential cross-section to the leading order QED prediction for each reaction as a function of $\cos\theta \in [-1, .1]$ for the CM energy $\sqrt{s} = 35\,\mathrm{GeV}$. Restrict the y axis between 0.5 and 1.5. Compare and comment on the quality of the theoretical description of the experimental data.

Solution: Figure 2 shows the plots of the ratios of the experimental differential cross-section for both $e^-e^+ \to \mu^-\mu^+$ and $e^-e^+ \to \tau^-\tau^+$. The asymmetry is clearly visible, which has a max of about $\sim 4\sigma$ discrepancy near $|\cos\theta| = 1$.

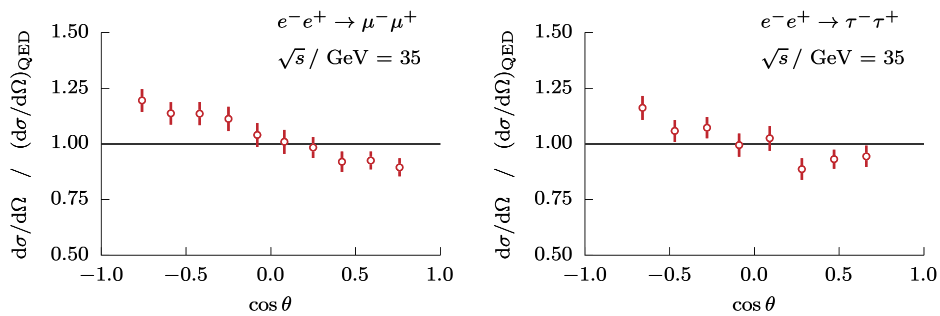


Figure 2: Plots of ratio of JADE experimental $s \cdot \mathrm{d}\sigma/\mathrm{d}\Omega$ to the QED theory result at $\mathcal{O}(\alpha^2)$ vs. $\cos\theta$ at $\sqrt{s} = 35\,\mathrm{GeV}$ for the reactions $e^-e^+ \to \mu^-\mu^+$ (left) and $e^-e^+ \to \tau^-\tau^+$ (right).

This clearly indicates that the leading QED form is not sufficient to describe the data. Since $\alpha \ll 1$, we do not naively expect that radiative corrections will drastically change the result. Further, since QED is parity invariant, and this is a direct annihilation reaction (meaning no t-channel contributions), this asymmetry in $\cos\theta$ is an indication that there is some parity violating contribution coming from some other interactions (here it is the weak interactions). We can gain some understanding of the magnitudes of other virtual particle contributions by adding an additional term, linear in $\cos\theta$, to the differential cross-section,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2}{4s}\left(1 + \cos^2\theta + \frac{8}{3}A\cos\theta\right) + \mathcal{O}(\alpha^2)\,,$$

where A is the *forward-backward asymmetry*. In general A can depend on the energy, $A = A(s)$. The factor of 8/3 arises from the general definition

$$A = \frac{\displaystyle\int_0^1 \mathrm{d}\cos\theta\,\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} - \int_{-1}^0 \mathrm{d}\cos\theta\,\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}}{\displaystyle\int_{-1}^1 \mathrm{d}\cos\theta\,\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}}\,.$$

The linear term vanishes when integrating over the entire domain of θ, but is the only contribution to the different of the forward and backward distributions.

A simple fit with this new model against the JADE data for $e^-e^+ \to \mu^-\mu^+$ (with $n_{\mathrm{d.o.f.}} = 10 - 1 = 9$ gives $A(\sqrt{s} = 35\,\mathrm{GeV}) = -0.11 \pm 0.01$ with a $\chi^2/n_{\mathrm{d.o.f.}} = 0.07$. See the fit output below.

```
  iter      chisq       delta/lim  lambda   A
0 5.5031142187e+02   0.00e+00  6.69e+00    1.000000e+00
1 5.2027323982e+00  -1.05e+07  6.69e-01   -6.970939e-03
2 6.6016451945e-01  -6.88e+05  6.69e-02   -1.075674e-01
3 6.6015998595e-01  -6.87e-01  6.69e-03   -1.076680e-01
```

```
iter      chisq       delta/lim  lambda   A

After 3 iterations the fit converged.
final sum of squares of residuals : 0.66016
rel. change during last iteration : -6.86728e-06

degrees of freedom    (FIT_NDF)                        : 9
rms of residuals      (FIT_STDFIT) = sqrt(WSSR/ndf)    : 0.270834
variance of residuals (reduced chisquare) = WSSR/ndf   : 0.0733511

Final set of parameters            Asymptotic Standard Error
=======================            ==========================
A               = -0.107668        +/- 0.0128       (11.88%)
```

