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PHYS 772 – The Standard Model of Particle Physics

Problem Set 8 – Solution

Due: Tuesday, April 08 at 4:00pm

Term: Spring 2025

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1. Can the following hadrons, in principle, exist within QCD? **(a)** qq, **(b)** $qq\bar{q}$, **(c)** $qq\bar{q}\bar{q}$, **(d)** gg, **(e)** qqg, **(f)** $q\bar{q}g$, **(g)** $qqqq\bar{q}$. **Hint:** Consider $\mathrm{SU}(3)_c$ symmetry transformations of observable hadrons. Gluons transform under the adjoint representation of $\mathrm{SU}(3)_c$.

Solution: Hadrons within QCD must be color neutral, that is a hadron h *must* belong to the $\mathbf{1}$ representation of $\mathrm{SU}(3)_c$. So, all we need to find is if the given combinations of quarks and gluons admit a singlet representation. Recall that quarks lie in the $\mathbf{3}$ of $\mathrm{SU}(3)_c$, antiquarks lie in the $\mathbf{3}^*$ of $\mathrm{SU}(3)_c$, and gluons lie in the $\mathbf{8}$ of $\mathrm{SU}(3)_c$.

So, for (a)

$$qq \to \mathbf{3} \times \mathbf{3} = \mathbf{3}^* + \mathbf{6} \not\supset \mathbf{1}\,,$$

therefore qq is **not** a valid hadron.

For (b), we have (recalling that $\mathbf{3} \times \mathbf{3}^* = \mathbf{1} + \mathbf{8}$),

$$qq\bar{q} \to \mathbf{3} \times \mathbf{3} \times \mathbf{3}^* = \mathbf{3} \times (\mathbf{1} \times \mathbf{8}) \not\supset \mathbf{1}\,,$$

since the $\mathbf{3} \times \mathbf{8} = \mathbf{3} + \mathbf{6}^* + \mathbf{15}$ which was found in Problem Set 7. Therefore, $qq\bar{q}$ is **not** a valid hadron.

For (c), $qq\bar{q}\bar{q}$ is

$$\begin{aligned} qq\bar{q}\bar{q} \to \mathbf{3} \times \mathbf{3} \times \mathbf{3}^* \times \mathbf{3}^* &= (\mathbf{3} \times \mathbf{3}^*) \times (\mathbf{3} \times \mathbf{3}^*)\,, \\ &= (\mathbf{1} + \mathbf{8}) \times (\mathbf{1} + \mathbf{8}) \supset \mathbf{1}\,. \end{aligned}$$

So, $qq\bar{q}\bar{q}$ **is** a valid hadron. These are *tetraquarks*, which candidates have been observed in the heavy quark sector, e.g., the $Z_c(3900)$.

For (d), gg, we need the product $\mathbf{8} \times \mathbf{8}$. From lecture, we worked out this product, and found it contains a singlet representation. Therefore,

$$gg \to \mathbf{8} \times \mathbf{8} \supset \mathbf{1}\,,$$

and thus is a valid hadron. These are *glueballs*, bound states of gluons. There is suspicion that higher mass states in the $J^{PC} = 0^{+}+$ and 2^{++} sectors contain strong mixing into these glueball states.

For (e), qqg, we have $\mathbf{3}\times\mathbf{3}\times\mathbf{8}=\mathbf{3}^*+\mathbf{3}^*+\mathbf{6}+\mathbf{6}+\mathbf{15}^*+\mathbf{15}^*+\mathbf{24}$ from Problem Set 7. So,

$$qqg \to \mathbf{3}\times\mathbf{3}\times\mathbf{8} \not\supset \mathbf{1}\,,$$

and thus is **not** a valid hadron.