Similarly, for $e^-e^+ \to \tau^-\tau^+$ ($n_{\mathrm{d.o.f.}} = 8 - 1 = 7$), we find $A = -0.08 \pm 0.02$ with a $\chi^2/n_{\mathrm{d.o.f.}} = 0.07$. See the fit output below.

```
     iter      chisq       delta/lim  lambda   A
   0 3.3441186905e+02   0.00e+00  5.96e+00    1.000000e+00
   1 4.6561732670e+00  -7.08e+06  5.96e-01    3.663427e-02
   2 5.3423349422e-01  -7.72e+05  5.96e-02   -8.363611e-02
   3 5.3422706976e-01  -1.20e+00  5.96e-03   -8.378645e-02
   4 5.3422706976e-01  -2.08e-10  5.96e-04   -8.378645e-02
iter      chisq       delta/lim  lambda   A

After 4 iterations the fit converged.
final sum of squares of residuals : 0.534227
rel. change during last iteration : -2.07819e-15

degrees of freedom    (FIT_NDF)                        : 7
rms of residuals      (FIT_STDFIT) = sqrt(WSSR/ndf)    : 0.276257
variance of residuals (reduced chisquare) = WSSR/ndf   : 0.0763182

Final set of parameters            Asymptotic Standard Error
=======================            ==========================
A               = -0.0837865       +/- 0.01639      (19.56%)
```

The two asymmetries are consistent with each other, indicating that the dominant contribution is independent of the lepton mass. Figure 3 shows the QED result, JADE data, and the fit result. Later in the course, we will see that the cause of this asymmetry is due to annihilation into a Z^0 boson.

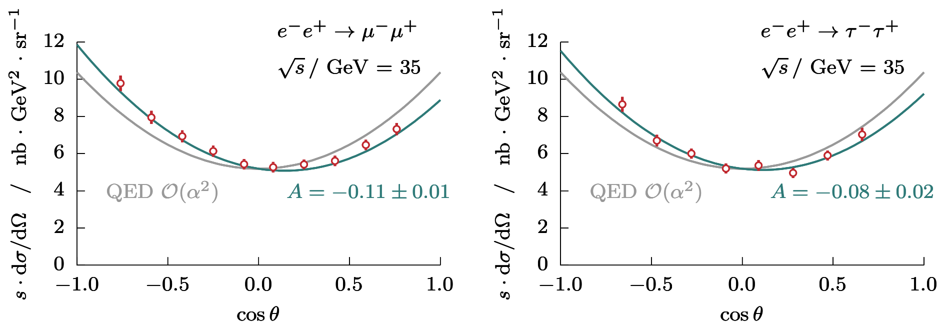


Figure 3: Plots of $s \cdot \mathrm{d}\sigma/\mathrm{d}\Omega$ vs. $\cos\theta$ for $\sqrt{s} = 35\,\mathrm{GeV}$ for the model cross-section including the asymmetry term $A\cos\theta$, compared with the JADE data for the reactions $e^-e^+ \to \mu^-\mu^+$ (left) and $e^-e^+ \to \tau^-\tau^+$ (right).

8. In supersymmetry (SUSY), each fermion has a scalar partner, and each gauge boson has a fermionic partner. For example, the supersymmetric partner of the muon is the spin-0 *smuon* ($\tilde{\mu}$), and the partner of the photon is the spin-1/2 *photino* ($\tilde{\gamma}$). These particles have yet to be discovered in nature, yet we can place bounds on some of their properties by performing precision experiments such as measuring the anomalous magnetic moment of the muon, $a_\mu \equiv (g_\mu - 2)/2$. In this problem, we will estimate bounds on the masses of these hypothetical particles.

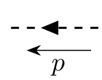
Let us consider a simple supersymmetric extension of the Standard Model which includes the smuon and the photino. The Lagrange density for this model is given by

$$\mathcal{L}_{\mathrm{SUSY}} = \mathcal{L}_{\mathrm{SM}} + \frac{i}{2}\bar{\chi}\overleftrightarrow{\partial\!\!\!/}\chi + \text{h.c.} - m_{\tilde{\gamma}}\bar{\chi}\chi$$
$$+ (D_\nu\varphi)^\dagger(D^\nu\varphi) - m_{\tilde{\mu}}^2\varphi^\dagger\varphi - q\varphi\bar{\psi}\chi + \text{h.c.}\,,$$

where $D_\nu = \partial_\nu + iqA_\nu$, A_ν is the photon field, ψ is the muon field, φ is the smuon field with mass term $m_{\tilde{\mu}}$, and χ is the photino field with mass term $m_{\tilde{\gamma}}$. The coupling q is the electric charge of the fields, e.g., $q = -e$ for the muon where e is the fundamental charge which is related to the fine-structure constant via $\alpha = e^2/4\pi \sim 1/137$. The smuon has the same electric charge as its Standard Model counterpart.

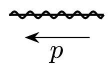
(a) Determine the Feynman rules for the SUSY model. That is, draw a diagram an associated factor for the smuon propagator, the photino propagator, and any interaction vertices with these two particles. **Hint:** You do not need to derive these using generating functionals, use your knowledge of other well-known field theories to determine the various quantities.

Solution: Using the Feynman rules for QED, Scalar QED, and Yukawa theory, we find the following rules. For the smuon propagator,



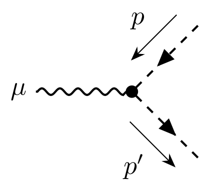
$$= \frac{i}{p^2 - m_{\tilde{\mu}}^2 + i\epsilon};$$

for the photino propagator



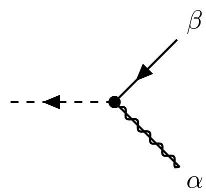
$$= \frac{i(\not{p} + m_{\tilde{\gamma}})}{p^2 - m_{\tilde{\gamma}}^2 + i\epsilon}\,;$$

the $\tilde{\mu}\tilde{\mu}\gamma$ vertex



$$= -iq(p + p')\,;$$

and for the $\tilde{\gamma}\tilde{\mu}\tilde{\mu}$ vertex is

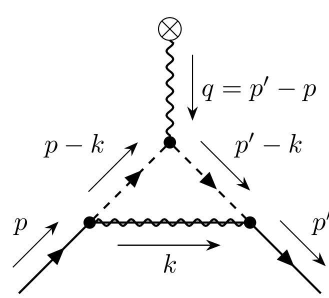


$$= -iq\,\delta_{\alpha\beta}\,;$$

(b) Draw the leading order contribution of this SUSY model to the $\mu\mu\gamma$ vertex function, $-iq\Gamma^\mu$, and write down the mathematical expression using the Feynman rules derived in part **(a)**. **Do Not** evaluate any integrals. Label all momenta and Lorentz indices. For the momenta, let p be the initial muon momentum, p' the final muon momentum, and q be the momentum transfer by the EM field, $q = p' - p$. The muon is on-mass shell, $p^2 = p'^2$. **Hint:** There is only a single contributing diagram.

Solution: Here we want the contribution to Γ^μ from the SUSY theory, which we define as $\delta\Gamma^\mu$. From the Feynman rules, we have

$$-iq\,\bar{u}(p')\,\delta\Gamma^\mu(p',p)\,u(p) =$$



,

where at leading order in α, the amplitude is

$$\delta\Gamma^\mu(p',p) = (-iq)^2 \int \frac{\mathrm{d}^4k}{(2\pi)^4}\, \frac{i(\not{k}+m_{\tilde{\gamma}})}{k^2-m_{\tilde{\gamma}}^2}\, \frac{i}{(p'-k)^2-m_{\tilde{\mu}}^2}\, \frac{i}{(p-k)^2-m_{\tilde{\mu}}^2}\,(p'+p-2k)^\mu\,,$$

$$= \frac{iq^2}{(2\pi)^4}\int \mathrm{d}^4k\, \frac{(p'+p-2k)^\mu(\not{k}+m_{\tilde{\gamma}})}{[k^2-m_{\tilde{\gamma}}^2][(p'-k)^2-m_{\tilde{\mu}}^2][(p-k)^2-m_{\tilde{\mu}}^2]}\,,$$

where we assume an implicit $+i\epsilon$ prescription in the denominators.