For (f), $q\bar{q}g$, we have from Problem Set 7, $\mathbf{3}\times\mathbf{3}^*\times\mathbf{8}=\mathbf{1}+\mathbf{8}+\mathbf{8}+\mathbf{8}+\mathbf{10}+\mathbf{10}^*+\mathbf{27}$. So,

$$q\bar{q}g \to \mathbf{3}\times\mathbf{3}^*\times\mathbf{8} \supset \mathbf{1}\,.$$

Therefore, $q\bar{q}g$ **is** a valid hadron. These are *hybrid* mesons, which had a substantial component from excited glue. The $\pi_1(1600)$ is an observed hybrid candidate.

2. Consider a non-abelian gauge field $A_\mu \equiv A^j_\mu T_j$, where $T_j \in \mathfrak{su}(N)$ are generators satisfying the Lie algebra $[T_j,T_k]=ic_{jkl}T_l$ with c_{jkl} being structure constants and $j,k,l=1,2,\ldots,N^2-1$. Under a local gauge transformation, $U=\exp(i\alpha^j(x)T_j)$ where $\alpha_j(x)\in\mathbb{R}$ for every j, the gauge fields transform as

$$A_\mu \to UA_\mu U^{-1}+\frac{i}{g}\left(\partial_\mu U\right)U^{-1}\,.$$

Show that under infinitesimal transformations, $\alpha^a(x)\ll 1$, the gauge fields transform as

$$A^j_\mu \to A^j_\mu - \frac{1}{g}\partial_\mu\alpha^j(x) - c_{jkl}\,\alpha^k A^l_\mu + \mathcal{O}(\alpha^2)\,.$$

Solution: Taking $\alpha^j(x)\ll 1$ for all $j=1,2,\ldots,N^2-1$, w can Taylor expand the exponential

$$U=\exp(i\alpha^j(x)T_j)=1+i\alpha^j(x)T_j+\mathcal{O}(\alpha^2)\,.$$

So, the gauge transformation is

$$\begin{aligned}
A^j_\mu T_j &\to UA^j_\mu T_j U^{-1}+\frac{i}{g}(\partial_\mu U)U^{-1}\,,\\
&=(1+i\alpha^j T_j+\mathcal{O}(\alpha^2))A^k_\mu T_k(1-i\alpha^l T_l+\mathcal{O}(\alpha^2))\\
&\quad+\frac{i}{g}\partial_\mu(1+i\alpha^j T_j+\mathcal{O}(\alpha^2))(1+i\alpha^k T_k+\mathcal{O}(\alpha^2))\,,\\
&=A^j_\mu T_j+i\alpha^k A^l_\mu(T_kT_l-T_lT_k)-\frac{1}{g}\partial_\mu\alpha^j T_j+\mathcal{O}(\alpha^2)\,,\\
&=A^j_\mu T_j+i\alpha^k A^l_\mu(ic_{klj}T_j)-\frac{1}{g}\partial_\mu\alpha^j T_j+\mathcal{O}(\alpha^2)\,,\\
&=\left(A^j_\mu-c_{jkl}\alpha^k A^l_\mu-\frac{1}{g}\partial_\mu\alpha^j+\mathcal{O}(\alpha^2)\right)T_j\,.
\end{aligned}$$

Therefore, the infinitesimal transformation gives

$$A^j_\mu \to A^j_\mu-\frac{1}{g}\partial_\mu\alpha^j-c_{jkl}\alpha^k A^l_\mu+\mathcal{O}(\alpha^2)\,.$$

3. The $SU(3)_c$ Yang-Mills Lagrange density for interacting gluon fields is given by $\mathcal{L}_{\rm YM} = -\frac{1}{2}\,\mathrm{tr}\,(G_{\mu\nu}G^{\mu\nu})$, where the field-strength tensor is defined as $G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig_s[A_\mu, A_\nu]$ with $A_\mu = A_\mu^a\,\lambda_a/2$ are the gluon gauge fields and λ_a are the Gell-Mann matrices. Write the Lagrange density as a free part $\mathcal{L}_{\rm YM}^{(\rm free)}$ and an interacting part $\mathcal{L}_{\rm YM}^{(\rm int)}$ which depends on the strong coupling g_s.