(c) Let's assume that $m_{\tilde{\mu}} \sim m_{\tilde{\gamma}} \sim \Lambda_{\text{SUSY}}$, where Λ_{SUSY} is a typical scale of SUSY interactions. One can show, assuming that $m_\mu \ll \Lambda_{\text{SUSY}}$, that the SUSY contribution to the muon anomaly a_μ is given by

$$\delta a_\mu^{\text{SUSY}} = \mathcal{C}\,\frac{\alpha}{4\pi}\,\frac{m_\mu}{\Lambda_{\text{SUSY}}}\,,$$

where $\mathcal{C}$ is a constant of $\mathcal{O}(1)$. Constrain the SUSY scale Λ_{SUSY} using the experimental and theoretical values of a_μ. Currently, the best experimental estimate for a_μ was recently measured to be $a_\mu^{(\text{ex.})} = 116\,592\,059(22)\times 10^{-11}$ by The Muon $g-2$ Collaboration (see D. P. Aguillard et al., Phys. Rev. Lett. **131**, 161802). Within the Standard Model, the best theoretical estimate for the anomaly is $a_\mu^{(\text{th.})} = 116\,591\,810(43)\times 10^{-11}$ from The Muon $g-2$ Theory Initiative (see Physics Reports 887 (2020) 1-166). Comment on why is the assumption $m_\mu \ll \Lambda_{\text{SUSY}}$ justifiable, and whether or not SUSY particles (within this model) can be detectable at the Large Hadron Collider.

Solution: To bound the SUSY scale, we note that the difference between the experimental and theoretical values is

$$\delta a_\mu = a_\mu^{(\text{ex.})} - a_\mu^{(\text{th.})}\,,$$

$$= 2.49(48)\times 10^{-9}\,.$$

We assume that $\mathcal{C}=1$, and estimate Λ_{SUSY} be requiring that $\delta a_\mu^{\text{SUSY}} \leq \delta a_\mu$,

$$\delta a_\mu \geq \frac{\alpha}{4\pi}\frac{m_\mu}{\Lambda_{\text{SUSY}}} \implies \Lambda_{\text{SUSY}} \geq \frac{\alpha}{4\pi}\frac{m_\mu}{\delta a_\mu}\,.$$

Since $\alpha \approx 1/137$, and $m_\mu \approx 105.7\,\text{MeV}$, we find

$$\Lambda_{\text{SUSY}} \geq \frac{\alpha}{4\pi}\frac{m_\mu}{\delta a_\mu}\,,$$

$$\geq 2.47\times 10^7\,\text{MeV}\,.$$

so, $\Lambda_{\text{SUSY}} \geq 25\,\text{TeV}$.

Since $m_\mu \approx 100\,\text{MeV}$, and we have not discovered SUSY particles up through scales 14 TeV (LHC energies), then the assumption that $m_\mu \ll \Lambda_{\text{SUSY}}$ is valid since we expect $\Lambda_{\text{SUSY}} > 14\,\text{TeV} \gg m_\mu$. Finding $\Lambda_{\text{SUSY}} \geq 25\,\text{TeV}$, the LHC today does not have the capabilities to discover particles within this SUSY theory.

(d) **Challenge (Optional):** Determine the constant $\mathcal{C}$ in $\delta a_\mu^{\rm SUSY}$ by evaluating the Feynman amplitude. To do this, perform the following steps

- Let P be the total momentum, $P = p + p'$, such that $P \cdot q = 0$. In the $q \to 0$ limit $P^2 = 4m_\mu^2$. Substitute $p = (P-q)/2$ and $p' = (P+q)/2$ in the Feynman integral. The vertex can then be written as a function $\Gamma^\mu(p', p) = \Gamma^\mu(P, q)$.
- Use the projection formula

$$\delta a_\mu = \frac{1}{12m_\mu^2}\,\mathrm{tr}\left[\left(m_\mu^2\gamma_\nu - P_\nu \not{P} - \frac{3}{2}m_\mu P_\nu\right)V^\nu + \frac{m_\mu}{4}\left(\frac{\not{P}}{2}+m_\mu\right)[\gamma_\nu,\gamma_\rho]\left(\frac{\not{P}}{2}+m_\mu\right)\delta V^{\rho,\nu}\right],$$

where $V^\nu = V^\nu(P) = \Gamma^\nu(P,0)$ and

$$\delta V^{\rho,\nu} = \delta V^{\rho,\nu}(P) = \left.\frac{\partial \Gamma^\nu}{\partial q_\rho}\right|_{q=0}.$$

Show that $\delta V^{\rho,\nu} = 0$ for this amplitude, and evaluate the remaining trace.
- Use the Feynman parameterization to combine the denominators of the Feynman integral. The relevant formula is

$$\frac{1}{A^2B} = 2\int_0^1 \mathrm{d}x\,\frac{x}{[x(A-B)+B]^3}.$$

- Perform the convergent Feynman integrals using the following formulae

$$\int \mathrm{d}^4k\,\frac{1}{[k^2+2P\cdot k - M^2]^3} = -\frac{i\pi^2}{2}\frac{1}{P^2+M^2},$$

$$\int \mathrm{d}^4k\,\frac{k^\mu}{[k^2+2P\cdot k - M^2]^3} = \frac{i\pi^2}{2}\frac{P^\mu}{P^2+M^2},$$

$$\int \mathrm{d}^4k\,\frac{k^2-(k\cdot p)^2/m^2}{[k^2+2P\cdot k - M^2]^3} = \frac{6i\pi^2 m^2\alpha^2}{P^2+M^2},$$

where $P = \alpha p$ with α being some factor independent of k.
- Simplify the expression by setting $m_{\tilde\gamma} = m_{\tilde\mu} \equiv \Lambda_{\rm SUSY}$, and assuming $m_\mu \ll \Lambda_{\rm SUSY}$. The remaining Feynman parameter integral over x can then be analytically evaluated.

Below I outline my approach. Recall the matrix element is

$$\delta\Gamma^\mu(p',p) = \frac{iq^2}{(2\pi)^4}\int \mathrm{d}^4k\,\frac{(p'+p-2k)^\mu(\not{k}+m_{\tilde\gamma})}{[k^2-m_{\tilde\gamma}^2][(p'-k)^2-m_{\tilde\mu}^2][(p-k)^2-m_{\tilde\mu}^2]}.$$

Now, $P = p'+p$, $q = p'-p$, so that $p = (P-q)/2$ and $p' = (P+q)/2$, so

$$\delta\Gamma^\mu(P,q) = \frac{iq^2}{(2\pi)^4}\int \mathrm{d}^4k\,\frac{(P-2k)^\mu(\not{k}+m_{\tilde\gamma})}{[k^2-m_{\tilde\gamma}^2][((P+q)/2-k)^2-m_{\tilde\mu}^2][(P-q)/2-k)^2-m_{\tilde\mu}^2]}.$$

Now, V^μ is

$$V^\mu(P) = \delta\Gamma^\mu(P,0)\,,$$
$$= \frac{iq^2}{(2\pi)^4}\int \mathrm{d}^4k\,\frac{(P-2k)^\mu(\not{k}+m_{\tilde{\gamma}})}{[k^2-m_{\tilde{\gamma}}^2][(P/2-k)^2-m_{\tilde{\mu}}^2]^2}\,,$$

and the $\delta V^\nu,\mu$ term is

$$\delta V^\nu,\mu(P) = \left.\frac{\partial\Gamma^\mu(P,q)}{\partial q_\nu}\right|_{q=0}\,,$$
$$= \frac{iq^2}{(2\pi)^4}\int \mathrm{d}^4k\,\frac{(P-2k)^\mu(\not{k}+m_{\tilde{\gamma}})}{[k^2-m_{\tilde{\gamma}}^2]}$$
$$\times \frac{\partial}{\partial q_\nu}\left.\frac{1}{[((P+q)/2-k)^2-m_{\tilde{\mu}}^2][(P-q)/2-k)^2-m_{\tilde{\mu}}^2]}\right|_{q=0}\,,$$
$$= 0\,,$$

which can be seen by using the product rule on the denominator, and recognizing that the second term is equivalent to the first except for and overall sign, thus they cancel.