Solution: Contracting the field strength tensors,

$$\begin{aligned} G_{\mu\nu}G^{\mu\nu} &= \Big(\partial_\mu A_\nu - \partial_\nu A_\mu + ig_s[A_\mu, A_\nu]\Big)\Big(\partial^\mu A^\nu - \partial^\nu A^\mu + ig_s[A^\mu, A^\nu]\Big)\,, \\ &= \left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) \\ &\quad + ig_s\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)[A^\mu, A^\nu] + ig_s[A_\mu, A_\nu]\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) \\ &\quad - g_s^2[A_\mu, A_\nu][A^\mu, A^\nu]\,. \end{aligned}$$

Now, we use that $A_\mu = A_\mu^a\,T_a$ where $T_a = \lambda_a/2$, so

$$\begin{aligned} G_{\mu\nu}G^{\mu\nu} = &\left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a\right)\left(\partial^\mu A^{\nu\,b} - \partial^\nu A^{\mu\,b}\right)T_aT_b \\ &+ ig_s\left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a\right)A^{\mu\,b}A^{\nu\,c}\,T_a[T_b, T_c] + ig_sA_\mu^aA_\nu^b\left(\partial_\mu A^{\nu\,c} - \partial_\nu A^{\mu\,c}\right)[T_a, T_b]T_c \\ &- g_s^2A_\mu^aA_\nu^bA^{\mu\,c}A^{\nu\,d}\,[T_a, T_b][T_c, T_d]\,. \end{aligned}$$

Furthermore, $[T_a, T_b] = if_{abc}T_c$, so

$$\begin{aligned} G_{\mu\nu}G^{\mu\nu} = &\left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a\right)\left(\partial^\mu A^{\nu\,b} - \partial^\nu A^{\mu\,b}\right)T_aT_b \\ &+ ig_s\left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a\right)A^{\mu\,b}A^{\nu\,c}\,T_a(if_{bcd}T_d) + ig_sA_\mu^aA_\nu^b\left(\partial_\mu A^{\nu\,c} - \partial_\nu A^{\mu\,c}\right)(if_{abd}T_d)T_c \\ &- g_s^2A_\mu^aA_\nu^bA^{\mu\,c}A^{\nu\,d}\,(if_{abe}T_e)(if_{cdf}T_f)\,. \end{aligned}$$

Now, taking the trace, we use $\mathrm{tr}(T_aT_b) = \mathrm{tr}(\lambda_a\lambda_b)/4 = \delta_{ab}/2$, so the Yang-Mills Lagrange density is

$$\begin{aligned} \mathcal{L}_{\rm YM} = &-\frac{1}{4}\left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a\right)\left(\partial^\mu A^{\nu\,a} - \partial^\nu A^{\mu\,a}\right) \\ &\quad - \frac{1}{4}g_sf_{bca}\left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a\right)A^{\mu\,b}A^{\nu\,c} - \frac{1}{4}g_sf_{abc}A_\mu^aA_\nu^b\left(\partial_\mu A^{\nu\,c} - \partial_\nu A^{\mu\,c}\right) \\ &\quad + \frac{1}{4}g_s^2f_{abe}f_{cde}\,A_\mu^aA_\nu^bA^{\mu\,c}A^{\nu\,d}\,, \\ = &-\frac{1}{4}\left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a\right)\left(\partial^\mu A^{\nu\,a} - \partial^\nu A^{\mu\,a}\right) \\ &\quad - \frac{1}{2}g_sf_{abc}\left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a\right)A^{\mu\,b}A^{\nu\,c} + \frac{1}{4}g_s^2f_{abe}f_{cde}\,A_\mu^aA_\nu^bA^{\mu\,c}A^{\nu\,d}\,, \\ \equiv &\;\mathcal{L}_{\rm YM}^{(\rm free)} + \mathcal{L}_{\rm YM}^{(\rm int)}\,, \end{aligned}$$

where we used $f_{bca} = f_{abc}$ from the antisymmetry properties of the structure constants. So, the free and interacting Lagrange densities are