Given this, we find that the muon anomaly is

$$\delta a_\mu^{\mathrm{SUSY}} = \frac{iq^2}{(2\pi)^4\,12m_\mu^2}\int \mathrm{d}^4k\,\frac{(P-2k)^\nu}{[k^2-m_{\tilde{\gamma}}^2][(P/2-k)^2-m_{\tilde{\mu}}^2]^2}$$
$$\times\,\mathrm{tr}\left[\left(m_\mu^2\gamma_\nu - P_\nu\not{P} - \frac{3}{2}m_\mu P_\nu\right)(\not{k}+m_{\tilde{\gamma}})\right]\,.$$

We evaluate the trace,

$$\mathrm{tr}\left[\left(m_\mu^2\gamma_\nu - P_\nu\not{P} - \frac{3}{2}m_\mu P_\nu\right)(\not{k}+m_{\tilde{\gamma}})\right] = m_\mu^2\,\mathrm{tr}(\gamma_\nu\not{k}) - P_\nu\,\mathrm{tr}(\not{P}\not{k}) - \frac{3}{2}m_\mu m_{\tilde{\gamma}}P_\nu\,\mathrm{tr}(I_4)\,,$$
$$= 4m_\mu^2 k_\nu - 4P_\nu P\cdot k - 6m_\mu m_{\tilde{\gamma}}P_\nu\,,$$

and then contract with $(P-2k)^\nu$,

$$(P-2k)^\nu\,\mathrm{tr}\left[\left(m_\mu^2\gamma_\nu - P_\nu\not{P} - \frac{3}{2}m_\mu P_\nu\right)(\not{k}+m_{\tilde{\gamma}})\right]$$
$$= (P-2k)^\nu\left[4m_\mu^2k_\nu - 4P_\nu P\cdot k - 6m_\mu m_{\tilde{\gamma}}P_\nu\right]\,,$$
$$= \left[4m_\mu^2(k\cdot P-2k^2) - 4(P^2-2k\cdot P)P\cdot k - 6m_\mu m_{\tilde{\gamma}}(P^2-2k\cdot P)\right]\,,$$
$$= \left[4m_\mu^2(k\cdot P-2k^2) - 4(4m_\mu^2-2k\cdot P)P\cdot k - 6m_\mu m_{\tilde{\gamma}}(4m_\mu^2-2k\cdot P)\right]\,,$$
$$= -8m_\mu^2k^2 + 8(P\cdot k)^2 - 12(P\cdot k)(m_\mu^2 - m_\mu m_{\tilde{\gamma}}) - 24m_\mu^3 m_{\tilde{\gamma}}\,,$$
$$= -12m_\mu^2\left[\frac{2}{3}\left(k^2 - \frac{(P\cdot k)^2}{m_\mu^2}\right) + \left(1-\frac{m_{\tilde{\gamma}}}{m_\mu}\right)(P\cdot k) + 2m_\mu m_{\tilde{\gamma}}\right]\,.$$

So, we have for $\delta a_\mu^{\rm SUSY}$

$$\delta a_\mu^{\rm SUSY} = -\frac{iq^2}{(2\pi)^4}\int \mathrm{d}^4k\,\frac{1}{[k^2-m_{\tilde\gamma}^2][(P/2-k)^2-m_{\tilde\mu}^2]^2} \times\left[\frac{2}{3}\left(k^2-\frac{(P\cdot k)^2}{m_\mu^2}\right)+\left(1-\frac{m_{\tilde\gamma}}{m_\mu}\right)(P\cdot k)+2m_\mu m_{\tilde\gamma}\right].$$