$$\mathcal{L}_{\rm YM}^{(\rm free)} = -\frac{1}{4}\left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a\right)\left(\partial^\mu A^{\nu\, a} - \partial^\nu A^{\mu\, a}\right)\,,$$

$$\mathcal{L}_{\rm YM}^{(\rm int)} = -\frac{1}{2} g_s f_{abc}\,\left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a\right) A^{\mu\, b} A^{\nu\, c} + \frac{1}{4} g_s^2\, f_{abe} f_{cde}\, A_\mu^a A_\nu^b A^{\mu\, c} A^{\nu\, d}\,.$$

4. We can learn about the structure of hadrons through interactions with electromagnetic probes. Consider elastic $e^- p \to e^- p$ for incident electron energies $E_e \gg m_p$. At leading order in the QED coupling, the process is dominated by one-photon exchange. The QED vertex for the proton can in general be written as

$$\Gamma_p^\mu(P', P) = \gamma^\mu F_1(Q^2) + \frac{i}{2m_p}\sigma^{\mu\nu} q_\nu F_2(Q^2)\,,$$

where P and P' are the initial and final momentum of the proton, respectively, and $q = P' - P$ is the momentum transfer by the photon with virtuality $Q^2 \equiv -q^2$. The form-factors F_1 and F_2 encode all the non-perturbative QCD interactions with the photon.

(a) Show that, in the initial proton rest frame, that the ratio of the final to initial electron energy is

$$\frac{E_e'}{E_e} = \left(1 + \frac{2E_e}{m_p}\sin^2\frac{\theta}{2}\right)^{-1}\,,$$

where E_e and E_e' are the initial and final electron energies, respectively, and θ is the scattering angle defined with respect to the incident electron momentum.

Solution: In the target system for relativistic electrons, $p_e = (E_e, \mathbf{p}_e)$, $p_e' = (E_e', \mathbf{p}_e')$ where $|\mathbf{p}_e| = E_e$, $|\mathbf{p}_e'| = E_e'$ and $\hat{\mathbf{p}}_e \cdot \hat{\mathbf{p}}_e' = \cos\theta$. The kinematics for the initial and final state protons are $P = (m_p, \mathbf{0})$ and $P' = (E_p', \mathbf{P}')$. Conservation of energy yields

$$E_e + m_p = E_e' + E_p' \implies (E_e + m_p - E_e')^2 = m_p^2 + \mathbf{P}'^2\,,$$

while conservation of momentum yields

$$\mathbf{p}_e = \mathbf{p}_e' + \mathbf{P}' \implies \mathbf{P}'^2 = (\mathbf{p}_e - \mathbf{p}_e')^2 = E_e^2 + E_e'^2 - 2E_e E_e' \cos\theta\,,$$

thus we find

$$\begin{aligned}(E_e + m_p - E_e')^2 &= m_p^2 + E_e^2 + E_e'^2 - 2E_e E_e'\cos\theta\,,\\ &\implies 2m_p E_e - 2m_p E_e' - 2E_e E_e' = -2E_e E_e'\cos\theta\,,\\ &\implies E_e - E_e' = \frac{E_e E_e'}{m_p}(1-\cos\theta)\,,\\ &\implies \frac{E_e}{E_e'} - 1 = \frac{2E_e}{m_p}\sin^2\frac{\theta}{2}\,,\end{aligned}$$

where we used the trigonometric identity $2\sin^2\theta/2 = 1 - \cos\theta$. Solving for E_e'/E_e we recover the expression desired.

(b) Show that, in the initial proton rest frame, that the differential cross-section is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2}{4E_e^2 \sin^4\frac{\theta}{2}} \frac{E_e'}{E_e} \left[\left(F_1^2 + \frac{Q^2}{4m_p^2} F_2^2 \right) \cos^2\frac{\theta}{2} + \frac{Q^2}{4m_p^2} \left(F_1 + F_2\right)^2 \sin^2\frac{\theta}{2} \right] ,$$

where Ω is the solid angle defined in the initial proton rest frame. **Hint:** Use the Gordon identity to rewrite the proton-photon vertex as

$$\Gamma^\mu = \gamma^\mu \left(F_1 + F_2\right) - \frac{(P' + P)^\mu}{2m_p} F_2 ,$$

for simpler trace relations.

Solution: The scattering amplitude at leading order in the QED coupling is

$$i\mathcal{M} = -(-ie)^2 \bar{u}(\mathbf{p}_e', r')\gamma_\mu u(\mathbf{p}_e, r) \frac{i}{Q^2} \bar{u}(\mathbf{P}', s')\Gamma_p^\mu u(\mathbf{P}, s) ,$$

so that the spin-averaged matrix element is

$$\begin{aligned}\langle |\mathcal{M}|^2 \rangle &= \frac{1}{4} \sum_{s,s'} \sum_{r,r'} |\mathcal{M}|^2 , \\ &= \frac{(4\pi\alpha)^2}{4Q^4} L_{\mu\nu} H^{\mu\nu} ,\end{aligned}$$

where the lepton tensor $L_{\mu\nu}$ is

$$L_{\mu\nu} = \mathrm{tr}\left[\gamma_\mu \not{p}_e \gamma_\nu \not{p}_e'\right] ,$$

and the hadron tensor $H^{\mu\nu}$ is

$$H^{\mu\nu} = \mathrm{tr}\left[\Gamma_p^\mu (\not{P} + m_p) \Gamma_p^\nu (\not{P}' + m_p)\right] .$$

Using the trace theorems, the lepton tensor evaluates to

$$L_{\mu\nu} = 4(p_e^\mu p_e'^\nu + p_e^\nu p_e'^\mu - g^{\mu\nu} p_e \cdot p_e') ,$$

with $p_e \cdot p_e' = -(p_e - p_e')^2/2 = Q^2/2$. Using the Gordon identity, the Hadronic tensor can be written as

$$\begin{aligned}H^{\mu\nu} &= \mathrm{tr}\left[\Gamma_p^\mu (\not{P} + m_p) \Gamma_p^\nu (\not{P}' + m_p)\right] , \\ &= \mathrm{tr}\left[\left(\gamma^\mu (F_1 + F_2) - \frac{(P' + P)^\mu}{2m_p} F_2\right)(\not{P} + m_p)\left(\gamma^\nu (F_1 + F_2) - \frac{(P' + P)^\nu}{2m_p} F_2\right)(\not{P}' + m_p)\right] .\end{aligned}$$

To perform the trace, we use `Feyncalc` and subsequently contract with the leptonic tensor to find

$$\langle |\mathcal{M}|^2 \rangle = \frac{4(4\pi\alpha)^2 m_p^2}{Q^4} \left[-\frac{Q^2}{2m_p^2}(F_1 + F_2)^2 + 4\left(F_1^2 + \frac{Q^2}{4m_p^2} F_2^2\right) E_e E_e' \cos^2\frac{\theta}{2} \right] ,$$

where we have additionally used $Q^2 = -q^2 = -(p_e - p_e')^2 = -2E_eE_e'(1-\cos\theta) = -4E_eE_e'\sin^2\theta/2$. Some additional manipulations lead to the form

$$\langle|\mathcal{M}|^2\rangle = \frac{(16\pi\alpha)^2 m_p^2}{Q^4}\, E_eE_e' \left[\left(F_1^2 + \frac{Q^2}{4m_p^2}F_2^2\right)\cos^2\frac{\theta}{2} + \frac{Q^2}{4m_p^2}\left(F_1+F_2\right)^2\sin^2\frac{\theta}{2}\right].$$