Next, we combine the denominators via the Feynman parameterization,

$$\frac{1}{A^2B} = 2\int_0^1 \mathrm{d}x\,\frac{x}{[x(A-B)+B]^3}\,,$$

where $A = (P/2-k)^2 - m_{\tilde\mu}^2 = k^2 - P\cdot k + m_\mu^2 - m_{\tilde\mu}^2$ and $B = k^2 - m_{\tilde\gamma}^2$. So, $A - B = -P\cdot k + m_\mu^2 + m_{\tilde\gamma}^2 - m_{\tilde\mu}^2$. Then,

$$\frac{1}{[(P/2-k)^2-m_{\tilde\mu}^2]^2[k^2-m_{\tilde\gamma}^2]} = 2\int_0^1\mathrm{d}x\,\frac{x}{[x(-P\cdot k+m_\mu^2+m_{\tilde\gamma}^2-m_{\tilde\mu}^2)+k^2-m_{\tilde\gamma}^2]^3}\,,$$
$$\equiv 2\int_0^1\mathrm{d}x\,\frac{x}{[k^2+2\hat P\cdot k-\hat M^2]^3}\,,$$

where we have defined $\hat P \equiv -xP/2$ and $\hat M^2 \equiv m_{\tilde\gamma}^2 - x(m_\mu^2+m_{\tilde\gamma}^2-m_{\tilde\mu}^2) = m_{\tilde\gamma}^2(1-x) - x(m_\mu^2 - m_{\tilde\mu}^2)$. So, the muon anomaly becomes

$$\delta a_\mu^{\rm SUSY} = -\frac{2iq^2}{(2\pi)^4}\int_0^1\mathrm{d}x\,x\int\mathrm{d}^4k\,\frac{1}{[k^2+2\hat P\cdot k-\hat M^2]^3} \times\left[\frac{2}{3}\left(k^2-\frac{(P\cdot k)^2}{m_\mu^2}\right)+\left(1-\frac{m_{\tilde\gamma}}{m_\mu}\right)(P\cdot k)+2m_\mu m_{\tilde\gamma}\right],$$
$$\equiv -\frac{2iq^2}{(2\pi)^4}\int_0^1\mathrm{d}x\,x\left[\frac{2}{3}\hat{\mathcal I}(\hat P,\hat M^2;P,m_\mu) + \left(1-\frac{m_{\tilde\gamma}}{m_\mu}\right)P^\nu\mathcal I_\mu(\hat P,\hat M^2)+2m_\mu m_{\tilde\gamma}\mathcal I(\hat P,\hat M^2)\right],$$

where we have defined

$$\mathcal I(\hat P,\hat M^2)\equiv\int\mathrm{d}^4k\,\frac{1}{[k^2+2\hat P\cdot k-\hat M^2]^3} = -\frac{i\pi^2}{2}\frac{1}{\hat P^2+\hat M^2}\,,$$
$$\mathcal I_\mu(\hat P,\hat M^2)\equiv\int\mathrm{d}^4k\,\frac{k_\mu}{[k^2+2\hat P\cdot k-\hat M^2]^3} = -\frac{i\pi^2 x}{4}\frac{P_\mu}{\hat P^2+\hat M^2}\,,$$
$$\hat{\mathcal I}(\hat P,\hat M^2,P,m_\mu)\equiv\int\mathrm{d}^4k\,\frac{(k^2-(k\cdot P)^2/m_\mu^2)}{[k^2+2\hat P\cdot k-\hat M^2]^3} = \frac{6i\pi^2m_\mu^2\alpha^2}{\hat P^2+\hat M^2} = \frac{3}{2}\frac{i\pi^2m_\mu^2x^2}{\hat P^2+\hat M^2}\,,$$

where in the second equalities we used the formulae given in the prompt and $\hat{P} = \alpha P = -(x/2)P$. Substituting these evaluations in $\delta a_\mu^{\mathrm{SUSY}}$, we find

$$\begin{aligned}\delta a_\mu^{\mathrm{SUSY}} &= -\frac{2iq^2(i\pi^2)}{(2\pi)^4}\int_0^1 \mathrm{d}x \left[m_\mu^2 x^2 - \left(1-\frac{m_{\tilde\gamma}}{m_\mu}\right)\frac{x}{4}P^2 - m_\mu m_{\tilde\gamma}\right]\frac{x}{\hat{P}^2+\hat{M}^2}\,,\\ &= \frac{q^2}{8\pi^2}\,m_\mu^2\int_0^1 \mathrm{d}x\left[x^2 - \left(1-\frac{m_{\tilde\gamma}}{m_\mu}\right)x - \frac{m_{\tilde\gamma}}{m_\mu}\right]\frac{x}{\hat{P}^2+\hat{M}^2}\,,\end{aligned}$$