Now, the differential cross section in the lab frame is given by

$$\begin{aligned}\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} &= \frac{1}{(8\pi)^2 m_p E_e}\,\frac{E_e'^2}{E_e'(E_e+m_p) - E_eE_e'\cos\theta}\,\langle|\mathcal{M}|^2\rangle\,,\\ &= \frac{1}{(8\pi)^2 m_p E_e}\,\frac{E_e'^2}{m_pE_e}\,\langle|\mathcal{M}|^2\rangle\,.\end{aligned}$$

where we used $m_p(E_e - E_e') = E_eE_e'(1-\cos\theta)$. Therefore, substituting the spin-averaged matrix element and using $Q^2 = -4E_eE_e'\sin^2\theta/2$, we find

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2}{4E_e^2\sin^4\frac{\theta}{2}}\left(\frac{E_e'}{E_e}\right)\left[\left(F_1^2 + \frac{Q^2}{4m_p^2}F_2^2\right)\cos^2\frac{\theta}{2} + \frac{Q^2}{4m_p^2}\left(F_1+F_2\right)^2\sin^2\frac{\theta}{2}\right],$$

as desired.

(c) Simplify the differential cross section for following limits: (i) the static source limit $m_p\to\infty$, and the (ii) structureless proton limit.

Solution: In the static source limit, $m_p\to\infty$, we find that $E_e = E_e'$ and $Q^2/m_p \to 0$, thus

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2}{4E_e^2\sin^4\frac{\theta}{2}}\cos^2\frac{\theta}{2}\,,$$

which is the Mott cross section with $F_1 = 1$ when $Q^2 = 0$ since $E_e = E_e'$. For a structureless proton, we have identically $F_1 = 1$ and $F_2 = 0$, thus

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2}{4E_e^2\sin^4\frac{\theta}{2}}\left(\frac{E_e'}{E_e}\right)\left[\cos^2\frac{\theta}{2} + \frac{Q^2}{4m_p^2}\sin^2\frac{\theta}{2}\right].$$

(d) The Mott cross section (modified for proton recoil) is that of an electron on a spinless target,

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\text{Mott}} = \frac{\alpha^2}{4E_e^2\sin^4\frac{\theta}{2}}\left(\frac{E_e'}{E_e}\right)\cos^2\frac{\theta}{2}\,.$$

Show that the electron-proton scattering cross section can bet written in the Rosenbluth form as

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\text{Mott}}\left[\frac{G_E^2 + \tau G_M^2}{1+\tau} + 2\tau G_M^2\tan^2\frac{\theta}{2}\right],$$

where $\tau = Q^2/4m_p^2$ and we have introduced the Sachs electric and magnetic form factors,

$$G_E = F_1 - \tau F_2\,,\qquad G_M = F_1 + F_2\,.$$

The Sachs form factors are often easier to measure, and offer interpretations for Fourier transforms of electromagnetic charge distributions. Show that $G_E(0) = 1$ (unit proton charge) and $G_M(0) = \mu_p$ (proton magnetic moment).

Solution: Given the definition of the Sachs' form-factors, we can solve for F_1 and F_2,

$$F_1 = \frac{G_E + \tau G_M}{1+\tau}\,, \qquad F_2 = \frac{G_M - G_E}{1+\tau}\,.$$

Squaring each and summing them,

$$F_1^2 = \frac{1}{(1+\tau)^2}(G_E^2 + \tau^2 G_M^2 + 2\tau G_E G_M)\,, \qquad F_2^2 = \frac{1}{(1+\tau)^2}(G_E^2 + G_M^2 - 2G_E G_M)\,,$$

$$\implies F_1^2 + \tau F_2^2 = \frac{1}{(1+\tau)^2}\left((1+\tau)G_E^2 + \tau(1+\tau)G_M^2\right) = \frac{G_E^2 + \tau G_M^2}{1+\tau}\,.$$

So, the differential cross section is given by

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\text{Mott}} \left[\frac{G_E^2 + \tau G_M^2}{1+\tau} + 2\tau G_M^2 \tan^2\frac{\theta}{2}\right]\,,$$

as desired.

Recall that $F_1(0) = 1$ and $F_2(0) = \mu_p - 1$. Thus, we have $G_E(0) = 1$ and $G_M(0) = \mu_p$ identically.