where we used $P^2 = 4m_\mu^2$. The denominator is written as

$$\begin{aligned}\hat{P}^2+\hat{M}^2 &= x^2P^2/4 + m_{\tilde\gamma^2}(1-x) - x(m_\mu^2 - m_{\tilde\mu^2})\,,\\ &= m_\mu^2\left[x^2 + \left(\frac{m_{\tilde\gamma}}{m_\mu}\right)^2(1-x) - x\left(1-\left(\frac{m_{\tilde\mu}}{m_\mu}\right)^2\right)\right]\,,\\ &\equiv m_\mu^2(x^2+\epsilon_{\tilde\gamma}^2(1-x) - x(1-\epsilon_{\tilde\mu}^2))\,,\end{aligned}$$

where we have defined the mass ratios $\epsilon_{\tilde\mu} \equiv m_{\tilde\mu}/m_\mu$ and $\epsilon_{\tilde\gamma} \equiv m_{\tilde\gamma}/m_\mu$. Thus, the contribution to a_μ is

$$\delta a_\mu^{(\mathrm{SUSY})} = \frac{\alpha}{2\pi}\int_0^1 \mathrm{d}x\,\frac{x^3-(1-\epsilon_{\tilde\gamma})x^2 - \epsilon_{\tilde\gamma}x}{x^2+\epsilon_{\tilde\gamma}^2(1-x)-x(1-\epsilon_{\tilde\mu}^2)}$$

where we used $q^2 = e^2$ and $\alpha = e^2/4\pi$.

We assume that $m_{\tilde\mu} = m_{\tilde\gamma} = \Lambda_{\mathrm{SUSY}}$, so that $\epsilon_{\tilde\mu} = \epsilon_{\tilde\mu} \equiv \epsilon_\Lambda = \Lambda_{\mathrm{SUSY}}/m_\mu$. So,

$$\delta a_\mu^{(\mathrm{SUSY})} = \frac{\alpha}{2\pi}\int_0^1 \mathrm{d}x\,\frac{x^3-(1-\epsilon_\Lambda)x^2-\epsilon_\Lambda x}{x^2+\epsilon_\Lambda^2(1-x)-x(1-\epsilon_\Lambda^2)}\,.$$

Next, it is reasonable to assume that $\Lambda_{\mathrm{SUSY}}/m_\mu = \epsilon_\Lambda \gg 1$. Therefore, since x is bounded in the integrand $0 \le x \le 1$, we approximate the integral as

$$\begin{aligned}\delta a_\mu^{(\mathrm{SUSY})} &= \frac{\alpha}{2\pi}\frac{1}{\epsilon_\Lambda}\int_0^1 \mathrm{d}x\,x(x-1)\,,\\ &= \frac{\alpha}{2\pi}\frac{1}{\epsilon_\Lambda}\left(\frac{x^3}{3}-\frac{x^2}{2}\right)\Bigg|_0^1\,,\\ &= \frac{\alpha}{2\pi}\frac{1}{\epsilon_\Lambda}\left(\frac{1}{3}-\frac{1}{2}\right)\,,\\ &= -\frac{\alpha}{12\pi}\,\frac{m_\mu}{\Lambda_{\mathrm{SUSY}}}\,.\end{aligned}$$

So, we conclude that the constant $\mathcal{C} = -1/3$. Let us revisit the bound on Λ_{SUSY},

$$\begin{aligned}\Lambda_{\text{SUSY}} &\geq \frac{\alpha}{12\pi}\,\frac{m_\mu}{\delta a_\mu}\,,\\ &\geq 8.22\times 10^6\,\text{MeV},\end{aligned}$$

or $\Lambda_{\text{SUSY}} \geq 8\,\text{TeV}$. Since the LHC operates at energies $\sim 14\,\text{TeV}$, these particles should in principle be detectable. However, to date no SUSY particles have been discovered, pushing the bound to $\Lambda_{\text{SUSY}} \geq 14\,\text{TeV}$.